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THE MATHEMATICS OF METAMATHEMATICS

WARSZAWA 1963

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Die Mathematiker sind eine Art Franzosen: redet man zu ihnen, so übersetzen sie es in ihre Sprache und dann ist es alsobald ganz etwas Anderes.

J. W. Goethe *

PREFACE

The title of this book is not meant as a pun, although it may, at first sight, appear to be so.

Metamathematics is a theory which deals with formalized mathematical theories. A formalized mathematical theory is, roughly speaking, a set of certain finite sequences of symbols, called formulas and terms, and of certain simple operations performed on those sequences. The formulas and terms are substitutes—formed by means of a few simple rules—of sentences and functions in an intuitive mathematical theory. Operations on formulas correspond to elementary steps of deduction in mathematical reasonings. The formulas corresponding to the axioms of the intuitive theory play a special part; they are the axioms of the formalized theory. The formulas which can be derived from the axioms by means of the accepted operations correspond to the theorems of the theory.

The set of all formulas and the set of all terms, when considered as sets of finite sequences with operations, can in turn be the subject of mathematical investigations employing more or less advanced auxiliary methods taken from mathematics. In the early period of the development of mathematical logic, the general tendency was to use only the most elementary methods possible, excluding all infinitistic methods. The precursor of that trend was Hilbert, who believed that in this way it would be possible to prove the consistency of mathematics. However, Gödel's results exposed the fiasco of Hilbert's finitistic methods as far as consistency is concerned. The use of the finitistic methods to investigate formalized theories is perhaps natural on account of the clearly finitistic character of the notion of formalized theory. In practice, however, the restriction of methods of proof to elementary finitistic ones complicates metamathematical investigations considerably. It also prevents a full recognition of the exact nature of formalized mathematical theories from the point of view of methods and ideas of modern mathematics. The use of the more advanced infinitistic methods makes it easier to explain the mathematical structure of formalized theories. The set of all terms of a formalized theory is an algebra, and generally an algebra with infinitely many operations. The set of all formulas of a formalized theory is also an algebra—in general, it is an algebra with infinite operations. After

* „Ferneres über Mathematik und Mathematiker", s. Werke, Grosse Weimarische Ausg. Abt. II, Bd. 11 (1893), s. 102.

the natural identification of equivalent formulas, the set of all formulas becomes a lattice: a Boolean algebra, a pseudo-Boolean algebra, a topological Boolean algebra, etc., depending on the type of the logic adopted in the theory. These algebras in turn are connected with the notions of a field of sets and a topological space. From this point of view it is natural to apply in metamathematics the methods of algebra, lattice theory, set theory and topology. The total sum of mathematical methods useful in metamathematics makes up what in the title of this book has been called the mathematics of metamathematics.

By means of infinitistic methods the meaning of many basic metamathematical theorems is made clear. The theorem on the completeness of the propositional calculus is seen to be exactly the same as Stone's theorem on the representation of Boolean algebras. The Gödel theorem on the completeness of the predicate calculus is a modification of the Stone representation theorem, taking into account some infinite operations in Boolean algebras. It is surprising that the Gödel completeness theorem can be obtained, for example, as a result of the Baire theorem on sets of the first category in topological spaces, etc.

The finitistic approach of Hilbert's school is completely abandoned in this book. On the contrary, the infinitistic methods, making use of the more profound ideas of mathematics, are distinctly favoured. This brings out clearly the mathematical structure of metamathematics. It also permits a greater simplicity and clarity in the proofs of the basic metamathematical theorems and emphasizes the mathematical contents of these theorems.

The title of this book is slightly inexact since not all mathematical methods used in metamathematics are exposed in it. Namely, Gödel's method of arithmetization has been omitted. The exact title of the book should be: Algebraic, lattice-theoretical, set-theoretical and topological methods in metamathematics. The arithmetization of metamathematics differs fundamentally from these methods and leads to different problems. That is why we did not consider it proper to include that subject. As a result, we have omitted that part of metamathematics which uses arithmetization in a natural way (the decision problem, the existence of undecidable sentences, etc.) and the theory of recursive functions, now being developed by a large number of mathematicians.

It is difficult to establish exactly who was the first to use infinitistic methods in metamathematics. The close relation between classical logic and the theory of Boolean algebras has been known for a long time. It was the investigations in logic of Boole himself that led to the notion which we now call the Boolean algebra. Stone's basic result concerning the representation of Boolean algebras permitted the broad application

of the theory of Boolean algebras to metamathematics. The method of treating the set of formulas or the set of equivalence classes of formulas as abstract algebras, due to Lindenbaum and Tarski, proved to be an essential research tool. It established a link between the metamathematics of the theories based on classical logic and the theory of Boolean algebras. The works of Stone and Tarski on the connection between the intuitionistic logic and pseudo-complemented lattices and the further works of McKinsey and Tarski on lattice-theoretical methods in intuitionistic and modal propositional calculi established an analogous link in the metamathematics of corresponding non-classical theories. Here another essential research approach is of chief importance: the interpretation of the formulas of propositional calculi as mappings in certain lattices. This interpretation is a generalization of the truth-table method, long used in logic. The extension of this method to the intuitionistic predicate calculus was first introduced by Mostowski for problems of non-deducibility of formulas. The method of interpretation of formulas as mappings, together with the method of identifying equivalent formulas and treating the set of equivalence classes as an abstract algebra has enabled us to give an algebraic-topological proof of the Gödel completeness theorem and of other basic theorems. The concept of the product of models modulo prime filters, introduced by Łoś and widely used by the Berkeley school, is another essential contribution to the mathematical concepts of metamathematics.

Research into infinitistic methods in metamathematics is now in full swing and is far from being completed. This book does not embrace the whole of the research conducted in this field. In particular it does not include Halmos's theory of polyadic algebras and that of cylindrical algebras worked out by Henkin, Tarski and Thompson. Neither does it deal with the theory of the languages with infinitely long formulas, which, through Hanf's latest results, has found application in mathematics itself, namely in the theory of prime filters in Boolean algebras (Tarski). Certain other applications of metamathematics to mathematics (the results of A. Robinson and others) and the general theory of models are also disregarded.

Moreover, the problems discussed in this book are far from being treated in an exhaustive manner. The purpose of this book is only to introduce the reader to the basic ideas of the infinitistic approach to metamathematics, especially into the methods directly connected with the authors' own work. The accompanying bibliography has no claims to completeness.

The book is written in an elementary way in the sense that it requires no knowledge of mathematics and metamathematics beyond that

of the basic notions of set theory: operations on sets, the notions of cardinal number and the transfinite induction. It does, however, assume a certain mathematical sophistication on the part of the reader.

All the mathematical knowledge necessary to understand the infinitistic methods of metamathematics is presented in Part One (Chapters I-IV). The reader will find there a brief exposition of the elementary ideas of topology and algebra, and an exposition of a part of lattice theory. The material contained in Part One was chosen entirely from the point of view of its application to metamathematics and it represents the minimum necessary to understand Parts Two and Three, which are concerned with the metamathematics of formalized theories based on classical logic or non-classical logics. Chapters III and IV can be omitted altogether by the reader interested only in classical logic. Chapter V constitutes an intuitive introduction to the technique of formalization of mathematical theories. It also contains a general definition of logic. Chapter VI is indispensable to the understanding of all the succeeding chapters. It contains the theory of the basic research tools in metamathematics: the interpretation of formulas as mappings and the construction of algebras by identification of equivalent formulas. Chapters VII and VIII deal with classical logic, Chapters IX and X with intuitionistic logic, and Chapter XI with positive and modal logic. Chapters IX and X can be read separately (after Chapters I-VI) irrespective of Chapters VII and VIII. In order to make things easier for the reader, we often give complete proofs in Chapters IX and X even when they are similar to the proofs of analogous theorems in classical logic. Positive and modal logics, on the other hand, have been treated rather superficially in Chapter XI and the proofs of theorems analogous to those of classical and intuitionistic logic have been omitted.

This book is addressed only to mathematicians and students of mathematics interested in the logical aspects of mathematics and the mathematical aspects of logic. For this reason the material illustrating logical problems given in the introductory Chapter V is taken only from mathematics. The philosophical aspects of logic and mathematics are completely omitted as foreign to the mathematical character of the book. Only in § 1 of Chapter IX is there a short summing up of the basic ideas which led to the rise of intuitionistic logic.

The inclusion of two chapters on intuitionism is not an indication of the authors' positive attitude towards intuitionistic ideas. Intuitionism, like other non-classical logics, has no practical application in mathematics. Nevertheless many authors devote their works to intuitionistic logic. On the other hand, the mathematical mechanism of intuitionistic logic is interesting: it is amazing that vaguely defined philosophical ideas

concerning the notion of existence in mathematics have led to the creation of formalized logical systems which, from the mathematical point of view, proved to be equivalent to the theory of lattices of open subsets of topological spaces. Finally, the formalization of intuitionistic logic achieved by Heyting and adopted in this book is not in agreement with the philosophical views of the founder of intuitionism, Brouwer, who opposed formalism in mathematics. Since in treating intuitionistic logic we have limited ourselves to problems which are directly connected with general algebraic, lattice-theoretical and topological methods employed in this book, we have not included the latest results of Beth and Kreisel concerning other notions of satisfiability than the algebraic notion of satisfiability which we have adopted.

We have given very little space to set-theoretical and semantic antinomies. We believe that antinomies should be relegated to the history of mathematics and that the material given to the reader should be in a form free from errors in the interpretation of the notion of set, etc. Similarly, in the theory of functions no one now uses the vague and inconsistent interpretation of the idea of function given hundreds of years ago.

The terminology employed in this book differs in several places from the terminology generally used. We have tried to unify the mathematical and metamathematical terminology in parallel problems. Thus, for example, in place of "the complete theory" we write "the maximal theory", since this notion is parallel to the notion of a maximal filter in lattice theory. Besides, the word "complete" is too often used with other meanings. We also call attention to the fact that the word "model" in this book has another more general meaning than that usually adopted in works on logic. Following Bourbaki we say "ordered set" in place of "partially ordered set".

Following Nöbeling, we use the term "topological Boolean algebra" in place of "closure algebra", which is employed by many authors. Because of its applications to intuitionistic logic, in this book the interior operation is first in importance, before the closure operation. For this reason we have adopted as the axioms of topological spaces and topological Boolean algebras the axioms dual to the well known Kuratowski axioms. Consequently, it has been difficult for us to employ the name "closure algebra" for the notions defined by the word "interior" and not by the word "closure". The name "topological Boolean algebra" has seemed more convenient also for the reason that it does not introduce any asymmetry in the duality of the basic notions of topology.

We use the same symbols $\cup$, $\cap$, $-$ to stand for set operations, lattice-theoretical operations and the corresponding propositional connectives in order to emphasize the close reciprocal relation between them.

This does not lead to misunderstanding anywhere and very much simplifies the translation of logical notions into the language of set theory and lattice theory.

Theorems within a given chapter are referred to by their numbers. Theorems quoted from other chapters have, in addition, Roman numerals specifying the chapters in which they appear. Similarly, formulas within a given section are referred to by their numbers, those from the same chapter but from other sections have the number of the paragraph added, and those from other chapters are marked by Roman numerals representing the chapter, the number of the section and that of the formula.

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H. Rasiowa *R. Sikorski*

Warsaw, 1962.

PART ONE

LATTICES

CHAPTER I

PRELIMINARY TOPOLOGICAL, ALGEBRAIC AND LATTICE-THEORETICAL NOTIONS

§ 1. Sets, mappings, Cartesian products. We assume that the reader is familiar with the fundamental notions from set theory (1). We recall here only the basic notation.

We write $a \in A$ if a is an *element of a set* A, and otherwise $a \notin A$. If every element of a set A belongs to a set B, we write $A \subset B$ and we say that A is a *subset* of B. The relation $\subset$ is called *inclusion*.

The *empty set* is denoted by 0.

For any sets A, B, the symbol $A \cup B$ ($A \cap B$) will denote the *union* (the *intersection*) of A and B, i.e. the set of all elements belonging to at least one of the sets (to both the sets) A, B. More generally, $\bigcup_{t \in T} A_t$ ($\bigcap_{t \in T} A_t$) will denote the *union* (the *intersection*) of sets A_t where $t \in T$, i.e. the set of all elements belonging to at least one of the sets (to each of the sets) A_t, $t \in T$.

If $A \cap B = 0$, the sets A, B are said to be *disjoint*.

The *difference* of sets A, B, i.e. the set of all elements of A which do not belong to B, will be denoted by $A - B$.

In applications we shall often consider only subsets of a fixed set X. The set X will then be called a *space*, and the difference $X - A$ (where $A \subset X$) will be called the *complement* of A and denoted by $-A$. Hence, if $A, B \subset X$, then $A - B = A \cap -B$.

The words *mapping*, *function*, *transformation* always have the same meaning. We write

$$f\colon X \to Y$$

to indicate that f is a mapping defined on X with values in Y. The set X is then called the *domain* of f. The set Y is called the *counter-domain* of f.

Usually, if f denotes a mapping, then $f(x)$ is used to denote the value of f at a point x. Sometimes we shall also write fx and f_x instead of $f(x)$.

(1) For a detail exposition of set theory see e.g. Fraenkel [2], Hausdorff [1], Kuratowski and Mostowski [1], Sierpiński [1], [3].

If f is a mapping of a set X into a set Y, and $A \subset X$, $B \subset Y$, then $f(A)$ denotes the set of all elements $f(x)$ where $x \in A$, and $f^{-1}(B)$ denotes the set of all elements $x \in X$ such that $f(x) \in B$. The sets $f(A)$ and $f^{-1}(B)$ are called the *image* of A and the *counter-image* of B, respectively. If $f(A) = B$, we say that f maps the set A *onto* the set B. If f is one-to-one, i.e. if $f(x_1) = f(x_2)$ implies $x_1 = x_2$, then f^{-1} denotes the mapping inverse to f, i.e. the mapping from $f(X)$ onto X, such that $f^{-1}(y) = x$ if and only if $f(x) = y$. If $f\colon X \to Y$ and $g\colon Y \to Z$, then $gf\colon X \to Z$ denotes the *superposition* of f and g, i.e. $gf(x) = g(f(x))$ for all $x \in X$. If f is a mapping defined on a set X, g is a mapping defined on a set $X_0 \subset X$ and

$$g(x) = f(x) \quad \text{for all} \quad x \in X_0,$$

then f is called an *extension of g over X* and g is called the *restriction of f to X_0*.

Functions defined on the set of all positive (or non-negative) integers are called *(infinite) sequences*. Functions defined on a set of integers $1, \ldots, m$ are called *finite sequences* or more precisely: *m-element sequences*. If a_n denotes the element assigned to an integer n, then the sequence is denoted by $\{a_n\}$ or—in the case of a finite sequence—by $\{a_1, \ldots, a_m\}$.

More generally, if for every t in a non-empty set T, a_t is an element of a set A, then the function which assigns the element a_t to every $t \in T$ will be denoted by $\{a_t\}_{t \in T}$ or simply by $\{a_t\}$.

The symbol $P_{t \in T} A_t$ will denote the *Cartesian product* of sets A_t ($t \in T$), i.e. the set of all mappings $a = \{a_t\}_{t \in T}$ such that $a_t \in A_t$ for every $t \in T$. In particular, $P_{n=1}^{m} A_n$ ($P_{n=1}^{\infty} A_n$) will denote the set of all m-element sequences (of all infinite sequences) $\{a_n\}$ such that $a_n \in A_n$ for $n = 1, 2, \ldots, m$ (for $n = 1, 2, \ldots$). Instead of $P_{n=1}^{m} A_n$ ($P_{n=1}^{\infty} A_n$) we shall also write $A_1 \times \ldots \times A_m$ ($A_1 \times A_2 \times \ldots$).

If all the sets are equal,

$$A_t = A \quad \text{for every} \quad t \in T,$$

we write A^T instead of $P_{t \in T} A_t$. In other words, A^T is the set of all mappings from T into A. We also write A^m and $A^{\aleph_0}$ instead of $A \times \ldots \times A$ (m times) and $A \times A \times \ldots$, respectively.

Mappings $f\colon A^m \to A$ will often be called *operations* in A.

If, for every $t \in T$, $\mathfrak{R}_t$ is a class of subsets of a set X_t, then $\overset{*}{P}_{t \in T} \mathfrak{R}_t$ will denote the class of all sets $P_{t \in T} A_t$ such that $A_t \in \mathfrak{R}_t$ for every $t \in T$. By definition, $\overset{*}{P}_{t \in T} \mathfrak{R}_t$ is a class of subsets of $P_{t \in T} X_t$.

We suppose that the reader is familiar with the notion of cardinal and ordinal numbers and of transfinite induction. The cardinal number of a set X (i.e. the power of X) will be denoted by $\overline{\overline{X}}$. The cardinal

number of *enumerable* or *countable* sets (i.e. of one-to-one images of the set of all positive integers) is denoted by $\aleph_0$.

§ 2. Topological spaces [1]. A *topological space* is a set X in which, with every set $A \subset X$, there is associated a set $\mathbf{I}A \subset X$ in such a way that

$$\mathbf{I}(A \cap B) = \mathbf{I}A \cap \mathbf{I}B , \tag{i_1}$$

$$\mathbf{I}A \subset A , \tag{i_2}$$

$$\mathbf{II}A = \mathbf{I}A , \tag{i_3}$$

$$\mathbf{I}X = X . \tag{i_4}$$

More precisely, a topological space is a pair $\{X, \mathbf{I}\}$ where X is a set and $\mathbf{I}$ satisfies (i_1)-(i_4).

An operation $\mathbf{I}$ satisfying (i_1)-(i_4) is called an *interior operation*. For every set $A \subset X$, the set $\mathbf{I}A$ is called the *interior* of the set A.

The set $-\mathbf{I}-A$ (i.e. the set $X-\mathbf{I}(X-A)$) is called the *closure* of A and denoted by $\mathbf{C}A$ ($A \subset X$). By definition,

$$\mathbf{C}A = -\mathbf{I}-A \quad \text{and} \quad \mathbf{I}A = -\mathbf{C}-A . \tag{1}$$

It follows immediately from (i_1)-(i_4) that the operation $\mathbf{C}$ satisfies the conditions [2]

$$\mathbf{C}(A \cup B) = \mathbf{C}A \cup \mathbf{C}B , \tag{c_1}$$

$$A \subset \mathbf{C}A , \tag{c_2}$$

$$\mathbf{CC}A = \mathbf{C}A , \tag{c_3}$$

$$\mathbf{C}0 = 0 . \tag{c_4}$$

Notice that, conversely, if with every set $A \subset X$ there is associated a set $\mathbf{C}A \subset X$ in such a way that conditions (c_1)-(c_4) are satisfied, then the operation $\mathbf{I}$ defined by the second of equations (1) satisfies conditions (i_1)-(i_4) and the first of equations (1) also holds.

Any operation $\mathbf{C}$ satisfying (c_1)-(c_4) is called a *closure operation*.

It follows immediately from (i_1) and (c_1) that, for arbitrary subsets A, B of a topological space X,

$$A \subset B \quad \text{implies} \quad \mathbf{I}A \subset \mathbf{I}B \text{ and } \mathbf{C}A \subset \mathbf{C}B . \tag{2}$$

A subset A of a topological space X is said to be *open (closed)* in X if $A = \mathbf{I}A$ (if $A = \mathbf{C}A$). By (1), a set A is open (closed) if and only if its complement $-A$ is closed (open).

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[1] For a detail exposition of the theory of topological spaces see e.g. Kelley [1], Kuratowski [3].

[2] Axioms (c_1)-(c_4) are due to Kuratowski [1].

By (i_3) (by (c_3)) the interior (the closure) of any set A is open (closed). It follows from (i_2), (c_4) and from (c_2), (i_4) that the empty set 0 and the whole space X are both open and closed.

Observe that if B is open, then for every set A

$$B \subset A \quad \text{if and only if} \quad B \subset \mathbf{I}A \tag{3}$$

on account of (i_2), (i_3) and (2). This implies that $\mathbf{I}A$ is the greatest open subset of A. Analogously, if B is closed, then for every set A

$$A \subset B \quad \text{if and only if} \quad \mathbf{C}A \subset B \tag{4}$$

on account of (c_2), (c_3) and (2). This implies that $\mathbf{C}A$ is the least closed set containing A.

By (i_1) (by (c_1)) the intersection (the union) of two open (closed) sets is open (closed). By an easy induction we infer that the intersection (union) of any finite sequence of open (closed) sets is open (closed).

If A is closed (open) and B is open (closed) then $A-B$ is closed (open) since $A-B = A \cap -B$.

The union $\bigcup_{t\in T} A_t$ (the intersection $\bigcap_{t\in T} A_t$) of any number of open (closed) sets A_t is open (closed). In fact, if A_t is open, then by (2), $A_t = \mathbf{I}A_t \subset \mathbf{I}\bigcup_{t\in T} A_t$ and consequently $\bigcup_{t\in T} A_t \subset \mathbf{I}\bigcup_{t\in T} A_t$; the converse inclusion follows from (i_2). The analogous statement for closed sets can be proved similarly or can be obtained from the case of open sets just proved when we pass to complements.

A class $\boldsymbol{B}$ of open subsets of X is said to be a *basis* of X if every open subset of X is the union of some sets belonging to $\boldsymbol{B}$.

A class $\boldsymbol{B}_0$ of open subsets of X is said to be a *subbasis* of X if the class $\boldsymbol{B}$ composed of the empty set 0, the whole space X, and of all the finite intersections $B_1 \cap \ldots \cap B_n$ where $B_1, \ldots, B_n \in \boldsymbol{B}_0$, is a basis of X. Of course, if a subbasis $\boldsymbol{B}_0$ contains 0 and X and if $B_1, B_2 \in \boldsymbol{B}_0$ implies $B_1 \cap B_2 \in \boldsymbol{B}_0$, then $\boldsymbol{B}$ is a basis of X.

The following simple theorem is often used to define an interior operation $\mathbf{I}$ in any set X.

2.1. *For every class $\boldsymbol{B}_0$ of subsets of a set X there exists exactly one interior operation $\mathbf{I}$ in X such that $\boldsymbol{B}_0$ is a subbasis of the topological space $\{X, \mathbf{I}\}$.*

Let $\boldsymbol{B}$ be the class composed of the empty set 0, the whole space X and of all the finite intersections $B_1 \cap \ldots \cap B_n$ where $B_1, \ldots, B_n \in \boldsymbol{B}_0$. For every set $A \subset X$, let $\mathbf{I}A$ be the union of all sets $B \in \boldsymbol{B}$ such that $B \subset A$. Properties (i_2), (i_3), (i_4) follow immediately from the definition of $\mathbf{I}$. Property (i_1) follows from the fact that $B_1 \cap B_2 \in \boldsymbol{B}$ for $B_1, B_2 \in \boldsymbol{B}$. Observe that $\boldsymbol{B}$ is a basis of the topological space X just defined. Thus $\boldsymbol{B}_0$ is a subbasis of X.

Conversely, if $\mathbf{I}$ is an interior operation in X such that $\boldsymbol{B}_0$ is a subbasis of $\{X, \mathbf{I}\}$, then the class $\boldsymbol{B}$ defined above is a basis. Consequently, the open set $\mathbf{I}A$ is the union of all sets $B \in \boldsymbol{B}$ such that $B \subset \mathbf{I}A$. But sets in $\boldsymbol{B}$ are open, and for open B the inclusion $B \subset \mathbf{I}A$ holds if and only if $B \subset A$ (see (3)). Thus $\mathbf{I}A$ is the union of all sets $B \in \boldsymbol{B}$ such that $B \subset A$, i.e. the interior operation $\mathbf{I}$ coincides with the interior operation defined in the first part of the proof of 2.1. This proves the uniqueness of $\mathbf{I}$.

For instance, let X be the set of all real numbers. Take as $\boldsymbol{B}_0$ the class of all intervals $a < \xi < b$, the empty set included. By 2.1 there exists only one interior operation $\mathbf{I}$ in X such that $\boldsymbol{B}_0$ is a subbasis. We shall always consider the set of all real numbers as a topological space with the interior operation $\mathbf{I}$. Observe that $\boldsymbol{B}$ is also a basis of X. A set A is open in X if and only if it is a finite or enumerable union of open intervals or if it is empty. Note that the enumerable class of all intervals $a < \xi < b$ with rational a and b (the empty set included) is also a basis for X.

Let X be any topological space. For arbitrary sets $A, B \subset X$ we have, by (i_1) and (1),

$$\mathbf{I}(A \cup B) \cap \mathbf{I}-B \subset \mathbf{I}A\,,$$

that is,

$$\mathbf{I}(A \cup B) \subset \mathbf{I}A \cup \mathbf{C}B\,. \tag{5}$$

Consequently, by (1) and (i_3),

$$\mathbf{I}(A \cup B) = \mathbf{II}(A \cup B) \subset \mathbf{I}(\mathbf{I}A \cup \mathbf{C}B)\,.$$

Replacing A and B in (5) by $\mathbf{C}B$ and $\mathbf{I}A$ respectively, we obtain

$$\mathbf{I}(\mathbf{I}A \cup \mathbf{C}B) \subset \mathbf{IC}B \cup \mathbf{CI}A\,.$$

Consequently

$$\mathbf{I}(A \cup B) \subset \mathbf{CI}A \cup \mathbf{IC}B\,. \tag{6}$$

A subset A of a topological space X is said to be *dense* provided $\mathbf{C}A = X$. A set A is said to be a *boundary* set provided its complement $-A$ is dense, i.e. $\mathbf{I}A = 0$.

A set A is said to be *nowhere dense* provided its closure $\mathbf{C}A$ is a boundary set, i.e. $\mathbf{IC}A = 0$.

If a set A is dense and $A \subset B$, then B is also dense, by (2). Consequently, each subset of a boundary set is a boundary set. Each subset of a nowhere dense set is nowhere dense. Each nowhere dense set is a boundary set. A closed set is nowhere dense if and only if it is a boundary set.

A is a boundary set if and only if it does not contain a non-empty open set B (i.e. if $B - A \neq 0$ for every open set $B \neq 0$). In fact, if B is

open and $B \subset A$, then $B = \mathbf{I}B \subset \mathbf{I}A$; hence, if $B \neq 0$, then $\mathbf{I}A \neq 0$, i.e. A is not a boundary set. Conversely, if A is not a boundary set, then $B = \mathbf{I}A$ is an open non-empty subset of A.

A set A is nowhere dense if and only if $\mathbf{C}A$ contains no open non-empty set, i.e. if $B - \mathbf{C}A \neq 0$ for every open set $B \neq 0$.

For instance, for every set A, the set $A - \mathbf{I}A$ is a boundary set. In fact, $\mathbf{I}(A - \mathbf{I}A) \subset A - \mathbf{I}A$ by (i_2). On the other hand, $\mathbf{I}(A - \mathbf{I}A) \subset \mathbf{I}A$ by (2). Hence $\mathbf{I}(A - \mathbf{I}A) = 0$.

Since $\mathbf{C}A - A = (-A) - \mathbf{I}(-A)$, we infer that, for every set A, the set $\mathbf{C}A - A$ is a boundary set. Hence, for every open set A, the set

$$\mathbf{C}A - A$$

is nowhere dense since it is a closed boundary set.

It follows immediately from (6) that the union of a boundary set and a nowhere dense set is boundary. Hence, by (c_1), the union of two (and consequently of an arbitrary finite number of) nowhere dense sets is nowhere dense.

A set A is said to be of the *first category* if it is the union of a sequence of nowhere dense sets. The union of any sequence of sets of the first category is also of the first category. Every subset of a set of the first category is also of the first category.

A topological space X is said to be *compact* if, for every indexed set $\{A_t\}_{t \in T}$ of open subsets, the equation $X = \bigcup_{t \in T} A_t$ implies the existence of a finite set $T_0 \subset T$ such that $X = \bigcup_{t \in T_0} A_t$. Replacing open sets by their complements we infer that a topological space X is compact if and only if, for any indexed set $\{B_t\}_{t \in T}$ of closed subsets, the equation $\bigcap_{t \in T} B_t = 0$ implies $\bigcap_{t \in T_0} B_t = 0$ for a finite set T_0 (in other words: if the intersection of every finite number of closed subsets B_t is non-empty, then the intersection of all B_t is non-empty).

If X is a compact space, the A_t are open subsets and B is a closed subset, then the inclusion $B \subset \bigcup_{t \in T} A_t$ implies $B \subset \bigcup_{t \in T_0} A_t$ for a finite set $T_0 \subset T$. In fact, we have $X = (-B) \cup \bigcup_{t \in T} A_t$. Therefore, by compactness, $X = (-B) \cup \bigcup_{t \in T_0} A_t$ for a finite set $T_0 \subset T$, i.e. $B \subset \bigcup_{t \in T_0} A_t$.

By passing to complements we infer from the last remark that if X is a compact space, A_t are closed subsets and B is an open subset, then the inclusion $\bigcap_{t \in T} A_t \subset B$ implies $\bigcap_{t \in T_0} A_t \subset B$ for a finite set $T_0 \subset T$.

A topological space X is said to be a T_0-*space* if, for every pair of distinct points x, y, there exists an open set containing exactly one of them.

A topological space X is said to be a T_1-*space* if, for every pair of distinct points x, y, there is an open set A such that $x \in A$ and $y \notin A$. A topological space X is a T_1-space if and only if every one-element

subset is closed (then, consequently, all finite subsets are closed). In fact, if (y) is closed, then $A = X-(y)$ is open and $x \in A$, $y \notin A$. On the other hand, if for every x $(x \neq y)$ there is an open set A_x such that $x \in A_x$ and $y \notin A_x$, then $(y) = X - \bigcup_x A_x$ is closed.

A topological space X is said to be a T_2-*space* or a *Hausdorff space* if, for every pair of distinct elements x, y, there exist two disjoint open sets A, B such that $x \in A$ and $y \in B$.

A topological space X is said to be *totally disconnected* if, for every pair of distinct points x, y, there exist two open disjoint sets A, B such that $x \in A$, $y \in B$ and $A \cup B = X$. The sets A and B are then simultaneously open and closed.

Obviously each totally disconnected space is a Hausdorff space, and each T_i-space is a T_{i-1}-space $(i = 1, 2)$.

A topological space is said to be *regular* provided that, for every open set A and every point $x \in A$, there exists an open set A_0 such that $x \in A_0$ and $\mathbf{C}A_0 \subset A$.

Every regular T_1-space X is a Hausdorff space. In fact, if $x, y \in X$ and $x \neq y$, then the set $A = X-(y)$ is open and $x \in A$. Let A_1 be an open set such that $x \in A_1$ and $\mathbf{C}A_1 \subset A$. The set $A_2 = X - \mathbf{C}A_1$ is open, $y \in A_2$ and $A_1 \cap A_2 = 0$.

Every compact Hausdorff space X is regular. Indeed, let A be an open set and $x \in A$. For every $y \in X-A$ there exist open sets A_y and A'_y such that $x \in A_y$ and $y \in A'_y$, $A_y \cap A'_y = 0$. Since the closed set $X-A$ is contained in the union of all sets A'_y, there exists a finite sequence $y_1, \ldots, y_m$ such that $X-A \subset A'_{y_1} \cup \ldots \cup A'_{y_m}$. The set $A_0 = A_{y_1} \cap \ldots \cap A_{y_m}$ is open and contains x. Since A_0 is a subset of the closed set $B = X-(A'_{y_1} \cup \ldots \cup A'_{y_m})$, we have $\mathbf{C}A_0 \subset \mathbf{C}B = B$, i.e. $\mathbf{C}A_0$ is disjoint from $A'_{y_1} \cup \ldots \cup A'_{y_m}$. Consequently $\mathbf{C}A_0 \cap (X-A) = 0$, i.e. $\mathbf{C}A_0 \subset A$.

A topological space is said to be *normal* provided that, for every pair B_1, B_2 of disjoint closed sets, there exist disjoint open sets A_1, A_2 such that $B_1 \subset A_1$ and $B_2 \subset A_2$.

Every normal T_1-space X is regular. In fact, if A is open and $x \in A$, then the sets $B_1 = (x)$ and $B_2 = X-A$ are closed and disjoint. If A_1, A_2 are disjoint open sets such that $B_1 \subset A_1$ and $B_2 \subset A_2$, then $x \in A_1$ and $\mathbf{C}A_1 \subset \mathbf{C}(X-A_2) = X-A_2 \subset A$.

Every compact Hausdorff space X is normal. In fact, let B_1, B_2 be two disjoint closed sets. Since X is regular, for every $x \in B_1$ there exists an open set A_x such that $x \in A_x$ and $\mathbf{C}A_x \subset X-B_2$. Since the closed set B_1 is contained in the union of all sets A_x, there exists a finite sequence $x_1, \ldots, x_n$ such that $B_1 \subset A_{x_1} \cup \ldots \cup A_{x_n}$. The open sets $A_1 = A_{x_1} \cup \ldots \cup A_{x_n}$ and $A_2 = X-(\mathbf{C}A_{x_1} \cup \ldots \cup \mathbf{C}A_{x_n})$ are disjoint and $B_1 \subset A_1$, $B_2 \subset A_2$.

2.2. *If X is a compact Hausdorff space, then every subset A of the first category is a boundary set, i.e. $X-A$ is dense* (1).

It suffices to prove that, for every open set $B_0 \neq 0$, there exists a point $x \in B_0 - A$.

By hypothesis, A is the union of a sequence $\{A_n\}$ of nowhere dense sets. The open set $B_0 - \mathbf{C}A_1$ is not empty, i.e. it contains a point x_1. Let B_1 be an open set such that $x_1 \in B_1$ and $\mathbf{C}B_1 \subset B_0 - \mathbf{C}A_1$. The open set $B_1 - \mathbf{C}A_2$ is not empty, i.e. it contains a point x_2. Let B_2 be an open set such that $x_2 \in B_2$ and $\mathbf{C}B_2 \subset B_1 - \mathbf{C}A_2$.

Continuing the procedure described above we define a sequence of open non-empty sets $\{B_n\}$ such that

$$\mathbf{C}B_n \subset B_{n-1} - \mathbf{C}A_n .$$

Since $\{\mathbf{C}B_n\}$ is a decreasing sequence of non-empty closed subsets of the compact space X, their intersection is not empty, i.e. there exists a point $x \in \mathbf{C}B_n$ for $n = 1, 2, \ldots$ Consequently $x \notin A$ and $x \in B_0$ since $x \in \mathbf{C}B_1 \subset B_0$.

Two topological spaces X and Y are said to be *homeomorphic* provided there exists a one-to-one mapping φ from X onto Y, which preserves the operation $\mathbf{I}$, i.e. such that

$$\varphi(\mathbf{I}A) = \mathbf{I}\varphi(A) \quad \text{for every set} \quad A \subset X . \tag{7}$$

Condition (7) is equivalent to the following:

$$\varphi^{-1}(\mathbf{I}B) = \mathbf{I}\varphi^{-1}(B) \quad \text{for every set} \quad B \subset Y . \tag{7'}$$

The proof of the equivalence can be obtained by the substitution $A = \varphi^{-1}(B)$ or $B = \varphi(A)$.

A one-to-one mapping φ from X onto Y satisfying (7) or (7') is said to be a *homeomorphism* of X onto Y. Of course, φ^{-1} is then a homeomorphism of Y onto X.

For instance, if $\boldsymbol{B}$ and $\boldsymbol{B}'$ are subbases in spaces X and Y respectively and if a one-to-one mapping φ of X onto Y has the property

$$\varphi(A) \in \boldsymbol{B}' \quad \text{if and only if} \quad A \in \boldsymbol{B} \tag{8}$$

or the equivalent property

$$\varphi^{-1}(B) \in \boldsymbol{B} \quad \text{if and only if} \quad B \in \boldsymbol{B}' , \tag{8'}$$

then φ is a homeomorphism.

Homeomorphic spaces X, Y have the same properties from the point of view of the notions introduced in this section. For instance,

(1) This is a modification of the well-known Baire theorem (see III, 11.1), due to Čech [1].

if φ is a homeomorphism of X onto Y and A is respectively open, closed, dense, nowhere dense, of the first category, etc. in the space X, so is $\varphi(A)$ in Y. If X is compact, so is Y. Notice that the homeomorphism φ also preserves the operation $\mathbf{C}$, i.e.

$$\varphi(\mathbf{C}A) = \mathbf{C}\varphi(A) \quad \text{for every set} \quad A \subset X \tag{9}$$

and

$$\varphi^{-1}(\mathbf{C}B) = \mathbf{C}\varphi^{-1}(B) \quad \text{for every set} \quad B \subset Y\,. \tag{9'}$$

Either of conditions (9), (9') is necessary and sufficient for a one-to-one mapping φ of X onto Y to be a homeomorphism.

If a topological space X is homeomorphic with a topological space Y, and Y is homeomorphic with a topological space Z, then X is homeomorphic with Z. For if φ is a homeomorphism from X onto Y and ψ is a homeomorphism from Y onto Z, then the superposition $\psi\varphi$ is a homeomorphism from X onto Z.

Let G be a subset of a topological space X. By 2.1, the class $\boldsymbol{B}_0$ of all sets

$$G \cap A\,, \quad \text{where } A \text{ is an open subset of } X\,, \tag{10}$$

defines an interior operation in G. Thus G becomes a topological space, $\boldsymbol{B}_0$ being a subbasis for G. To point out that the interior operation in G is defined in the way just described we shall say that G is a *topological subspace* of the topological space X. It follows easily from the definition that $\boldsymbol{B}_0$ is not only a subbasis for G but it is the class of all open subsets of the topological space G. In other words, a set $B \subset G$ is open in the topological subspace G if and only if there exists a set $A \subset X$ open in X such that $B = G \cap A$. By passing to complements we infer that a set $B \subset G$ is closed in G if and only if there exists a set $A \subset X$ closed in X, such that $B = G \cap X$.

If $\boldsymbol{B}$ is a subbasis (basis) of the space X, then the class of all sets

$$G \cap A\,, \quad \text{where} \quad A \in \boldsymbol{B}\,, \tag{11}$$

is a subbasis (basis) of the topological space G. It is easy to verify that the last condition is also sufficient for G to be a topological subspace of X.

For a detail investigation of the notion of topological subspaces, see III, § 2.

The following theorem will be useful in VIII, § 23:

2.3. *If a topological subspace G of a topological Hausdorff space X is compact, then G is closed in X.*

We have to show that the set $X - G$ is open in X. Thus it suffices to prove that for every point $x \in X - G$ there exists a set G_x open in X

such that $x \in G_x \subset X-G$ (for this implies that $X-G$ is the union of the open sets G_x, $x \in X-G$).

Let $x \in X-G$. Since X is a Hausdorff space, for every point $y \in G$ there exist disjoint open sets A_y and B_y such that $y \in A_y$ and $x \in B_y$. By definition, G is the union of all sets $G \cap A_y$ ($y \in G$). Since the sets $G \cap A_y$ are open subsets of the compact space G, there exists a finite set $(y_1, \ldots, y_n)$ such that

$$G = (G \cap A_{y_1}) \cup \ldots \cup (G \cap A_{y_n}), \quad \text{i.e.} \quad G \subset A_{y_1} \cup \ldots \cup A_{y_n}.$$

The set $G_x = B_{y_1} \cap \ldots \cap B_{y_n}$ has the required property.

§ 3. Equivalence relations. A binary relation $x \sim y$ defined for arbitrary elements x, y of a non-empty set A is an *equivalence relation* in A if it is reflexive, symmetric and transitive, i.e., for arbitrary elements $x, y, z \in A$:

(e_1) $$x \sim x,$$

(e_2) $$\text{if } x \sim y, \text{ then } y \sim x,$$

(e_3) $$\text{if } x \sim y \text{ and } y \sim z, \text{ then } x \sim z.$$

Let $\sim$ be an equivalence relation in A. For every $x \in A$, let $|x|$ denote the set of all $y \in A$ such that $x \sim y$. The sets $|x|$, where $x \in A$, are called the *equivalence classes* of the equivalence relation $\sim$. The set of all equivalence classes $|x|$ ($x \in A$) will be denoted by $A/\sim$.

It is easy to verify that the equivalence classes have the following properties:

(a_1) $$x \in |x|,$$

(a_2) $$\text{if } x_1 \sim x_2, \text{ then } |x_1| = |x_2|,$$

(a_3) if $x_1 \sim x_2$ does not hold, then the classes $|x_1|$ and $|x_2|$ are disjoint.

Thus the equivalence relation $\sim$ determines a decomposition of the set A into disjoint non-empty subsets (viz. into the equivalence classes of $\sim$) such that two elements $x, y \in A$ belong to the same subset if and only if $x \sim y$.

As we have observed, by passing from elements $x, y, \ldots$ to the equivalence classes $|x|, |y|, \ldots$ the equivalence relation is transformed into the usual equality. This method is very often applied in mathematics and it is usually called the *identification of equivalent elements*.

It is especially often applied in the construction of certain mathematical notions, e.g. in the usual definition of integers treated as pairs of natural numbers, in the definition of rational numbers treated as pairs

of integers, etc. In the first case, we consider the set of all pairs $\{m, n\}$ of natural numbers with the equivalence relation

$$\{m_1, n_1\} \sim \{m_2, n_2\} \quad \text{if and only if} \quad m_1 + n_2 = m_2 + n_1 ;$$

integers are the equivalence classes of $\sim$, and usually we say that integers are obtained by identification of equivalent pairs $\{m, n\}$. Similarly, in the second case, we consider the set of all pairs $\{m, n\}$ of integers $(n \neq 0)$ with the equivalence relation

$$\{m_1, n_1\} \sim \{m_2, n_2\} \quad \text{if and only if} \quad m_1 n_2 = m_2 n_1 ;$$

rational numbers are equivalence classes of $\sim$, i.e. they are obtained by identification of pairs equivalent in the sense mentioned above.

In this book, the symbols $\sim$, $\approx$, $\sim$ will always denote equivalence relations. Then the symbols $|x|$, $\|x\|$, $|x|$ will always denote the equivalence classes of the equivalence relations $\sim$, $\approx$, $\sim$, respectively.

An equivalence relation $\sim$ defined in a set A is said to be *consistent with a mapping* $f\colon A^m \to B$ if, for all elements $a_1, \ldots, a_m, a'_1, \ldots, a'_m \in A$, the conditions

$$a_1 \sim a'_1, \ \ldots, \ a_m \sim a'_m \tag{1}$$

imply

$$f(a_1, \ldots, a_m) = f(a'_1, \ldots, a'_m) . \tag{2}$$

If $\sim$ is consistent with f, then by writing

$$f'(|a_1|, \ldots, |a_m|) = f(a_1, \ldots, a_m) \tag{3}$$

we define a mapping $f'\colon (A/\sim)^m \to B$. This definition is correct since, by (2), the right-hand side of (3) does not depend on the choice of elements $a_1, \ldots, a_m$ from the equivalence classes $|a_1|, \ldots, |a_m|$. The mapping f' is said to be the *mapping corresponding to* f.

For the case in which $B = A$, i.e. where f is an operation in A, the above remark can be generalized as follows. An equivalence relation $\sim$ defined in a set A is said to be a *congruence with respect to an operation* $f\colon A^m \to A$ if conditions (1) imply

$$f(a_1, \ldots, a_m) \sim f(a'_1, \ldots, a'_m) . \tag{4}$$

If $\sim$ is a congruence in A with respect to an operation $f\colon A^m \to A$, then by writing

$$f'(|a_1|, \ldots, |a_m|) = |f(a_1, \ldots, a_m)| \tag{5}$$

we define an operation f' in $A/\sim$. This definition is correct since, by (4), the right-hand side of (5) does not depend on the choice of elements

$a_1, \ldots, a_m$ from the equivalence classes $|a_1|, \ldots, |a_m|$. The mapping f': $(A/\sim)^m \to A/\sim$ is said to be the *operation corresponding to the operation* f. Very often f' is denoted by the same symbol f.

§ 4. Abstract algebras [1]. Let A be a non-empty set. According to § 1, every mapping o: $A^m \to A$, i.e. every mapping

$$a = o(a_1, \ldots, a_m), \quad \text{where} \quad a, a_1, \ldots, a_m \in A,$$

is called an *m-argument operation* in A. The number m is called the *number of arguments* of the operation o. The case of $m = 0$ is also admitted: by a *0-argument operation* o in A we understand a constant element $o \in A$.

If $m = 1$, i.e. if o is a unary operation, then instead of $o(a)$ we shall often write oa. If $m = 2$, i.e. if o is a binary operation, then instead of $o(a_1, a_2)$ we shall usually write $a_1 o a_2$.

A set $A' \subset A$ is said to be *closed* under an m-argument operation o in A provided $o(a_1, \ldots, a_m) \in A'$ for all $a_1, \ldots, a_m \in A'$.

By an *abstract algebra*, or simply: *algebra*, we shall understand any pair

$$\{A, \{o_\varphi\}_{\varphi \in \Phi}\} \tag{1}$$

where A is a non-empty set and, for every $\varphi \in \Phi$, o_φ is an operation in A. The cardinal of the set Φ may be arbitrary, finite or infinite. In particular, the set Φ may be empty; then the algebra (1) is simply the set A without any operation.

The most frequent case is that in which Φ is the set of integers $1, \ldots, s$. Then the algebra (1) will also be denoted by

$$\{A, o_1, \ldots, o_s\}. \tag{2}$$

For instance, every group A can be interpreted as an abstract algebra $\{A, \cdot\}$ with one binary operation, $\cdot$, viz. group multiplication. Algebraic rings A are abstract algebras $\{A, +, \cdot\}$ with two binary operations: addition, $+$, and multiplication, $\cdot$. Every linear space A over an algebraic field F can be interpreted as an abstract algebra (1) where Φ is the set F augmented by an element, say ψ, and

$$o_\psi \text{ is the addition in } A,$$

$$o_\varphi a = \varphi a \quad \text{for every} \quad \varphi \in F.$$

An algebra (1) is said to be *degenerate* if A contains only one element, say a. Then all operations o_φ in A are defined by the equation

$$a = o_\varphi(a, \ldots, a).$$

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[1] For the basic concepts of modern algebra see e.g. van der Waerden [1], Birkhoff-MacLane [1], Birkhoff [2], [3].

To simplify terminology and notation, we shall not strictly distinguish between an abstract algebra (1) or (2) and the set A of its elements, and often we shall talk about the abstract algebra A instead of (1) or (2).

A subset A' of an abstract algebra A is said to be a *subalgebra* of A if A' is closed under all operations in A. Every subalgebra A' of A will always be considered as an abstract algebra with the same operations restricted to A'.

Note that if A' is a subalgebra of A and o is a zero-argument operation in A, i.e. a fixed element $o \in A$, then $o \in A'$.

The intersection of any class of subalgebras of A is also a subalgebra of A. Consequently, for every non-empty set A_0 of elements of A, there is a least subalgebra A' containing A_0 (viz. A' is the intersection of all subalgebras containing A_0). This least subalgebra A' is called the *subalgebra generated by* A_0, and A_0 is called a *set of generators* of A'. According to this definition, a set $A_0 \subset A$ is said to *generate* A or to be a *set of generators* of A if the whole algebra A is the only subalgebra containing A_0.

An abstract algebra $\{A, \{o_\varphi\}_{\varphi \in \Phi}\}$ and another abstract algebra

$$\{B, \{o'_\varphi\}_{\varphi \in \Phi'}\} \tag{3}$$

are said to be *similar* provided $\Phi' = \Phi$ and, for every $\varphi \in \Phi$, the operations o_φ and o'_φ have the same number of arguments. In particular, an algebra $\{A, o_1, \ldots, o_s\}$ is similar to an algebra

$$\{B, o'_1, \ldots, o'_{s'}\} \tag{4}$$

if and only if $s = s'$ and if, for every fixed integer $j = 1, \ldots, s$, the operations o_j and o'_j have the same number of arguments.

For instance, every subalgebra A' of an algebra A is similar to A.

Corresponding operations in similar algebras will very often be denoted by the same symbols. For instance, if (3) is similar to (1), we shall often write

$$\{B, \{o_\varphi\}_{\varphi \in \Phi}\} \tag{5}$$

instead of (3). Analogously, if (4) is similar to (2), we shall write

$$\{B, o_1, \ldots, o_s\} \tag{6}$$

instead of (4).

A mapping h of an abstract algebra (1) into a similar abstract algebra (5) is said to be a *homomorphism* provided it preserves all operations, i.e.

$$h\big(o_\varphi(a_1, \ldots, a_m)\big) = o_\varphi\big(h(a_1), \ldots, h(a_m)\big) \quad \text{for} \quad a_1, \ldots, a_m \in A,$$

where m is the number of arguments of o_φ.

The following simple statements will often be used without explicit reference.

4.1. *If* $\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{C}$ *are similar algebras and*

$$h\colon \boldsymbol{A} \to \boldsymbol{B}, \quad g\colon \boldsymbol{B} \to \boldsymbol{C}$$

are homomorphisms, then the superposition

$$gh\colon \boldsymbol{A} \to \boldsymbol{C}$$

is also a homomorphism.

4.2. *If* h *is a homomorphism of an algebra* $\boldsymbol{A}$ *into a similar algebra* $\boldsymbol{B}$, *then the image* $h(\boldsymbol{A})$ *of* $\boldsymbol{A}$ *is a subalgebra of* $\boldsymbol{B}$.

4.3. *If a homomorphism* h *of* $\boldsymbol{A}$ *into* $\boldsymbol{B}$ *maps a set of generators of the algebra* $\boldsymbol{A}$ *onto a set of generators of the algebra* $\boldsymbol{B}$, *then* h *maps* $\boldsymbol{A}$ *onto the algebra* $\boldsymbol{B}$.

The easy proof of 4.1 and 4.2 is left to the reader.

To prove 4.3 let us observe that $h(\boldsymbol{A})$ is a subalgebra of $\boldsymbol{B}$ containing a set of generators of $\boldsymbol{B}$. By the definition of generators, we have $\boldsymbol{B} = h(\boldsymbol{A})$.

4.4. *Let* $\boldsymbol{A}_0$ *be a set of generators of an algebra* $\boldsymbol{A}$, *and let* h_1, h_2 *be homomorphisms from* $\boldsymbol{A}$ *into a similar algebra* $\boldsymbol{B}$. *If*

$$h_1(a) = h_2(a) \quad \text{for all} \quad a \in \boldsymbol{A}_0,$$

then $h_1 = h_2$, *i.e.*

$$h_1(a) = h_2(a) \quad \text{for all} \quad a \in \boldsymbol{A}.$$

In other words,

If a mapping $g\colon \boldsymbol{A}_0 \to \boldsymbol{B}$ *can be extended to a homomorphism* h *from* $\boldsymbol{A}$ *into* $\boldsymbol{B}$, *then this extension* h *is unique.*

To prove 4.4 let us observe that the set of all $a \in \boldsymbol{A}$ such that $h_1(a) = h_2(a)$ is a subalgebra of $\boldsymbol{A}$. Since this subalgebra contains all generators in $\boldsymbol{A}_0$, it coincides with the whole algebra $\boldsymbol{A}$.

It follows immediately from 4.4 that if $\boldsymbol{A}_0$ generates $\boldsymbol{A}$, h is a homomorphism from $\boldsymbol{A}$ into itself and

(7) if $h(a) = a$ for all $a \in \boldsymbol{A}_0$, then $h(a) = a$ for all $a \in \boldsymbol{A}$.

If a homomorphism is one-to-one, it is called an *isomorphism*. If there exists an isomorphism h from an algebra $\boldsymbol{A}$ onto a similar algebra $\boldsymbol{B}$, then the algebras $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isomorphic*. The inverse mapping h^{-1} is then an isomorphism from $\boldsymbol{B}$ onto $\boldsymbol{A}$. For instance, all similar degenerate algebras are isomorphic.

It follows from 4.1 that the superposition of two isomorphisms is also an isomorphism. Thus if an algebra A is isomorphic to an algebra B, and B is isomorphic to an algebra C, then A and C are isomorphic.

4.5. *Let f be a one-to-one mapping from a set A_0 of generators of an algebra A onto a set B_0 of generators of a similar algebra B. If f can be extended to a homomorphism h: $A \to B$ and f^{-1} can be extended to a homomorphism g: $B \to A$, then h is an isomorphism from A onto B and $h^{-1} = g$.*

By hypothesis, $gh(a) = a$ for all $a \in A_0$. Thus, by 4.1 and (7),

$$gh(a) = a \quad \text{for all} \quad a \in A .$$

Similarly $hg(b) = b$ for every $b \in B_0$ and consequently, by 4.1 and (7),

$$hg(b) = b \quad \text{for every} \quad b \in A .$$

This proves that h is a one-to-one mapping from A onto B and $g = h^{-1}$.

Let $\mathfrak{K}$ be a class of similar algebras. An algebra $A \in \mathfrak{K}$ is said to be *$\mathfrak{K}$-free* (or: *free in the class $\mathfrak{K}$*) if it contains a set A_0 such that A_0 generates A and that every mapping f: $A_0 \to B$, where B is any algebra in $\mathfrak{K}$, can be extended to a homomorphism h: $A \to B$. Then A_0 is said to be a *set of $\mathfrak{K}$-free generators* for A.

4.6. *If A and B are $\mathfrak{K}$-free algebras with sets A_0 and B_0 of $\mathfrak{K}$-free generators respectively, and the sets A_0, B_0 have the same cardinal, then A and B are isomorphic. More precisely, every one-to-one mapping f from A_0 onto B_0 can be extended to an isomorphism from A onto B.*

In fact, f can be extended to a homomorphism h: $A \to B$, and f^{-1} can be extended to a homomorphism g: $B \to A$. By 4.5, h is an isomorphism from A onto B.

An equivalence relation $\sim$ defined in an abstract algebra A (see (1)) is said to be a *congruence* if it is a congruence with respect to all operations in A, i.e. if for every $\varphi \in \Phi$ and for arbitrary elements $a_1, \ldots, a_m$, $b_1, \ldots, b_m \in A$, the conditions

$$a_1 \sim b_1, \ \ldots, \ a_m \sim b_m$$

imply

$$o_\varphi(a_1, \ldots, a_m) \sim o_\varphi(b_1, \ldots, b_m) ,$$

m being the number of arguments of o_φ.

If $\sim$ is a congruence in A, then, writing

$$o_\varphi(|a_1|, \ldots, |a_m|) = |o_\varphi(a_1, \ldots, a_m)| , \tag{8}$$

we define the corresponding operation o_φ in the set $A/\sim$ of all equivalence classes $|a|$ $(a \in A)$ on account of § 3, (5). Thus

$$\{A/\sim, \{o_\varphi\}_{\varphi \in \Phi}\} \tag{9}$$

is also an abstract algebra similar to $\boldsymbol{A}$. We call (9) the *quotient algebra* of $\boldsymbol{A}$ and $\sim$. It follows immediately from (8) that the mapping

$$h(a) = |a| \tag{10}$$

is a homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{A}/\sim$. The homomorphism (10) is called the *natural homomorphism* from $\boldsymbol{A}$ onto $\boldsymbol{A}/\sim$.

Of course, if the algebra $\boldsymbol{A}$ has only a finite number of operations and we use the notation (2) instead of (1), then we write

$$\{\boldsymbol{A}/\sim, o_1, \ldots, o_s\} \tag{11}$$

instead of (9).

The notion of congruence in an algebra is closely related to the notion of homomorphism. On the one hand, as we have observed, every congruence $\sim$ determines a homomorphism (10). On the other hand,

4.7. *If h is a homomorphism of an algebra $\boldsymbol{A}$ onto an algebra $\boldsymbol{B}$, then writing*

$$a_1 \sim a_2 \quad \text{if and only if} \quad h(a_1) = h(a_2)$$

we define a congruence $\sim$ in $\boldsymbol{A}$. The algebra $\boldsymbol{B}$ is then isomorphic to $\boldsymbol{A}/\sim$; viz. the mapping

$$h_0(|a|) = h(a) \qquad (a \in \boldsymbol{A}) \tag{12}$$

is an isomorphism from $\boldsymbol{A}/\sim$ onto $\boldsymbol{B}$.

The proof of 4.7 is by an easy verification.

The next theorem shows that the result of two successive formations of quotient algebras can be obtained directly by a single formation of quotient algebras.

4.8. *Let $\sim$ be a congruence defined in a quotient algebra $\boldsymbol{A}/\sim$, and let $\approx$ be the relation in the algebra $\boldsymbol{A}$ defined as follows:*

$$a_1 \approx a_2 \quad \text{if and only if} \quad |a_1| \sim |a_2| . \tag{13}$$

The relation $\approx$ is a congruence in the algebra $\boldsymbol{A}$, and the equation

$$h_0(||a||) = ||a|| \quad \text{for} \quad a \in \boldsymbol{A} \tag{14}$$

defines an isomorphism from $\boldsymbol{A}/\approx$ onto $(\boldsymbol{A}/\sim)/\sim$.

Let h be the superposition of the natural homomorphisms from $\boldsymbol{A}$ onto $\boldsymbol{A}/\sim$ and from $\boldsymbol{A}/\sim$ onto $(\boldsymbol{A}/\sim)/\sim$, i.e. $h(a) = ||a||$. By (13),

$$a_1 \approx a_2 \quad \text{if and only if} \quad h(a_1) = h(a_2) .$$

By the first part of 4.7, $\approx$ is a congruence in $\boldsymbol{A}$. The final part of 4.8 follows from the second part of 4.7.

4.9. *Let $\sim$ be a congruence in an algebra A, and let h be a homomorphism from A into a similar algebra B, such that*

$$\text{if} \quad a_1 \sim a_2 , \quad \text{then} \quad h(a_1) = h(a_2) .$$

Then the formula

$$h_0(|a|) = h(a) \qquad (a \in A)$$

defines a homomorphism h_0: $A/\sim \to B$.

Theorem 4.9 follows easily from the definition of homomorphism and from § 3, (3).

Let $\{A_n\}_{n \in N}$ be an indexed set of similar algebras

$$\left[A_n, \{o_\varphi\}_{\varphi \in \Phi}\right]_{n \in N} . \tag{15}$$

The Cartesian product

$$A = P_{n \in N} A_n \tag{16}$$

will always be considered as an abstract algebra

$$\left[A, \{o_\varphi\}_{\varphi \in \Phi}\right] \tag{17}$$

similar to (15), with the following definition of operation o_φ:

$$o_\varphi(\{a_{1,n}\}_{n \in N}, \ldots, \{a_{m,n}\}_{n \in N}) = \{o_\varphi(a_{1,n}, \ldots, a_{m,n})\}_{n \in N} , \tag{18}$$

where $\{a_{1,n}\}_{n \in N}, \ldots, \{a_{m,n}\}_{n \in N}$ are any points in A and m is the number of arguments of o_φ. Roughly speaking we perform the operation o_φ on elements in A by performing this operation on coordinates of these elements.

The algebra (17) just defined will be called the *product of algebras* (15).

4.10. *If A is the product of similar algebras A_n ($n \in N$) and, for every $n \in N$, h_n is a homomorphism from a similar algebra A' into A_n, then the formula*

$$h(a) = \{h_n(a)\}_{n \in N} \quad \text{for} \quad a \in A'$$

defines a homomorphism from A' into A.

This follows directly from the definition of homomorphism and from the definition of the operations in A.

The following generalization of the notion of abstract algebra will be useful in this book.

Let A be a non-empty set. By a *generalized operation* in A we shall understand any mapping

$$O: \mathfrak{D} \to A , \tag{19}$$

where $\mathfrak{D}$ is a class of non-void subsets of A. Thus O assigns, to every set $S \in \mathfrak{D}$, an element OS in A. The class $\mathfrak{D}$ is called the *domain of the generalized operation O* under consideration according to the general terminology assumed for mappings. Sets $S \in \mathfrak{D}$ are called *admissible sets*

for the operation O. It is important to note that sets S in $\mathfrak{D}$ can be infinite. To underline this fact we also call O an *infinite operation* in A (operations defined on p. 12 and 22 will be called *finite operations*). If $S \in \mathfrak{D}$ is given in the form of an indexed set $\{a_t\}_{t \in T}$, then we write

$$O_{t \in T} a_t \tag{20}$$

instead of OS.

By a *generalized abstract algebra* we shall understand any finite sequence $\{A, o_1, \ldots, o_s, O_1, \ldots, O_r\}$ where A is a non-empty set, $o_1, \ldots, o_s$ are finite operations in A, and $O_1, \ldots, O_r$ are infinite (i.e. generalized) operations in A. (The restriction to a finite number of finite and infinite operations is not necessary but it suffices for the purpose of this book).

If the class of all non-void subsets of A is the common domain for all the generalized operations $O_1, \ldots, O_r$, then

$$\{A, o_1, \ldots, o_s, O_1, \ldots, O_r\}$$

is said to be a *complete generalized algebra*.

An example of a generalized complete algebra is given by the unit interval A of real numbers $0 \leqslant x \leqslant 1$ with a binary operation: multiplication, and two infinite operations: inf and sup which assign to every set $S \subset A$ the numbers $\sup S = \text{l. u. b.}\, S$ and $\inf S = \text{g. l. b.}\, S$. If A is supposed to be the set of all rational numbers in the unit interval, with the same finite and infinite operations, then the generalized algebra A is not complete, since the domain of sup is composed only of those sets $S \subset A$ such that $\text{l. u. b.}\, S$ is rational, and similarly for inf.

A subset A' of a set A is said to be *closed* with respect to a generalized operation O in A if, for every set $S \subset A'$,

$$S \in \mathfrak{D} \quad \text{implies} \quad OS \in A',$$

$\mathfrak{D}$ being the domain of O in A.

The definition of a subalgebra of a generalized algebra is the same as that on p. 23. A set $A' \subset A$ is said to be a *subalgebra* of a generalized algebra $\{A, o_1, \ldots, o_s, O_1, \ldots, O_r\}$ if A' is closed under all finite and infinite operations in A. Then $\{A', o_1, \ldots, o_s, O_1, \ldots, O_r\}$ is also a generalized algebra, the operations in A' being those in A, restricted to A'.

A set $A_0 \subset A$ is said to *generate* a generalized algebra A if A is its least subalgebra containing A_0. Then A_0 is also said to be a *set of generators* for A.

Two generalized abstract algebras $\{A, o_1, \ldots, o_s, O_1, \ldots, O_r\}$ and $\{B, o_1', \ldots, o_{s'}', O_1', \ldots, O_{r'}'\}$ are said to be *similar* if the algebras $\{A, o_1, \ldots, o_s\}$, $\{B, o_1', \ldots, o_{s'}'\}$ are similar (in consequence $s' = s$) and if $r' = r$. If B is similar to A, then the corresponding operations in B will be denoted by the same symbols as in A.

Let $\{A, o_1, \ldots, o_s, O_1, \ldots, O_r\}$ and $\{B, o_1, \ldots, o_s, O_1, \ldots, O_r\}$ be similar generalized algebras. A mapping h: $A \to B$ is said to be a *homomorphism* if h is a homomorphism from $\{A, o_1, \ldots, o_s\}$ into $\{B, o_1, \ldots, o_s\}$ (in the sense defined on p. 23), and if h preserves all infinite operations, i.e.

$$h(O_i S) = O_i h(S) \quad (i = 1, \ldots, r) \tag{21}$$

for every set S admissible for O_i in A. Identity (21) should be understood as follows: if S is admissible for O_i in A, then $h(S)$ is admissible for O_i in B, and (21) holds.

It is easy to verify that, under the above definitions, theorems 4.1 and 4.4 are valid also for homomorphisms between generalized algebras.

For the purposes of this book it is convenient to generalize the notion of free algebras as follows. Let A_0 be a set of generators of a generalized algebra A, and let $\mathfrak{K}$ be a class of complete algebras similar to A. We say that A is a *generalized $\mathfrak{K}$-free algebra* (or: a *generalized free algebra for the class* $\mathfrak{K}$) and A_0 is a set of free generators of A if, for every algebra $B \in \mathfrak{K}$, every mapping g: $A_0 \to B$ can be extended to a homomorphism h: $A \to B$ (this extension is unique, by 4.4).

In this definition we require neither that A be in $\mathfrak{K}$ nor that A be complete. Consequently theorem 4.6 is not valid for generalized $\mathfrak{K}$-free algebras.

A generalized algebra $\{A', o_1', \ldots, o_s', O_1', \ldots, O_r'\}$ is said to be an *extension* of a similar generalized algebra $\{A, o_1, \ldots, o_s, O_1, \ldots, O_r\}$ provided A is a subset of A' and

a) for all $a_1, \ldots, a_{m_j} \in A$

$$o_j(a_1, \ldots, a_{m_j}) = o_j'(a_1, \ldots, a_{m_j}) \quad (j = 1, \ldots, s);$$

b) if a set $S \subset A$ is admissible for O_i, then it is also admissible for O_i' and

$$O_i S = O_i' S \quad (i = 1, \ldots, r).$$

Note that if $r = 0$, then A' is an extension of A if and only if A is a subalgebra of A'.

Every generalized algebra $\{A, o_1, \ldots, o_s, O_1, \ldots, O_r\}$ can be extended to a complete algebra $\{A', o_1', \ldots, o_s', O_1', \ldots, O_r'\}$, in various ways. Note the following method of extension: Let a_0 be a fixed element, $a_0 \notin A$, and let $A' = A \cup (a_0)$. Operations in A' are defined as follows:

$$o_j'(a_1, \ldots, a_{m_j}) = \begin{cases} o_j(a_1, \ldots, a_{m_j}) & \text{if} \quad a_1, \ldots, a_{m_j} \in A, \\ a_0 & \text{otherwise,} \end{cases}$$

$$O_i' S = \begin{cases} O_i S & \text{if} \quad S \subset A \text{ and } S \text{ is admissible for } O_i, \\ a_0 & \text{otherwise.} \end{cases} \tag{22}$$

In other words, the result of a finite or infinite operation in A' is equal to a_0 unless the operation concerned is feasible in A.

According to the convention on similar algebras on p. 28, if A' is an extension of A, the corresponding operations in A' and A will often be denoted by the same symbols.

If O is an infinite operation in a set A, $\mathfrak{D}$ is the domain of O and $\mathfrak{Q}$ is a subset of $\mathfrak{D}$, then $O|\mathfrak{Q}$ will denote the operation O restricted to sets $S \in \mathfrak{Q}$ only. By definition, $\mathfrak{Q}$ is the domain of $O|\mathfrak{Q}$.

Let $\{A, o_1, \ldots, o_s, O_1, \ldots, O_r\}$ be a generalized algebra and, for $i = 1, \ldots, r$, let $\mathfrak{Q}_i$ be a subset of the domain of O_i. Then $\{A, o_1, \ldots, o_s, O_1|\mathfrak{Q}_1, \ldots, O_r|\mathfrak{Q}_r\}$ is also a generalized algebra. Of course, the algebra $\{A, o_1, \ldots, o_s, O_1, \ldots, O_r\}$ is an extension of the algebra $\{A, o_1, \ldots, o_s, O_1|\mathfrak{Q}_1, \ldots, O_r|\mathfrak{Q}_r\}$.

The notion of products of abstract algebras can also be generalized to the case of generalized algebras.

Let $\{A_n\}_{n \in N}$ be an indexed set of similar generalized algebras

$$\{A_n, o_1, \ldots, o_s, O_1, \ldots, O_r\}. \tag{23}$$

The Cartesian product

$$A = P_{n \in N} A_n \tag{24}$$

will always be considered to be a generalized algebra

$$\{A, o_1, \ldots, o_s, O_1, \ldots, O_r\} \tag{25}$$

similar to (23), with the following definition of operations.

If o is one of the finite operations $o_1, \ldots, o_s$, then

$$o(a_1, \ldots, a_m) = \{o(a_{1,n}, \ldots, a_{m,n})\}_{n \in N} \tag{26}$$

for any points $a_1 = \{a_{1,n}\}_{n \in N}, \ldots, a_m = \{a_{m,n}\}_{n \in N}$ in A, m being the number of arguments of o (this definition is a repetition of (18)).

If O is one of the infinite operations $O_1, \ldots, O_r$ and for every $n \in N$, S_n is a set (subset of A_n) admissible for O in the algebra A_n, then the set

$$S = P_{n \in N} S_n \tag{27}$$

is admissible for the operation O in A, and

$$OS = \{OS_n\}_{n \in N}. \tag{28}$$

By definition, if $\mathfrak{D}_n$ is the domain of O in A_n, then $\overset{*}{P}_{n \in N} \mathfrak{D}_n$ (see p. 12) is the domain of O in $A = P_{n \in N} A_n$.

The generalized algebra (25) just defined will be called the *product of generalized algebras* (23) [1]. Note that, by definition, only sets (27) are admissible in the product A of all A_n.

Since admissible sets in generalized algebras will usually be given in the form of indexed sets, we shall repeat below the definitions (27) and (28) in terms of indexed sets.

Let O be one of the infinite operations $O_1, \ldots, O_r$. Suppose that, for every $n \in N$, the indexed set $\{a_{t,n}\}_{t \in T_n}$ is admissible for the operation O in the algebra A_n. Let

$$T = P_{n \in N}\, T_n \tag{29}$$

and, for every $t = \{t_n\}_{n \in N} \in T$, let

$$a_t = \{a_{t_n,n}\}_{n \in N} \in A\ . \tag{30}$$

The indexed set

$$\{a_t\}_{t \in T} \tag{31}$$

just defined is admissible for the operation O in A, and

$$O_{t \in T} a_t = \{O_{t_n \in T_n} a_{t_n,n}\}_{n \in N}\ . \tag{32}$$

Note that theorem 4.10 also holds for generalized algebras.

Since only sets (27) (i.e. sets (31)) are admissible for infinite operations in the product A of A_n, the algebra A is not, in general, complete even when all the factors A_n are complete algebras. However, it is possible to introduce another kind of product of generalized algebras, called the complete product, which is a complete algebra if all the factors are complete.

The *complete product of algebras* A_n, $n \in N$, is an algebra (25) with the same definition (24) of the set of its elements and with the same definition (26) of finite operations, but with another definition of infinite operations. We quote this definition below in terms of indexed sets.

Let O be one of the operations $O_1, \ldots, O_r$ and let $\{a_t\}_{t \in T}$ be an indexed set of elements $a_t = \{a_{t,n}\}_{n \in N}$ in A. The indexed set $\{a_t\}_{t \in T}$ is said to be *admissible* for the operation O in the complete product if and only if, for every $n \in N$, the indexed set $\{a_{t,n}\}_{t \in T}$ of elements of A_n is admissible for the operation O in the algebra A_n. Then, by definition,

$$O_{t \in T} a_t = \{O_{t \in T}\, a_{t,n}\}_{n \in N}\ . \tag{33}$$

Observe that the complete product of all A_n is an extension of the product of all A_n.

Complete products of generalized algebras will not play any essential part in this book.

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[1] See Sikorski [15].

§ 5. Ordered sets. A binary relation $x \leqslant y$ defined in a set A is said to be an *ordering* [1] *in* A if it is reflexive, antisymmetric and transitive, i.e. if for arbitrary $x, y, z \in A$

$$(o_1) \qquad x \leqslant x\,,$$

$$(o_2) \qquad \text{if } x \leqslant y \text{ and } y \leqslant x\,, \text{ then } x = y\,,$$

$$(o_3) \qquad \text{if } x \leqslant y \text{ and } y \leqslant z\,, \text{ then } x \leqslant z\,.$$

Instead of $x \leqslant y$ we also write $y \geqslant x$. We then say that y is *greater* than x, or: x is *less* than y, or: x is *included* in y.

For instance, the inclusion $\subset$ is an ordering in the set of all subsets of a fixed space. Another example is the relation $f \leqslant g$ defined in the natural way in the set of all real functions on a space X:

$$f \leqslant g \quad \text{if and only if} \quad f(x) \leqslant g(x) \quad \text{for every} \quad x \in X\,.$$

A non-empty set A with a fixed ordering $\leqslant$ is said to be *ordered.* More exactly: an ordered set is a pair $\{A, \leqslant\}$ where A is a non-empty set and $\leqslant$ is an ordering in A.

Observe that every subset of an ordered set is also an ordered set with the same ordering $\leqslant$.

An element a of an ordered set A is said to be *greatest (least)* provided $x \leqslant a$ $(x \geqslant a)$ for every $x \in A$. Every ordered set A contains at most one greatest element and at most one least element. In fact, if a_1 and a_2 are the greatest (least) elements of A, then, by definition, $a_1 \leqslant a_2$ and $a_2 \leqslant a_1$, and consequently $a_1 = a_2$ by (o_2).

An element a of an ordered set A is said to be *maximal* (*minimal*) if there is no element b in A such that $a \leqslant b$ $(a \geqslant b)$ and $a \neq b$. An ordered set can contain many maximal or minimal elements. For instance, if the ordering relation is defined as follows:

$$(1) \qquad x \leqslant y \quad \text{if and only if} \quad x = y$$

then every element is simultaneously maximal and minimal.

Let S be a non-empty set of elements of an ordered set A. An element a_0 is said to be an *upper* (*lower*) *bound* of S provided $a \leqslant a_0$ $(a \geqslant a_0)$ for all $a \in S$. If the set of all upper (lower) bounds of S contains the least (greatest) element, it is called the *least upper bound* (the *greatest lower bound*) *of* S and denoted by $\mathrm{l.u.b.}\,S$ $(\mathrm{g.l.b.}\,S)$. In practice the set S will be given in the form of an indexed set $\{a_t\}_{t \in T}$ where $T \neq 0$. Then we shall write $\mathrm{l.u.b.}_{t \in T}\, a_t$ $(\mathrm{g.l.b.}_{t \in T}\, a_t)$ instead of $\mathrm{l.u.b.}\,S$ $(\mathrm{g.l.b.}\,S)$.

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[1] Many authors write "partial ordering" instead of "ordering", and "ordering" instead of "linear ordering".

By definition, the equation $a_0 = \text{l.u.b.}_{t \in T} a_t$ ($a_0 = \text{g.l.b.}_{t \in T} a_t$) holds if and only if the two following conditions are satisfied:

(a) $a_t \leqslant a_0$ ($a_0 \leqslant a_t$) for every $t \in T$;

(b) if $a_t \leqslant b$ ($b \leqslant a_t$) for every $t \in T$, then $a_0 \leqslant b$ ($b \leqslant a_0$).

The example of ordering (1) shows that $\text{l.u.b.}_{t \in T} a_t$ ($\text{g.l.b.}_{t \in T} a_t$) does not always exist, even when T contains only two elements. The case in which every two-element subset of A has the least upper bound and the greatest lower bound is especially important and will be discussed later (§ 6).

An ordering $\leqslant$ defined in a set A is said to be a *linear ordering* provided it also satisfies the following additional condition:

(o_4) for arbitrary $x, y \in A$, either $x \leqslant y$ or $y \leqslant x$.

A set A with a fixed linear ordering relation is said to be a *chain*. Any set of real numbers or of ordinals is an example of a chain, the ordering being the usual one. In a chain, the notions of greatest (least) element and of maximal (minimal) element coincide. Every subset of a chain is also a chain with the same ordering $\leqslant$.

The following theorem will be called the *induction principle for ordered sets* [1]:

5.1. *If every chain of elements of an ordered set A has an upper bound in A, then A contains a maximal element. More precisely, for every $a_0 \in A$ there exists a maximal element $a \geqslant a_0$.*

We define by induction a transfinite increasing sequence $\{a_\xi\}$ of elements of A as follows:

a_0 is the element mentioned in 5.1;

if ξ is an isolated ordinal, i.e. $\xi = \eta + 1$, then a_ξ is an element such that $a_\xi \geqslant a_\eta$ and $a_\xi \neq a_\eta$;

if ξ is a limit ordinal, then a_ξ is an upper bound of the chain of all a_η where $\eta < \xi$.

The transfinite induction breaks off on an ordinal ξ_0. Of course a_{ξ_0} is a maximal element of A and $a_0 \leqslant a_{\xi_0}$.

A binary relation $x \leqslant y$ defined in a set A is said to be a *quasi-ordering* in A provided it satisfies only conditions (o_1) and (o_3). A set A with a fixed quasi-ordering $\leqslant$ in A is said to be *quasi-ordered*. More precisely, a quasi-ordered set is a pair $\{A, \leqslant\}$ where $\leqslant$ is a quasi-ordering in A.

5.2. *Let $\leqslant$ be a quasi-ordering in a set A. Write*

(1) $x \sim y$ *if and only if* $x \leqslant y$ *and* $y \leqslant x$.

[1] Kuratowski [2], Zorn [1].

The relation $\sim$ *just defined is an equivalence relation in* A. *For any* $x, y \in A$, *let*

$$|x| \leqslant |y| \quad \textit{if and only if} \quad x \leqslant y . \tag{2}$$

The relation $\leqslant$ *defined by* (2) *in* $A/\sim$ *is an ordering in* $A/\sim$.

The reflexivity of $\sim$ follows from (o_1). The symmetry of $\sim$ follows immediately from definition (1). The transitivity of $\sim$ follows from (o_3). Thus $\sim$ is an equivalence relation in A.

It follows from (o_3) that if $x \leqslant y$ and $x \sim x_1$, $y \sim y_1$, then $x_1 \leqslant y_1$. This implies that definition (2) of the relation $\leqslant$ in $A/\sim$ is correct, i.e. that for any $a = |x| \in A/\sim$ and $b = |y| \in A/\sim$, the inequality $a \leqslant b$ does not depend on the choice of representatives x, y of a and b. That $\leqslant$ is reflexive and transitive in $A/\sim$ follows immediately from the same properties of $\leqslant$ in A. If $|x| \leqslant |y|$ and $|y| \leqslant |x|$, then $x \leqslant y$ and $y \leqslant x$ by (2). Hence $x \sim y$ by (1), i.e. $|x| = |y|$. This proves the antisymmetry of $\leqslant$ in $A/\sim$. Thus (2) defines an ordering $\leqslant$ in $A/\sim$.

§ 6. Lattices [1]. An ordering $\leqslant$ in a set A is said to be a *lattice ordering* if, for each $a, b \in A$, the elements

$$\text{l.u.b.}(a, b) \quad \text{and} \quad \text{g.l.b.}(a, b)$$

exist. Then the ordered set A is said to be a *lattice*; the least upper bound of $a, b \in A$ will be denoted by

$$a \cup b$$

and called the *join* of elements a, b, and the greatest lower bound of $a, b \in A$ will be denoted by

$$a \cap b$$

and called the *meet* of a, b.

The following properties of join and meet result immediately from the definition:

(1)	$a \cup a = a$,	$a \cap a = a$,
(2)	$a \leqslant a \cup b$,	$a \cap b \leqslant a$,
(3)	$b \leqslant a \cup b$,	$a \cap b \leqslant b$,
(4)	$a \leqslant c$ and $b \leqslant c$ imply $a \cup b \leqslant c$,	$c \leqslant a$ and $c \leqslant b$ imply $c \leqslant a \cap b$,
(5)	$a \leqslant b$ if and only if $a \cup b = b$,	$b \leqslant a$ if and only if $a \cap b = b$,
(6)	$a \leqslant c$, $b \leqslant d$ imply $a \cup b \leqslant c \cup d$,	$a \leqslant c, b \leqslant d$ imply $a \cap b \leqslant c \cap d$.

[1] For a detail exposition of lattice theory, see Birkhoff [3]; cf. also Hermes [1]

The next properties of join and meet can also be deduced easily from the definition:

$$(l_1) \qquad a \cup b = b \cup a\,, \qquad\qquad a \cap b = b \cap a\,,$$

$$(l_2) \qquad a \cup (b \cup c) = (a \cup b) \cup c\,, \qquad a \cap (b \cap c) = (a \cap b) \cap c\,,$$

$$(l_3) \qquad (a \cap b) \cup b = b\,, \qquad\qquad a \cap (a \cup b) = a\,.$$

Identities (1) are called the *idempotent laws*. Identities (l_1) are the *commutative laws*. Identities (l_2) are the *associative laws*. Identities (l_3) are called the *absorption laws*.

The associative laws enable us to write finite joins and meets

$$a_1 \cup \ldots \cup a_n\,, \qquad a_1 \cap \ldots \cap a_n \qquad (n = 1, 2, \ldots)$$

without any use of parentheses (of course if $n = 1$, then both expressions are said to be equal to a_1). Sometimes we shall denote these finite joins and meets by the shorter symbols $\bigcup_{i=1}^{n} a_i$ and $\bigcap_{i=1}^{n} a_i$, respectively. By definition

$$\bigcup_{i=1}^{n} a_i = a_1 \cup \ldots \cup a_n = \text{l.u.b.}\,(a_1, \ldots, a_n)\,,$$

$$\bigcap_{i=1}^{n} a_i = a_1 \cap \ldots \cap a_n = \text{g.l.b.}\,(a_1, \ldots, a_n)\,.$$

Every lattice $\boldsymbol{A}$ will always be considered as an abstract algebra $\{\boldsymbol{A}, \cup, \cap\}$. Identities (l_1), (l_2), (l_3) can be used as a set of axioms characterizing $\cup$ and $\cap$. Namely the following theorem holds:

6.1. *Suppose that* $\{\boldsymbol{A}, \cup, \cap\}$ *is an abstract algebra with two binary operations* $\cup, \cap$ *satisfying identities* (l_1), (l_2), (l_3). *Then, for all* $a, b \in \boldsymbol{A}$,

$$(7) \qquad a \cup b = b \quad \textit{if and only if} \quad a \cap b = a\,.$$

Writing $a \leqslant b$ *if and only if one of equations* (7) *holds, we define an ordering* $\leqslant$ *in* $\boldsymbol{A}$. *It is a lattice ordering; viz. for all* $a, b \in \boldsymbol{A}$

$$(8) \qquad \text{l.u.b.}\,(a, b) = a \cup b\,, \qquad \text{g.l.b.}\,(a, b) = a \cap b\,.$$

If $a \cup b = b$, then $a \cap b = a$ by the second absorption law. Conversely, if $a \cap b = a$, then $a \cup b = b$ by the first absorption law. Thus equations (7) are equivalent.

Applying the second of axioms (l_3) and later the first of axioms (l_3) and (l_1) we get the first of identities (1):

$$a \cup a = a \cup \big(a \cap (a \cup b)\big) = a\,.$$

Hence $a \leqslant a$. If $a \leqslant b$ and $b \leqslant a$, then $a = b$ since

$$a = a \cup b = (a \cap b) \cup b = b$$

by the first absorption law. If $a \leqslant b$ and $b \leqslant c$, then $a \leqslant c$ since

$$a = a \cap b = a \cap (b \cap c) = (a \cap b) \cap c = a \cap c$$

by the second associative law. Thus $\leqslant$ is an ordering in A.

It follows directly from the second absorption law that $a \leqslant a \cup b$. Replacing here a by b, and conversely, we infer that $b \leqslant a \cup b$ on account of (l_1). On the other hand, if $a \leqslant c$ and $b \leqslant c$, i.e. $a \cup c = c$ and $b \cup c = c$, then, by (l_1), (l_2) and the first of identities (1) just proved, we have

$$(a \cup b) \cup c = (a \cup b) \cup (c \cup c) = (a \cup c) \cup (b \cup c) = c \cup c = c\,,$$

i.e. $a \cup b \leqslant c$. This proves the first of equations (8). Replacing everywhere $\cup, \cap, \leqslant$ by $\cap, \cup, \geqslant$, respectively, in the above proof, we obtain the proof of the second of equations (8).

In the sequel we shall consider lattices as abstract algebras $\{A, \cup, \cap\}$. Observe that by replacing everywhere in (l_1)-(l_3) the signs $\cup, \cap$ by $\cap, \cup$, respectively, we obtain a set of axioms equivalent to (l_1)-(l_3). In fact, (l_1) and (l_2) are invariant under this substitution, and the axioms obtained from (l_3) are equivalent to (l_3) because of (l_1). Thus the operations $\cup$ and $\cap$ play a symmetrical role. Consequently every theorem proved for $\cup$ and $\cap$ remains true if $\cup$ and $\cap$ are replaced by $\cap$ and $\cup$, respectively. This fact is called the *duality principle* for lattices. If $\cup$ and $\cap$ are replaced in (7) by $\cap$ and $\cup$ respectively, then these equations become equivalent to $a \geqslant b$. Therefore, if we apply the duality principle to statements concerning the lattice ordering, then $\leqslant$ should be everywhere replaced by $\geqslant$ and conversely. We have applied the duality principle once: in the proof of the second of equations (8) in theorem 6.1.

If elements of a lattice A are subsets of a space X, and $\cup$ and $\cap$ coincide with the set-theoretical union and intersection, then A is said to be a *set lattice*. Thus a set lattice is a non-empty class A of subsets of a space X such that the union and intersection of two sets belonging to A also belong to A. The ordering determined in a set lattice by theorem 6.1 coincides with the set-theoretical inclusion $\subset$.

Every chain and the ordered set of all real functions on a space X (see § 5) are also examples of lattices.

According to the general definition of subalgebras (see § 4), a subset A' of a lattice A is a *sublattice* of A if it is closed under operations $\cup$ and $\cap$, i.e. if

$$a \cup b \in A' \quad \text{and} \quad a \cap b \in A' \quad \text{for arbitrary} \quad a, b \in A'.$$

Each sublattice A' of A is a lattice since axioms (l_1)-(l_3) also hold in A'. The ordering in A' coincides with the ordering in A.

For instance, the set A' of all elements greater (or: less) than a given element $a \in A$ is a sublattice of A.

A homomorphism (in the sense of § 4) of a lattice $\boldsymbol{A}$ into a lattice $\boldsymbol{B}$, i.e. a mapping h of $\boldsymbol{A}$ into $\boldsymbol{B}$ such that

$$h(a \cup b) = h(a) \cup h(b) \quad \text{and} \quad h(a \cap b) = h(a) \cap h(b) \quad \text{for} \quad a, b \in \boldsymbol{A}$$

will also be called a *lattice homomorphism*. The homomorphic image $h(\boldsymbol{A})$ of $\boldsymbol{A}$ is also a lattice, namely a sublattice of $\boldsymbol{B}$.

Every lattice homomorphism h preserves the ordering in lattices, i.e.

$$a \leqslant b \quad \text{implies} \quad h(a) \leqslant h(b)\,. \tag{9}$$

6.2. *In order that a one-to-one mapping h of a lattice $\boldsymbol{A}$ onto a lattice $\boldsymbol{B}$ be an isomorphism (in the sense defined in § 4) it is necessary and sufficient that h and h^{-1} be increasing mappings, i.e. that*

$$h(a) \leqslant h(b) \quad \textit{if and only if} \quad a \leqslant b \quad (a, b \in \boldsymbol{A})\,. \tag{9'}$$

This follows from the fact that the join and the meet are uniquely determined by the lattice ordering.

In each lattice $\boldsymbol{A}$ the notions of the greatest (least) element and of the maximal (minimal) element coincide. Indeed, $a \leqslant a \cup x$ for every $x \in \boldsymbol{A}$. If a is a maximal element, then this implies that $a = a \cup x$. Since $x \leqslant a \cup x$, we have $x \leqslant a$ for every $x \in \boldsymbol{A}$, i.e. a is the greatest element. On the other hand, in any ordered set the greatest element is maximal. By duality we obtain the analogous statement for minimal elements.

A lattice $\boldsymbol{A}$ may or may not contain a greatest or a least element. The simplest examples are given by intervals of real numbers with or without end points. The greatest (least) element of a lattice $\boldsymbol{A}$, if it exists, will be called the *unit element* (the *zero element*) *of* $\boldsymbol{A}$ and will be denoted by $\vee$ (by $\wedge$), or by $\vee_{\boldsymbol{A}}$ (by $\wedge_{\boldsymbol{A}}$) if necessary. By definition

$$x \leqslant \vee \quad \text{and} \quad x \geqslant \wedge\,, \tag{10}$$

and

$$x \cup \vee = \vee\,, \quad x \cap \vee = x\,, \tag{11}$$

$$x \cup \wedge = x\,, \quad x \cap \wedge = \wedge \tag{12}$$

for every $x \in \boldsymbol{A}$. Of course, applying the duality principle to statements containing the sign $\vee$ or $\wedge$ it is necessary to replace $\vee$ by $\wedge$ everywhere, and conversely.

Observe that if we add to a lattice $\boldsymbol{A}$ a new element $\vee$ (or: $\wedge$) and we define the operations $\cup$ and $\cap$ on $\vee$ (on $\wedge$) by means of (11) (of (12)) where $x \in \boldsymbol{B} = \boldsymbol{A} \cup (\vee)$ ($x \in \boldsymbol{B} = \boldsymbol{A} \cup (\wedge)$), then we get a lattice $\boldsymbol{B}$ and $\vee$ ($\wedge$) is the unit element (the zero element) of $\boldsymbol{B}$. The statement remains true if we add two elements $\vee$, $\wedge$ simultaneously to $\boldsymbol{A}$ in this way.

Let $\{A_n\}_{n\in N}$ be an indexed set of lattices considered as abstract algebras

$$\{A_n, \cup, \cap\}.$$

By the general definition in § 4, p. 27, the product of all A_n is an algebra

$$\{A, \cup, \cap\}$$

where $A = P_{n\in N} A_n$ and the operations $\cup$, $\cap$ in A are defined as follows:

$$\begin{aligned} \{a_n\}_{n\in N} \cup \{b_n\}_{n\in N} &= \{a_n \cup b_n\}_{n\in N}, \\ \{a_n\}_{n\in N} \cap \{b_n\}_{n\in N} &= \{a_n \cap b_n\}_{n\in N}, \end{aligned} \tag{13}$$

for any $\{a_n\}_{n\in N}$, $\{b_n\}_{n\in N}$ in A.

6.3. *The product A of lattices A_n is a lattice. Moreover,*

$$\{a_n\}_{n\in N} \leqslant \{b_n\}_{n\in N} \text{ if and only if } a_n \leqslant b_n \text{ for every } n \in N. \tag{14}$$

The proof of the first part of 6.3 follows from the easy verification that all the axioms (l_1)-(l_3) hold in A. It is based on the commutativity of $\{\ \}$ with $\cup$ and $\cap$. As an example we shall verify the first of the axioms (l_1). We have

$$\{a_n\}_{n\in N} \cup \{b_n\}_{n\in N} = \{a_n \cup b_n\}_{n\in N} = \{b_n \cup a_n\}_{n\in N} = \{b_n\}_{n\in N} \cup \{a_n\}_{n\in N}.$$

To prove the second part, let us observe that $\{a_n\}_{n\in N} \leqslant \{b_n\}_{n\in N}$ if and only if $\{a_n\}_{n\in N} \cup \{b_n\}_{n\in N} = \{a_n \cup b_n\}_{n\in N} = \{b_n\}_{n\in N}$, i.e. if and only if $a_n \cup b_n = b_n$ for every $n \in N$.

§ 7. Infinite joins and meets. Generalizing the definition of the join and meet of two elements (and, consequently, of a finite number of elements) we introduce the following definition: Let S be a non-empty, finite or infinite subset of a lattice A. If l.u.b.S exists, it is called the *join* of elements in S and is denoted by $\bigcup S$. If the g.l.b.S exists, it is called the *meet* of elements in S and is denoted by $\bigcap S$. Usually the set S will be given in the form of an indexed set $\{a_t\}_{t\in T}$, $T \neq 0$. Then the (infinite, in general) join and meet of all elements a_t will be denoted by the symbols

$$\bigcup_{t\in T} a_t \quad \text{and} \quad \bigcap_{t\in T} a_t \tag{1}$$

respectively and sometimes by the more exact symbols

$$\bigcup^{A}_{t\in T} a_t \quad \text{and} \quad \bigcap^{A}_{t\in T} a_t \tag{1'}$$

respectively.

As a rule we shall use the simpler notation (1). However, sometimes it is necessary to use the notation (1'). For instance, sometimes we shall examine, together with a lattice A, various sublattices A' of A. Then

it is necessary to add the superscript A or A' to $\bigcup$ and $\bigcap$ in order to distinguish joins and meets in A from joins and meets in A', respectively. Joins (meets) in a lattice A and in a sublattice A' do not coincide, in general. In fact, if $a_t \in A'$ for every $t \in T$, then the join $\bigcup^{A'}_{t\in T} a_t$ (if it exists) is the least upper bound of all a_t in the lattice A' and it is an upper bound of all a_t in the lattice A. However, it is not always the least upper bound of all a_t in A. Consequently we have

$$\bigcup^{A}_{t\in T} a_t \leqslant \bigcup^{A'}_{t\in T} a_t$$

and, by duality,

$$\bigcap^{A}_{t\in T} a_t \geqslant \bigcap^{A'}_{t\in T} a_t$$

whenever the infinite joins or meets concerned exist. In general, the signs $\leqslant, \geqslant$ cannot be replaced here by $=$. For instance, if A is the lattice of all real numbers, $a_t = t$ for $0 < t < 1$, and A' is the sublattice composed of all a_t and the number 2, then $\bigcup^{A}_{t} a_t = 1$ and $\bigcup^{A'}_{t} a_t = 2$.

If the set T contains only a finite number of elements $t_1, \ldots, t_n$, then by definition

$$\bigcup_{t\in T} a_t = a_{t_1} \cup \ldots \cup a_{t_n} \quad \text{and} \quad \bigcap_{t\in T} a_t = a_{t_1} \cap \ldots \cap a_{t_n}$$

and then the superscript ^{A} at the signs $\bigcup$ and $\bigcap$ is superfluous because the join and meet do not depend on the choice of a sublattice.

The next theorem contains fundamental properties of infinite joins and meets in arbitrary lattices.

7.1. *In any lattice A, if the infinite joins and meets concerned exist, then*

(2) $a_{t_0} \leqslant \bigcup_{t\in T} a_t$ *for every* $t_0 \in T$;

(3) $\bigcap_{t\in T} a_t \leqslant a_{t_0}$ *for every* $t_0 \in T$;

(4) *if* $a_t \leqslant b$ *for every* $t \in T$, *then* $\bigcup_{t\in T} a_t \leqslant b$;

(5) *if* $b \leqslant a_t$ *for every* $t \in T$, *then* $b \leqslant \bigcap_{t\in T} a_t$;

(6) $\bigcap_{t\in T} a_t \leqslant \bigcup_{t\in T} a_t$;

(7) *if* $a_t \leqslant b_t$ *for every* $t \in T$, *then* $\bigcup_{t\in T} a_t \leqslant \bigcup_{t\in T} b_t$;

(8) *if* $a_t \leqslant b_t$ *for every* $t \in T$, *then* $\bigcap_{t\in T} a_t \leqslant \bigcap_{t\in T} b_t$.

If the join $\bigcup_{t\in T} a_t$ exists, then the join $\bigcup_{t\in T}(a \cup a_t)$ also exists and

(9) $$\bigcup_{t\in T}(a \cup a_t) = a \cup \bigcup_{t\in T} a_t .$$

If the meet $\bigcap_{t\in T} a_t$ exists, then the meet $\bigcap_{t\in T}(a \cap a_t)$ also exists and

(10) $$\bigcap_{t\in T}(a \cap a_t) = a \cap \bigcap_{t\in T} a_t .$$

If all the joins or meets concerned exist, then

$$\bigcup_{s\in S}\bigcup_{t\in T} a_{s,t} = \bigcup_{t\in T}\bigcup_{s\in S} a_{s,t}\ ; \tag{11}$$

$$\bigcap_{s\in S}\bigcap_{t\in T} a_{s,t} = \bigcap_{t\in T}\bigcap_{s\in S} a_{s,t}\ ; \tag{12}$$

$$\bigcup_{s\in S}\bigcap_{t\in T} a_{s,t} \leqslant \bigcap_{t\in T}\bigcup_{s\in S} a_{s,t}\ ; \tag{13}$$

$$\bigcup_{t\in T} a_t \cup \bigcup_{t\in T} b_t = \bigcup_{t\in T}(a_t \cup b_t)\ ; \tag{14}$$

$$\bigcap_{t\in T} a_t \cap \bigcap_{t\in T} b_t = \bigcap_{t\in T}(a_t \cap b_t)\ ; \tag{15}$$

$$\bigcap_{t\in T} a_t \cup \bigcap_{t\in T} b_t \leqslant \bigcap_{t\in T}(a_t \cup b_t)\ ; \tag{16}$$

$$\bigcup_{t\in T}(a_t \cap b_t) \leqslant \bigcup_{t\in T} a_t \cap \bigcup_{t\in T} b_t\ . \tag{17}$$

(2) and (4) are another formulation of the definition of the infinite join, and (3) and (5) are another formulation of the definition of the infinite meet. (6) follows easily, for example from (2) and (3).

Defining $b = \bigcup_{t\in T} b_t$ we have $a_t \leqslant b_t \leqslant b$ by (2) and consequently $\bigcup_{t\in T} a_t \leqslant b$ by (4). Thus (7) is proved. By duality we obtain (8) using (3) and (5).

Let $b = a \cup \bigcup_{t\in T} a_t$. Since $a_t \leqslant a \cup a_t$, we get $\bigcup_{t\in T} a_t \leqslant \bigcup_{t\in T}(a \cup a_t)$ by (7). Since $a \leqslant a \cup a_{t_0} \leqslant \bigcup_{t\in T}(a \cup a_t)$ (see (2)), we get $b \leqslant \bigcup_{t\in T}(a \cup a_t)$ (see § 6 (4)). On the other hand, $a \cup a_t \leqslant b$ for every $t \in T$ because of (2) and § 6 (6). Hence $\bigcup_{t\in T}(a \cup a_t) \leqslant b$ by (4), which completes the proof of (9). By duality we prove (10).

Let $b_t = \bigcup_{s\in S} a_{s,t}$. Since $a_{s,t} \leqslant b_t$ by (2), we get $\bigcup_{t\in T} a_{s,t} \leqslant \bigcup_{t\in T} b_t$ because of (7). Hence, by (4), $\bigcup_{s\in S}\bigcup_{t\in T} a_{s,t} \leqslant \bigcup_{t\in T} b_t$, i.e.

$$\bigcup_{s\in S}\bigcup_{t\in T} a_{s,t} \leqslant \bigcup_{t\in T}\bigcup_{s\in S} a_{s,t}\ .$$

By symmetry, we obtain the converse inequality, which proves (11). By duality we prove (12).

Let $b_t = \bigcup_{s\in S} a_{s,t}$. By (8), $\bigcap_{t\in T} a_{s,t} \leqslant \bigcap_{t\in T} b_t$. By (4), $\bigcup_{s\in S}\bigcap_{t\in T} a_{s,t} \leqslant \bigcap_{t\in T} b_t$, which proves (13).

Since $a_t \leqslant a_t \cup b_t$ for every $t \in T$, we have $\bigcup_{t\in T} a_t \leqslant \bigcup_{t\in T}(a_t \cup b_t)$ by (7). By the same argument, $\bigcup_{t\in T} b_t \leqslant \bigcup_{t\in T}(a_t \cup b_t)$. Consequently

$$\bigcup_{t\in T} a_t \cup \bigcup_{t\in T} b_t \leqslant \bigcup_{t\in T}(a_t \cup b_t)\ .$$

On the other hand, $a_{t_0} \cup b_{t_0} \leqslant \bigcup_{t\in T} a_t \cup \bigcup_{t\in T} b_t$ for every $t_0 \in T$ by (2). Hence, by (4), we obtain the converse inequality

$$\bigcup_{t\in T}(a_t \cup b_t) \leqslant \bigcup_{t\in T} a_t \cup \bigcup_{t\in T} b_t$$

which proves (14). The proof of (15) is by duality.

Since $a_t \leqslant a_t \cup b_t$ for every $t \in T$, we have $\bigcap_{t\in T} a_t \leqslant \bigcap_{t\in T}(a_t \cup b_t)$. By the same argument, $\bigcap_{t\in T} b_t \leqslant \bigcap_{t\in T}(a_t \cup b_t)$. Hence (16) follows. The proof of (17) is by duality.

Let A' be a sublattice of a lattice A and let $a_t \in A'$ for every $t \in T$. If the element $\bigcup^{A}_{t \in T} a_t$ exists and belongs to A', then $\bigcup^{A'}_{t \in T} a_t$ also exists and

$$\bigcup^{A}_{t \in T} a_t = \bigcup^{A'}_{t \in T} a_t . \tag{18}$$

Analogously $\bigcap^{A}_{t \in T} a_t \in A'$ implies

$$\bigcap^{A}_{t \in T} a_t = \bigcap^{A'}_{t \in T} a_t . \tag{19}$$

If, for every indexed set $\{a_t\}_{t \in T}$ of elements of a sublattice A' of a lattice A, the existence $\bigcup^{A}_{t \in T} a_t$ implies $\bigcup^{A}_{t \in T} a_t \in A'$ and the existence of $\bigcap^{A}_{t \in T} a_t$ implies $\bigcap^{A}_{t \in T} a_t \in A'$, then A' is said to be a *complete sublattice* of A.

Let h be a lattice homomorphism from a lattice A into a lattice B.

Suppose that $a_t \in A$ for every $t \in T$ and that the join

$$a = \bigcup^{A}_{t \in T} a_t \tag{20}$$

exists. Since $a_t \leqslant a$ for every $t \in T$, we have $h(a_t) \leqslant h(a)$ for every $t \in T$ (see § 6, (9)), and consequently

$$h(a) \geqslant \bigcup^{B}_{t \in T} h(a_t) , \tag{21}$$

provided the join on the right-hand side exists.

The sign $\geqslant$ cannot, in general, be replaced here by the sign $=$. If

$$h(a) = \bigcup^{B}_{t \in T} h(a_t) ,$$

i.e. if

$$h\left(\bigcup^{A}_{t \in T} a_t\right) = \bigcup^{B}_{t \in T} h(a_t) ,$$

then we say that the *homomorphism h from A into B preserves the* (infinite) *join* (20).

Suppose now that $a_t \in T$ for every $t \in T$ and that the meet

$$a = \bigcap^{A}_{t \in T} a_t \tag{22}$$

exists. Since $a \leqslant a_t$ for every $t \in T$, we have $h(a) \leqslant h(a_t)$ for every $t \in T$ (see § 6, (9)), and consequently

$$h(a) \leqslant \bigcap^{B}_{t \in T} h(a_t) , \tag{23}$$

provided the join on the right-hand side exists. The sign $\leqslant$ cannot, in general, be replaced here by the sign $=$. If

$$h(a) = \bigcap^{B}_{t \in T} h(a_t) ,$$

i.e. if

$$h\left(\bigcap^{A}_{t \in T} a_t\right) = \bigcap^{B}_{t \in T} h(a_t) ,$$

then we say that the *homomorphism h from A into B preserves the* (infinite) *meet* (22).

For instance, each isomorphism h of a lattice A *onto* a lattice B preserves all infinite joins and meets. This follows from the fact that infinite joins and meets are defined only by means of the relation $\leqslant$, and h preserves this relation in both directions (see 6.2). The above remarks fail to apply, in general, to isomorphisms *into*.

It should be emphasized that in expressions like "a homomorphism h preserves a join (meet)" we must always indicate which lattice is assumed as the counter-domain of h. For instance, if h is an isomorphism from A onto a proper sublattice $B' = h(A)$ of a lattice B, then h considered as a homomorphism from A into B' preserves all joins and meets in A but h conceived as a homomorphism from A into B does not, in general, have this property. As an example of this type we shall repeat the example from p. 39. Let B be the lattice of all real numbers, let $a_t = t$ for $0 < t < 1$, let A be the sublattice formed of all a_t and the number 2, and let h be the identity mapping on A. Then 2 is the join of all a_t in A, $h(2)$ is the join of all $h(a_t)$ in $B' = h(A) = A$, but $h(2)$ is not the join of all $h(a_t)$ in the lattice B. Thus h considered as a homomorphism from A into B does not preserve the infinite join $2 = \bigcup_t^A a_t$.

A lattice A is said to be *complete* [1] if the join $\bigcup S$ and the meet $\bigcap S$ exist for every non-empty set $S \subset A$.

For instance, the lattice of all real numbers (including $\pm\infty$) is complete. The lattice of all rational numbers is not complete. The lattice of all subsets of a fixed space is complete and the infinite joins and meets always coincide with the set-theoretical unions and intersections. Every finite lattice is complete.

Every complete lattice has the unit element and the zero element. Namely, the join (the meet) of all elements is the unit element (the zero element).

Conversely, if a lattice A has the unit element and the meet $\bigcap S'$ exists for every non-empty set $S' \subset A$, then A is complete. Indeed, if S is any non-empty subset of A, let S' be the set of all $b \in A$ such that $a \leqslant b$ for every $a \in S$. The set S' is not empty since $1 \in S'$. It is easy to prove that the meet $\bigcap S'$ is the join $\bigcup S$.

Similarly we prove that if a lattice A has the zero element and the join $\bigcup S'$ exists for every non-empty set $S' \subset A$, then A is complete.

It is easy to verify that every complete sublattice of a complete lattice is a complete lattice.

---

[1] Birkhoff [1].

Every lattice A can be considered as a generalized algebra

$$\{A, \cup, \cap, \bigcup, \bigcap\} \tag{24}$$

with two binary operations $\cup$ and $\cap$ (the join and meet) and two infinite operations $\bigcup$ and $\bigcap$ (the infinite join and the infinite meet). The domain of $\bigcup$ (of $\bigcap$) is the class of all sets $S \subset A$ such that $\bigcup S$ ($\bigcap S$) exists. Thus if A is a complete lattice, then (24) is a complete algebra in the sense defined in § 4, p. 28. A set $A' \subset A$ is a complete sublattice of a lattice A if and only if $\{A', \cup, \cap, \bigcup, \bigcap\}$ is a subalgebra of $\{A, \cup, \cap, \bigcup, \bigcap\}$ in the sense defined in § 4, p. 28.

Sometimes it is necessary to restrict the domain of $\bigcup$ and $\bigcap$ to less classes $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ of sets $S \subset A$. We then write

$$\{A, \cup, \cap, \bigcup | \mathfrak{Q}_1, \bigcap | \mathfrak{Q}_2\} \tag{25}$$

instead of (24), according to the convention in § 4, p. 30. Usually the joins and meets of sets in $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ respectively will be written in the form of indexed sets:

$$\begin{aligned} a_s &= \bigcup_{t \in T'_s}^{A} a_{s,t} \qquad (s \in S'), \\ b_s &= \bigcap_{t \in T''_s}^{A} b_{s,t} \qquad (s \in S''). \end{aligned} \tag{Q}$$

A lattice homomorphism h from A into a lattice B will be called a **Q**-*homomorphism* from A into B provided it preserves all the joins and meets in (Q). In other words, h is called a **Q**-homomorphism from A into B if it is a homomorphism from the generalized algebra $\{A, \cup, \cap, \bigcup | \mathfrak{Q}_1, \bigcap | \mathfrak{Q}_2\}$ into the generalized algebra $\{B, \cup, \cap, \bigcup, \bigcap\}$, in the sense defined in § 4, p. 29. The same terminology is applied to isomorphisms: by a **Q**-*isomorphism from* A *into* B we shall understand any one-to-one **Q**-homomorphism, i.e. any isomorphism preserving all the joins and meets in (Q).

As we have proved in 6.3, the product $\{A, \cup, \cap\}$ of any indexed set of lattices $\{A_n, \cup, \cap\}_{n \in N}$, is a lattice. Now we shall prove that:

7.2. *For any indexed set of elements* $a_t = \{a_{t,n}\}_{n \in N}$ *of elements of* $A = P_{n \in N} A_n$,

$$\begin{aligned} \bigcup_{t \in T} a_t &= \{\bigcup_{t \in T} a_{t,n}\}_{n \in N}, \\ \bigcap_{t \in T} a_t &= \{\bigcap_{t \in T} a_{t,n}\}_{n \in N}, \end{aligned} \tag{26}$$

provided all the joins or meets on the right-hand side exist.

Thus if all A_n *are complete lattices,* A *is also.*

We shall verify only the first of identities (26). The proof of the second is by duality.

For brevity, let $b_n = \bigcup_{t \in T} a_{t,n}$ and $b = \{b_n\}_{n \in N}$. Since $a_{t,n} \leqslant b_n$ for every $n \in N$, we have $a_t \leqslant b$ for all $t \in T$ (see § 6, (14)). On the other hand, suppose that, for every $t \in T$, $a_t \leqslant c$ for an element $c = \{c_n\}_{n \in N}$. Thus $a_{t,n} \leqslant c_n$ for every $t \in T$ and consequently $b_n \leqslant c_n$. This proves that $b \leqslant c$, by § 6, (14). Thus b is the join of all elements a_t.

7.3. *Suppose that, for every* $n \in N$,

$$\{A_n, \cup, \cap, \bigcup | \mathfrak{Q}_{1,n}, \bigcap | \mathfrak{Q}_{2,n}\} \tag{27}$$

is a lattice with infinite joins and meets restricted to classes $\mathfrak{Q}_{1,n}, \mathfrak{Q}_{2,n}$. *The lattice*

$$\{\boldsymbol{P}_{n \in N} A_n, \cup, \cap, \bigcup | \overset{*}{\boldsymbol{P}}_{n \in N} \mathfrak{Q}_{1,n}, \bigcap | \overset{*}{\boldsymbol{P}}_{n \in N} \mathfrak{Q}_{2,n}\}$$

is the product of generalized algebras (27) (in the sense defined in § 4, p. 31).

It suffices to verify that the operations defined in the product of all A_n by means of § 4, (28), coincide with the infinite join and meet. This follows directly from 7.2.

§ 8. Filters and ideals. A non-empty set ∇ of elements of a lattice A is said to be a *filter* in A provided that, for any elements $a, b \in A$,

(f) $\quad a \cap b \in \nabla \quad$ *if and only if* $\quad a \in \nabla$ *and* $b \in \nabla$.

In other words, a non-empty set ∇ of elements of a lattice A is a filter if and only if the following two conditions are satisfied:

(f_1) $\quad$ *if* $a, b \in \nabla$, *then* $a \cap b \in \nabla$;

(f_2) $\quad$ *if* $a \in \nabla$ *and* $a \leqslant b$, *then* $b \in \nabla$.

The condition (f_2) can be replaced here by the following:

(f_2') $\quad$ *if* $a \in \nabla$ *and* $b \in A$, *then* $a \cup b \in \nabla$.

The notion dual to filter is called *ideal*. The definition of the ideal is obtained by replacing $\cup, \cap, \leqslant$ everywhere by $\cap, \cup, \geqslant$, respectively, in the definition of filter: a non-empty set Δ of elements of a lattice A is said to be an *ideal* [1] in A provided, for any elements $a, b \in A$,

(i) $\quad a \cup b \in \Delta \quad$ *if and only if* $\quad a \in \Delta$ *and* $b \in \Delta$.

In other words, a non-empty set Δ of elements of a lattice A is an ideal if and only if the following two conditions are satisfied:

(i_1) $\quad$ *if* $a, b \in \Delta$, *then* $a \cup b \in \Delta$;

(i_2) $\quad$ *if* $a \leqslant b$ *and* $b \in \Delta$, *then* $a \in \Delta$.

---

[1] Stone [1]. Filters are called *dual ideals* by many authors.

Condition (i_2) can be replaced here by the following:

(i_2') *if* $a \in \Delta$ *and* $b \in A$, *then* $a \cap b \in \Delta$.

For instance, the whole lattice A is a filter (an ideal) in A.

For every fixed element $a_0 \in A$, the set of all elements $a \geqslant a_0$ ($a \leqslant a_0$) is a filter (an ideal) called the *principal filter (ideal) generated by* a_0.

If the lattice A has the unit element $\vee$ (the zero element $\wedge$), then the set composed of $\vee$ only (of $\wedge$ only) is a filter (ideal) called the *unit filter* (the *zero ideal*). It is the principal filter (ideal) generated by $\vee$ (by $\wedge$).

If h is a homomorphism from a lattice A into a lattice B containing the unit element $\vee$ (the zero element $\wedge$), then the set $h^{-1}(\vee)$ (the set $h^{-1}(\wedge)$) is a filter (an ideal) in A, provided it is not empty.

More generally, if h is a homomorphism from a lattice A into a lattice B, and ∇ is a filter (Δ is an ideal) in B, then the set $h^{-1}(\nabla)$ (the set $h^{-1}(\Delta)$) is a filter (an ideal) in A provided it is not empty.

If A' is a sublattice of a lattice A and ∇ is a filter (Δ is an ideal) in A, then the common part $A' \cap \nabla$ ($A' \cap \Delta$) is a filter (an ideal) in A' provided it is not empty.

It follows directly from (f_2) (from (i_2)) that if A has the unit element $\vee$ (the zero element $\wedge$), then $\vee \in \nabla$ ($\wedge \in \Delta$) for every filter ∇ (ideal Δ) in A.

It is easy to verify that the intersection of any non-empty class of filters (ideals) in A is a filter (an ideal) in A provided it is not empty. The intersection is certainly non-empty if A has the unit element (the zero element).

For every non-empty set A_0 of elements in A there exists a least filter ∇ (a least ideal Δ) containing A_0. Namely the filter ∇ (the ideal Δ) is the intersection of all filters (ideals) containing A_0. The least filter ∇ (ideal Δ) is said to be *filter (ideal) generated by the set* A_0. If A has the unit element $\vee$ (the zero element $\wedge$), then the hypothesis that A_0 is not empty can be omitted: if A_0 is empty, then the unit filter (the zero ideal) composed only of $\vee$ (of $\wedge$) is the least filter (ideal) containing A_0. In other words, the filter (ideal) composed only of $\vee$ (of $\wedge$) is the filter (ideal) generated by the empty set.

8.1. *The filter (ideal) generated by a non-empty set* A_0 *of a lattice* A *is the set of all elements* $a \in A$ *such that* $a \geqslant a_1 \cap \ldots \cap a_n$ (*such that* $a \leqslant a_1 \cup \ldots \cup a_n$) *for some elements* $a_1, \ldots, a_n \in A_0$.

In fact, the set of all elements a satisfying this condition is contained in every filter which contains A_0. On the other hand, this set is also a filter (since it satisfies (f_1) and (f_2)) and it contains A_0. Thus this set is the least filter containing A_0.

By duality, we prove the statement for ideals.

8.2. *For any fixed element $a_0 \in A$ and a filter ∇ in A, the class of all elements a such that*

$$a \geqslant a_0 \cap c \quad \text{for an element} \quad c \in \nabla$$

is the least filter containing a_0 and ∇, i.e. the filter generated by the set $(a_0) \cup \nabla$.

This follows easily from 8.1 because of (f_1).

By duality, we prove the statement for ideals.

If the set A_0 contains only one element a_0, then the filter (ideal) generated by A_0 is the principal filter (ideal) generated by a_0. More generally, if A_0 consists of a finite number of elements $a_1, \ldots, a_n$, then the filter (ideal) generated by A_0 is the principal filter (ideal) generated by the element $a_0 = a_1 \cap \ldots \cap a_n$ $(a_0 = a_1 \cup \ldots \cup a_n)$.

A filter (an ideal) in A is said to be *proper* if it is a proper subset of the lattice A. If A has the zero element $\wedge$ (the unit element $\vee$), then a filter ∇ (an ideal Δ) is proper if and only if $\wedge \notin \nabla$ (if $\vee \notin \Delta$). This follows easily from (f_2) (from (i_2)).

Any non-empty class of filters (ideals) in a lattice A will be considered as an ordered set, the ordering relation being the set-theoretical inclusion. Thus by a *chain of filters* we mean a non-empty class of filters such that, for any filters ∇_1, ∇_2 in this class, either $\nabla_1 \subset \nabla_2$ or $\nabla_2 \subset \nabla_1$. By duality we obtain the notion of a *chain of ideals*.

8.3. *The union of any chain of filters (ideals) in a lattice A is a filter (an ideal) in A. The union of any chain of proper filters (ideals) in a lattice A having the zero (unit) element is a proper filter (ideal).*

The proof of the first part of 8.3 is by an easy verification. If all filters in the chain are proper, then they do not contain $\wedge$. Consequently the union does not contain $\wedge$, i.e. it is a proper filter.

By duality we prove the dual statement for ideals.

A filter (an ideal) is said to be *maximal* in A provided it is proper and it is not any proper subset of a proper filter (ideal) in A. In other words, a proper filter (ideal) is maximal if and only if it is a maximal element in the ordered set of all proper filters (ideals) in A.

8.4. *If a lattice A has the zero element (the unit element), then every proper filter ∇_0 (ideal Δ_0) in A is contained in a maximal filter (ideal).*

Let A be the ordered set of all proper filters in A. By the second part of 8.3, every chain of elements in A has an upper bound in A. By the induction principle 5.1 for ordered sets, for every given filter $\nabla_0 \in A$ there exists a maximal element $\nabla \in A$, i.e. a maximal filter in A, such that $\nabla_0 \subset \nabla$.

By duality, we prove the dual statement for ideals.

8.5. *If a lattice A has the zero element $\wedge$ (the unit element $\vee$), then for every element $a_0 \neq \wedge$ ($a_0 \neq \vee$) in A there exists a maximal filter ∇ (maximal ideal Δ) in A such that $a_0 \in \nabla$ ($a_0 \in \Delta$).*

Theorem 8.5 follows immediately from 8.4 where ∇_0 (Δ_0) is the principal filter (ideal) generated by a_0.

According to the general definition concerning abstract algebras (see § 4, p. 22), a lattice A is said to be *degenerate* if it has only one element (this element is then simultaneously the unit and the zero of A).

8.6. *In every non-degenerate lattice A having the zero (unit) element there exists a maximal filter (ideal).*

This follows immediately from 8.5 where a_0 is an element different from $\wedge$.

By duality we prove the dual statement for ideals.

A filter ∇ (an ideal Δ) is said to be *prime* provided it is proper and the condition $a \cup b \in \nabla$ implies that either $a \in \nabla$ or $b \in \nabla$ (the condition $a \cap b \in \Delta$ implies that either $a \in \Delta$ or $b \in \Delta$).

For instance, if A is the interval $0 \leqslant x \leqslant 1$ of real numbers, then the intervals $0 \leqslant x < x_0$ where $0 < x_0 \leqslant 1$, and the intervals $0 \leqslant x \leqslant x_0$ where $0 \leqslant x_0 < 1$ are prime ideals. The intervals $x_0 < x \leqslant 1$ where $0 \leqslant x_0 < 1$, and the intervals $x_0 \leqslant x \leqslant 1$ where $0 < x_0 \leqslant 1$ are prime filters. The interval $0 \leqslant x < 1$ is a maximal ideal and the interval $0 < x \leqslant 1$ is a maximal filter. There exist no other prime or maximal ideals or filters in this lattice.

8.7. *Let Δ and ∇ be two disjoint sets whose union is the whole lattice A. The set Δ is a prime ideal if and only if ∇ is a prime filter.*

Suppose that Δ is a prime ideal. If $a, b \in \nabla$, then $a \notin \Delta$ and $b \notin \Delta$. The ideal Δ being prime, the element $a \cap b$ does not belong to Δ, i.e., $a \cap b \in \nabla$. If $b \in \nabla$ and $a \geqslant b$, then $b \notin \Delta$ and, by (i_2), $a \notin \Delta$, i.e., $a \in \nabla$. Thus ∇ is a filter. ∇ is proper since Δ is non-empty. If $a \cup b \in \nabla$ then $a \cup b \notin \Delta$, and, by (i_1) at least one of the elements a, b does not belong to Δ, i.e. either $a \in \nabla$ or $b \in \nabla$. This proves that ∇ is prime. By duality we obtain that if ∇ is a prime filter, then Δ is a prime ideal.

Theorem 8.7 shows that there exists a natural one-to-one correspondence between prime ideals and prime filters of a lattice A. We obtain this correspondence by associating with every prime ideal Δ its complement $\nabla = -\Delta = A - \Delta$ to the lattice A. By 8.7 the set-theoretical complement $-\Delta$ is a prime filter and each prime filter ∇ is of the form $\nabla = -\Delta$ where Δ is a prime ideal. The practical aspect of this correspondence is that, in some sense, it is the same whether we speak about prime ideals or about prime filters. This fact will be used in the representation theorem to be proved later (§ 9).

Observe that the unit filter in a lattice A is maximal if and only if the lattice A has exactly two elements. The unit filter in a lattice A having at least two elements is prime if and only if the lattice A has the following property:

(1) if $a \cup b = V$, then either $a = V$ or $b = V$.

Of course, every two-element lattice has property (1) but there also exist lattices having this property and containing more than two elements (see the example quoted on p. 47).

§ 9. Distributive lattices. A lattice A is said to be *distributive* if, for all $a, b, c \in A$,

(l_4) $a \cap (b \cup c) = (a \cap b) \cup (a \cap c), \quad a \cup (b \cap c) = (a \cup b) \cap (a \cup c)$.

Observe that if one of the *distributive laws* (l_4) is satisfied for arbitrary $a, b, c \in A$, then the remaining dual law is also satisfied. Consequently, to prove that a lattice A is distributive it suffices to show that one of the identities (l_4) holds in A.

Indeed, suppose that the first axiom (l_4) holds. Then

$$(a \cup b) \cap (a \cup c) = \big((a \cup b) \cap a\big) \cup \big((a \cup b) \cap c\big) = a \cup \big((a \cap c) \cup (b \cap c)\big)$$
$$= \big(a \cup (a \cap c)\big) \cup (b \cap c) = a \cup (b \cap c)$$

by (l_1), (l_2), (l_3) and the first distributive law (l_4). By duality we obtain the second part of our remark.

Chains, set lattices and the lattice of all real functions defined on a space are examples of distributive lattices. The lattice formed of the following five points on the plane:

(1) $p_0 = (-1, 0), \quad p_1 = (0, -1), \quad p_2 = (0, 0), \quad p_3 = (0, 1), \quad p_4 = (1, 0)$

with the operations determined by the following ordering:

$(x_1, y_1) \leqslant (x_2, y_2)$. if and only if either $(x_1, y_1) = (x_2, y_2)$ or $x_1 < x_2$

is not distributive. In fact,

$$p_1 \cap (p_2 \cup p_3) = p_1 \cap p_4 = p_1$$

and

$$(p_1 \cap p_2) \cup (p_1 \cap p_3) = p_0 \cup p_0 = p_0 \neq p_1 .$$

Observe that a lattice A is distributive if and only if, for every element $a_0 \in A$, the mapping

(2) $$h_1(x) = a_0 \cap x$$

is a homomorphism of A onto the sublattice of all elements $y \leqslant a_0$, and the mapping

$$h_2(x) = a_0 \cup x \tag{3}$$

is a homomorphism of A onto the sublattice of all elements $y \geqslant a_0$.

9.1. *In a distributive lattice, each maximal filter ∇ (maximal ideal Δ) is prime.*

Suppose that ∇ is not prime, i.e. there exist elements a, b such that

$$a \cup b \in \nabla, \quad a \notin \nabla, \quad b \notin \nabla.$$

By 8.2 the set ∇_1 of all elements x such that

$$x \geqslant a \cap c \quad \text{for an element} \quad c \in \nabla$$

is a filter.

The filter ∇_1 is proper since $b \notin \nabla_1$. In fact, the hypothesis $b \in \nabla_1$ implies that $b \geqslant a \cap c$ for an element $c \in \Delta$. Since $a \cup b \in \nabla$ and $c \cup b \in \nabla$, we infer that

$$b = (a \cap c) \cup b = (a \cup b) \cap (c \cup b) \in \nabla$$

in contradiction to our assumptions.

Obviously $\nabla \subset \nabla_1$ and $\nabla_1 \neq \nabla$ since $a \in \nabla_1$ and $a \notin \nabla$. Thus the filter ∇ is not maximal.

By duality we obtain the proof for ideals.

Example (1) shows that theorem 9.1 is false without the hypothesis of distributivity. The set ∇ composed of points p_3, p_4 is a maximal filter. It is not prime because $p_1 \cup p_2 \in \nabla$ and $p_1 \notin \nabla$, $p_2 \notin \nabla$.

9.2. *For arbitrary elements a, b of a distributive lattice A, if the relation $b \leqslant a$ does not hold, then there exists a prime filter ∇ (prime ideal Δ) such that*

$$a \notin \nabla \quad \textit{and} \quad b \in \nabla \tag{4}$$

(such that $a \in \Delta$ and $b \notin \Delta$).

Let A be the ordered set of all filters ∇ in A such that (4) holds. The set A is not empty because it contains the principal filter generated by b. By the first part of 8.3 any chain of filters in A has an upper bound in A. By the induction principle 5.1 for ordered sets, there exists a maximal element ∇_0 in A. Since ∇_0 also satisfies (4), it suffices to prove that the filter ∇_0 is prime.

Suppose that ∇_0 is not prime, i.e. there exist elements $a_1, a_2 \in A$ such that

$$a_1 \cup a_2 \in \nabla_0 \quad \text{but} \quad a_1 \notin \nabla_0 \text{ and } a_2 \notin \nabla_0. \tag{5}$$

Let V_i be the filter generated by $(a_i) \cup V_0$, $i = 1, 2$. One of the filters V_1 and V_2 does not contain the element a. For if $a \in V_1$ and $a \in V_2$, then by 8.2 there exist elements $c_1, c_2 \in V_0$ such that

$$a \geqslant a_1 \cap c_1 \quad \text{and} \quad a \geqslant a_2 \cap c_2 .$$

Assuming $c = c_1 \cap c_2$, we have $c \in V_0$ and

$$a \geqslant a_1 \cap c , \quad a \geqslant a_2 \cap c .$$

Consequently

$$a \geqslant (a_1 \cap c) \cup (a_2 \cap c) = (a_1 \cup a_2) \cap c \in V_0 .$$

This implies $a \in V_0$, which is impossible.

Let i be the one of the integers $1, 2$ for which $a \notin V_i$. The filter V_i satisfies (4), i.e. $V_i \in A$. By definition, $V_0 \subset V_i$. Since $a_i \in V_i$ but $a_i \notin V_0$, we have $V_i \neq V_0$. This contradicts the hypothesis that V_0 is a maximal element in the set A. Theorem 9.2 is proved (the proof for ideals is by duality).

Observe that each sublattice of a distributive lattice is also distributive. If there exists a homomorphism of a distributive lattice $\boldsymbol{A}$ onto a distributive lattice $\boldsymbol{B}$, then $\boldsymbol{B}$ is also distributive. In particular, every lattice isomorphic to a set lattice is distributive.

Conversely, every distributive lattice is isomorphic to a set lattice [1].

To prove this we assume the following notation: $\mathfrak{F}(\boldsymbol{A})$ denotes the set of all prime filters of a lattice $\boldsymbol{A}$, $h(a)$—the set of all $V \in \mathfrak{F}(\boldsymbol{A})$ such that $a \in V$, and $\boldsymbol{P}(\boldsymbol{A})$—the class of all sets $h(a)$, $a \in \boldsymbol{A}$.

9.3. *If $\boldsymbol{A}$ is a distributive lattice, then $\boldsymbol{P}(\boldsymbol{A})$ is a set lattice and h is an isomorphism of $\boldsymbol{A}$ onto $\boldsymbol{P}(\boldsymbol{A})$* [2].

If $a, b \in \boldsymbol{A}$, $a \neq b$, then one of the relations $a \leqslant b$, $b \leqslant a$ does not hold. By theorem 9.2, there is a prime filter which belongs to exactly one of the sets $h(a)$, $h(b)$. Thus $h(a) \neq h(b)$, i.e. the mapping h is one-to-one.

It suffices to prove that

$$h(a \cup b) = h(a) \cup h(b) , \tag{6}$$

$$h(a \cap b) = h(a) \cap h(b) \tag{7}$$

for $a, b \in \boldsymbol{A}$, where the signs $\cup$, $\cap$ on the right-hand sides of (6) and (7) denote, respectively, the set-theoretical union and intersection.

If $V \in h(a \cup b)$, i.e. $a \cup b \in V$, then either $a \in V$ or $b \in V$ since V is prime. Consequently, either $V \in h(a)$ or $V \in h(b)$ which proves that $V \in h(a) \cup h(b)$. Conversely, if $V \in h(a) \cup h(b)$, i.e. either $a \in V$ or $b \in V$, then, by (f_2'), $a \cup b \in V$, i.e. $V \in h(a \cup b)$. This proves (6).

[1] Birkhoff [1].

[2] Stone [4].

If $V \in h(a \cap b)$, i.e. $a \cap b \in V$, then, by (f), $a \in V$ and $b \in V$, i.e. $V \in h(a) \cap h(b)$. Conversely, if $V \in h(a) \cap h(b)$, i.e. $a \in V$ and $b \in V$, then $a \cap b \in V$ by (f), i.e. $V \in h(a \cap b)$. This proves (7).

In the sequel h, $\mathfrak{T}(A)$ and $\boldsymbol{P}(A)$ will be called the *Stone isomorphism*, the *Stone space* and the *Stone lattice* of A.

We shall consider the Stone space $\mathfrak{T}(A)$ as a topological space, the class $\boldsymbol{P}(A)$ being assumed as a subbasis (see 2.1).

9.4. *The Stone space $\mathfrak{T}(A)$ of any distributive lattice A is a T_0-space. If A has the unit V, then $\mathfrak{T}(A)$ [1] is compact.*

If $V_1, V_2 \in \mathfrak{T}(A)$ and $V_1 \neq V_2$, then there exists an element $a \in A$ which belongs to precisely one of the sets V_1 and V_2, say $a \in V_1$ and $a \notin V_2$. Consequently $V_1 \in h(a)$ and $V_2 \notin h(a)$, i.e. $h(a)$ is an open set containing exactly one of the points V_1, V_2 of the space $\mathfrak{T}(A)$. Thus $\mathfrak{T}(A)$ is a T_0-space.

Suppose that the unit V of A exists and $\mathfrak{T}(A)$ is the union of certain open sets G_t, $t \in T$. By definition of the topology in $\mathfrak{T}(A)$, every set G_t is the union of some sets $h(a_u)$ where u ranges over a set U_t. Consequently $\mathfrak{T}(A)$ is the union of all sets $h(a_u)$ where u ranges over the union U of all sets U_t, $t \in T$. It suffices to prove that

$$\mathfrak{T}(A) = h(a_{u_1}) \cup \ldots \cup h(a_{u_m})$$

for some $u_1, \ldots, u_m \in U$, because this equation implies

$$\mathfrak{T}(A) = G_{t_1} \cup \ldots \cup G_{t_m}, \quad \text{where} \quad u_j \in U_{t_j} \quad \text{for} \quad j = 1, \ldots, m.$$

Suppose the contrary, i.e. that for all $u_1, \ldots, u_m \in U$

$$h(a_{u_1} \cup \ldots \cup a_{u_m}) = h(a_{u_1}) \cup \ldots \cup h(a_{u_m}) \neq \mathfrak{T}(A) = h(V).$$

Since h is one-to-one, we infer that

$$a_{u_1} \cup \ldots \cup a_{u_m} \neq V \quad \text{for} \quad u_1, \ldots, u_m \in U,$$

i.e. (see 8.1) that the ideal Δ_0 generated by all a_u ($u \in U$) does not contain the unit element V. By 8.4, the proper ideal Δ_0 is contained in a maximal ideal Δ which is prime because of 9.1. By 8.7 the set V of all $a \in A$ such that $a \notin \Delta$ is a prime filter in A, i.e. $V \in \mathfrak{T}(A)$. By definition, $a_u \notin V$, i.e. $V \notin h(a_u)$ for every $u \in U$. This is impossible since $\mathfrak{T}(A)$ is the union of all sets $h(a_u)$, $u \in U$.

Notice that, by the duality principle and the one-to-one correspondence between prime ideals and prime filters (see § 8), theorems 9.3 and 9.4 remain true if $\mathfrak{T}(A)$ is assumed to be the set of all prime ideals of A, $h(a)$ is defined as the set of all prime ideals Δ such that $a \notin \Delta$, and $\boldsymbol{P}(A)$ is, as previously, the class of all sets $h(a)$.

[1] Stone [4].

9.5. *The power of* $\mathfrak{S}(A)$ *is* $\leqslant 2^{\mathfrak{m}}$, *where* $\mathfrak{m}$ *is the cardinal of* A.

$2^{\mathfrak{m}}$ is the cardinal of the set of all subsets of the set A. Since every prime filter is a subset of A, the cardinal of the set of all prime filters is $\leqslant 2^{\mathfrak{m}}$.

§ 10. Complement and pseudo-complement. It follows from § 9 that the notions of (finite) join and meet in any lattice A are abstract analogues of the notions of union and intersection of subsets of a fixed space X. If the lattice A has the unit element or the zero element, they are abstract analogues of the whole space and the empty set respectively (see § 6, (10), (11), and (12)).

The set-theoretical complement $-A$ of a subset A of a space X can be defined as the greatest subset of X which is disjoint from A, or as the least subset of X whose union with A gives the whole space X. Each of the two remarks suggests a definition of complement of elements in lattices. Unfortunately, the two definitions are not, in general, equivalent. Therefore we have to define two notions of complement of an element a in a lattice A.

Suppose that a lattice A has the zero element. An element $c \in A$ is said to be the $\cap$-*complement* of an element a in A if c is the greatest element such that $a \cap c = \wedge$ (i.e. if it is the greatest element in the set of all $x \in A$ such that $a \cap x = \wedge$).

Suppose that a lattice A has the unit element $\vee$. An element $c \in A$ is said to be the $\cup$-*complement* of an element a in A if c is the least element such that $a \cup c = \vee$ (i.e. if c is the least element in the set of all $x \in A$ such that $a \cup x = \vee$).

It follows from the definition that the $\cap$-complement ($\cup$-complement) of a, if it exists, is uniquely determined by a. The notions of $\cap$-complement and $\cup$-complement are dual.

If an element $a \in A$ has simultaneously the $\cap$-complement c_1 and the $\cup$-complement c_2, then they are not, in general, equal. If the lattice A is distributive, then

$$c_1 \leqslant c_2$$

since

$$c_1 = c_1 \cap \vee = c_1 \cap (a \cup c_2) = (c_1 \cap a) \cup (c_1 \cap c_2) = \wedge \cup (c_1 \cap c_2) = c_1 \cap c_2 .$$

An element $c \in A$ is said to be the *complement* of an element $a \in A$ if it is simultaneously the $\cap$-complement and the $\cup$-complement of a. Then

$$a \cap c = \wedge \quad \text{and} \quad a \cup c = \vee . \tag{1}$$

Conversely,

10.1. *If a lattice A is distributive and* (1) *holds, then c is the complement of a.*

Indeed, suppose that $a \cap x = \Lambda$. Then

$$x = x \cap (a \cup c) = (x \cap a) \cup (x \cap c) = x \cap c ,$$

i.e. $x \leqslant c$. This proves that c is the $\cap$-complement of a. Similarly, if $a \cup x = V$, then

$$x = x \cup (a \cap c) = (x \cup a) \cap (x \cup c) = x \cup c ,$$

i.e. $x \geqslant c$. This proves that c is the $\cup$-complement of a. Thus c is the complement of a.

Only the notion of complement and the notion of $\cap$-complement will play an essential part in this book. Therefore we shall introduce special notation only for these notions. The $\cap$-complement (in particular, the complement) of an element a in a lattice A will be denoted, if it exists, by

$$-a .$$

We always assume that the sign "$-$" binds more strongly than "$\cup$" and "$\cap$", i.e. expressions like

$$-a \cup -b , \quad -a \cap b$$

should be understood as

$$(-a) \cup (-b) , \quad (-a) \cap b$$

respectively, etc.

The $\cap$-complement $-a$ will also be called the *pseudo-complement* of a in A.

By definition,

(2) $$x \leqslant -a \quad \text{if and only if} \quad a \cap x = \Lambda .$$

In particular,

(3) $$a \cap -a = \Lambda .$$

In any lattice with the unit element V and the zero element Λ, V is the complement of Λ and Λ is the complement of V, in symbols:

(4) $$V = -\Lambda , \quad \Lambda = -V .$$

If elements a, b have the pseudo-complements $-a, -b$, then

(5) $$a \leqslant b \quad \text{implies} \quad -b \leqslant -a$$

since then $a \cap -b = (a \cap b) \cap -b = a \cap (b \cap -b) = a \cap \Lambda = \Lambda$.

10.2. *Suppose that the lattice is distributive. If elements* a, b *have complements* $-a, -b$, *then*

$$a \leqslant b \quad \text{if and only if} \quad -b \leqslant -a . \tag{6}$$

If an element a *has the complement* $-a$, *then* $-a$ *also has a complement, and*

$$--a = a , \tag{7}$$

i.e. a *is the complement of* $-a$. *If elements* a, b *have complements, then also the elements* $a \cup b$ *and* $a \cap b$ *have complements, and*

$$-(a \cup b) = -a \cap -b , \quad -(a \cap b) = -a \cup -b . \tag{8}$$

If $-a$ is the complement of a, then

$$a \cap -a = \Lambda \quad \text{and} \quad a \cup -a = V ,$$

i.e.

$$-a \cap a = \Lambda \quad \text{and} \quad -a \cup a = V .$$

Hence it follows by 10.1 and (1) (where a is replaced by $-a$) that a is the complement of $-a$. This proves (7).

One part of equivalence (6) is contained in (5). If $-b \leqslant -a$, then, by (7) and (5), $a = --a \leqslant --b = b$. Thus (6) is true.

Since

$$(a \cup b) \cap (-a \cap -b) = (a \cap -a \cap -b) \cup (b \cap -a \cap -b) = \Lambda \cup \Lambda = \Lambda$$

and

$$(a \cup b) \cup (-a \cap -b) = (a \cup b \cup -a) \cap (a \cup b \cup -b) = V \cap V = V ,$$

we infer from 10.1 and (1) (where a is replaced by $a \cup b$) that the first of identities (8) holds. By duality we obtain the second.

§ 11. Relative pseudo-complement. Difference. It is useful to generalize the notions of pseudo-complement and complement as follows.

Let a, b be elements of a lattice $\boldsymbol{A}$. An element $c \in \boldsymbol{A}$ is said to be the *pseudo-complement of* a *relative to* b (or: *modulo* b) if c is the greatest element such that $a \cap c \leqslant b$. The pseudo-complement of a relative to b is denoted, if it exists, by the symbol

$$a \Rightarrow b .$$

By definition, for every $x \in \boldsymbol{A}$,

$$x \leqslant a \Rightarrow b \quad \text{if and only if} \quad a \cap x \leqslant b . \tag{1}$$

This equivalence, which can be considered to be another definition of $a \Rightarrow b$, will be the basis for the investigation of the relative pseudo-complement.

Since $a \cap b \leqslant b$, we infer from (1) that

$$b \leqslant a \Rightarrow b \tag{2}$$

provided $a \Rightarrow b$ exists.

The element $a \Rightarrow b$ does not always exists. For instance, $a \Rightarrow a$ exists if and only if the lattice $\boldsymbol{A}$ has the unit element V. Then

$$a \Rightarrow a = V \,. \tag{3}$$

If $\boldsymbol{A}$ has the unit element, then

$$a \leqslant b \quad \text{if and only if} \quad a \Rightarrow b = V \tag{4}$$

and

$$V \Rightarrow b = b \,. \tag{5}$$

If $\boldsymbol{A}$ has the zero element Λ, then

$$-a = a \Rightarrow \Lambda \,. \tag{6}$$

This equation should be understood as follows: if one of the sides of (6) exists, the remaining one does also and they are equal. Thus the relative pseudo-complement can be considered as a generalization of the notion of pseudo-complement introduced in § 10.

The hypothesis that the relative pseudo-complement exists has rather strong consequences. In the following two theorems we shall show that it implies some properties of a distributive character.

11.1. *If* $a \Rightarrow \big((a \cap b) \cup (a \cap c)\big)$ *exists, then*

$$a \cap (b \cup c) = (a \cap b) \cup (a \cap c) \,.$$

For brevity let $d = (a \cap b) \cup (a \cap c)$. Since $a \cap b \leqslant d$ and $a \cap c \leqslant d$, we have, by (1),

$$b \leqslant a \Rightarrow d \quad \text{and} \quad c \leqslant a \Rightarrow d \,.$$

Consequently $b \cup c \leqslant a \Rightarrow d$ which implies, by (1),

$$a \cap (b \cup c) \leqslant d \,.$$

On the other hand, $a \cap b \leqslant a \cap (b \cup c)$ and $a \cap c \leqslant a \cap (b \cup c)$. Hence

$$d \leqslant a \cap (b \cup c)$$

which proves 11.1.

Similarly we can prove the following infinite analogue of 11.1:

11.2. *If the join* $\bigcup_{t \in T} a_t$ *exists and if* $a \Rightarrow c$ *exists for every* $c \in \boldsymbol{A}$, *then the infinite join* $\bigcup_{t \in T}(a \cap a_t)$ *also exists and*

$$a \cap \bigcup_{t \in T} a_t = \bigcup_{t \in T}(a \cap a_t) \,.$$

For brevity let $b = \bigcup_{t \in T} a_t$. We have

$$a \cap a_t \leqslant a \cap b \quad \text{for every} \quad t \in T .$$

On the other hand, if

$$a \cap a_t \leqslant c \quad \text{for every} \quad t \in T ,$$

then, by (1),

$$a_t \leqslant a \Rightarrow c \quad \text{for every} \quad t \in T ,$$

and consequently $b \leqslant a \Rightarrow c$, i.e., by (1),

$$a \cap b \leqslant c .$$

This proves that $a \cap b = \bigcup_{t \in T} a \cap a_t$.

It follows from (2) that the definition of the pseudo-complement $a \Rightarrow b$ of a relative to b can also be formulated as follows: it is the greatest element $c \geqslant a \cap b$ such that $a \cap c = a \cap b$.

Suppose that the lattice A under consideration has the unit element $\wedge$. We shall say that $a \Rightarrow b$ is the *complement of a relative to b* (or: *modulo b*) if it is the least element $c \geqslant a \cap b$ such that $a \cup c = \wedge$.

To justify the names adopted, let us introduce the following notation. For every element $g \in A$, the symbol A^g will denote the sublattice of all elements $x \geqslant g$ (i.e. the principal filter generated by b). The lattice A^g has the zero element, viz. g is the zero element of A^g. If A has the unit element $\vee$, then $\vee$ is also the unit element of A^g.

11.3. *An element c is the pseudo-complement of a relative to b in A if and only if c is the pseudo-complement of a in the lattice $A^{a \cap b}$. An element c is the complement of a relative to b if and only if c is the complement of a in the lattice $A^{a \cap b}$.*

This follows immediately from the definition of relative pseudo-complement and complement.

Note that if $a \geqslant b$, then c is the pseudo-complement (complement) of a relative to b if and only if c is the pseudo-complement (complement) of a in A^b.

11.4. *If an element a of a distributive lattice A has the complement $-a$, then for every $b \in A$ the complement $a \Rightarrow b$ of a relative to b exists, and*

$$a \Rightarrow b = -a \cup b . \tag{7}$$

By 11.3 it suffices to prove that the element $-a \cup b$ is the complement of a in the lattice $A^{a \cap b}$, i.e. (see 10.1) that

$$a \cap (-a \cup b) = a \cap b \quad \text{and} \quad a \cup (-a \cup b) = \vee .$$

The proof of these identities requires only a simple calculation.

We shall now define notions dual to the relative pseudo-complement and relative complement. They are called the pseudo-difference and difference respectively.

An element c is said to be the *pseudo-difference of b and a* if it is the least element such that $a \cup c \geqslant b$. The pseudo-difference of b and a is denoted, if it exists, by the symbol

$$b - a .$$

By definition, for every $x \in A$,

(1′) $$x \geqslant b - a \quad \text{if and only if} \quad a \cup x \geqslant b .$$

Since $a \cup b \geqslant b$, we infer from (1′) that

(2′) $$b \geqslant b - a$$

provided $b - a$ exists.

The element $b - a$ does not always exist. For instance, $a - a$ exists if and only if the lattice A has the zero element $\wedge$. Then

(3′) $$a - a = \wedge .$$

If A has the zero element, then

(4′) $$a \geqslant b \quad \text{if and only if} \quad b - a = \wedge$$

and

(5′) $$b - \wedge = b .$$

If A has the unit element, then

(6′) $$\vee - a \text{ is the } \cup\text{-complement of } a .$$

It follows from (2′) that the definition of the pseudo-difference $b - a$ of b and a can be formulated as follows: it is the least element $c \leqslant a \cup b$ such that $a \cup c = a \cup b$.

Suppose that the lattice A has the zero element $\wedge$. We shall say that $b - a$ is the *difference* of b and a if it is the greatest element $c \leqslant b$ such that $a \cap c = \vee$.

To explain the notions just defined, let us introduce the following notation. For every element $g \in A$, the symbol A_g will denote the sublattice of all elements $x \leqslant g$ (i.e. the principal ideal generated by b). The lattice A_g has the unit element, viz. g is the unit element of A_g. If A has the zero element $\wedge$, then $\wedge$ is also the zero element of A_g.

An element c is the pseudo-difference of b and a if and only if c is the $\cup$-complement of a in the lattice $A_{a \cup b}$. An element c is the difference of b and a if and only if c is the complement of a in the lattice $A_{a \cup b}$.

If $a \leqslant b$, then c is the pseudo-difference (difference) of b and a if and only if c is the $\cup$-complement (complement) of a in the lattice A_b.

If an element a of a distributive lattice A has the complement $-a$, then the difference $b-a$ exists and

$$b-a = b \cap -a \,. \tag{7'}$$

The proof of all the above statements for $b-a$ can be obtained from proofs of the dual statements for $a \Rightarrow b$ by using the duality principle.

It follows from the above remarks that $b-a$ is a lattice-theoretical substitute for the notion of difference of sets. This explains the names of $b-a$. The operation $a \Rightarrow b$ has no analogue in set theory. From the point of view of applications to mathematical logic, the operation $a \Rightarrow b$ is much more important than $b-a$. Therefore the next two sections will be exclusively devoted to an investigation of lattices in which the operation $a \Rightarrow b$ is always feasible. Of course, by duality we can make an analogous investigation of lattices in which $b-a$ is always feasible. This is omitted because such lattices will not play any essential part in this book.

Except for expressions of the form $a-b$ (seldom used in this book), the sign "$-$" (if conceived as a lattice-theoretical operation) will denote exclusively the pseudo-complementation defined in § 10 and, in particular, the complementation in lattices. Similarly as in the case of $\cup$ and $\cap$, we always assume that the sign "$-$" binds more strongly than $\Rightarrow$, i.e. expressions like $-a \Rightarrow b$ should be understood as $(-a) \Rightarrow b$, etc.

§ 12. Relatively pseudo-complemented lattices. Pseudo-Boolean algebras. A lattice A is said to be *relatively pseudo-complemented* [1] if $a \Rightarrow b$ exists for all elements $a, b \in A$. By § 11, (3), every relatively pseudo-complemented lattice has the unit element. However, it does not, in general, have the zero element. Every pseudo-complemented lattice with the zero element is called a *pseudo-Boolean algebra* [2].

We recall the fundamental property of $\Rightarrow$ in any pseudo-complemented lattice (see § 11, (1))

$$c \leqslant a \Rightarrow b \quad \text{if and only if} \quad a \cap c \leqslant b \,. \tag{1}$$

If A is a pseudo-Boolean algebra, then $-a$ exists for every $a \in A$, viz. (see § 11, (6))

$$-a = a \Rightarrow \Lambda \,. \tag{2}$$

---

[1] See Birkhoff [3].

[2] For examination of dual lattices called *Brouwerian algebras*, see e.g. McKinsey and Tarski [2].

Obviously, every relatively pseudo-complemented lattice can be conceived as an algebra

$$\{A, \cup, \cap, \Rightarrow\}$$

with three binary operations. Similarly, every pseudo-Boolean algebra A can be conceived as an algebra

$$\{A, \cup, \cap, \Rightarrow, -\}$$

with three binary operations $\cup$, $\cap$, $\Rightarrow$ and one unary operation $-$.

As an example of a pseudo-Boolean algebra we quote here the set-lattice $\mathfrak{G}(X)$ of all open subsets of a topological space X. For any subsets A, B, C of X, the inclusion $A \cap C \subset B$ holds if and only if $C \subset (X-A) \cup B$ (where $X-A$ is the set-theoretical complement of the set A). If C is open, this inclusion is equivalent to $C \subset \mathbf{I}\big((X-A) \cup B\big)$ (see § 2, (3), p. 14). Thus for any $A, B \in \mathfrak{G}(X)$, the set $C = \mathbf{I}\big((X-A) \cup B\big)$ is the greatest open set C satisfying $A \cap C \subset B$, i.e.

$$A \Rightarrow B = \mathbf{I}\big((X-A) \cup B\big). \tag{3}$$

Also the class $\mathfrak{G}_0(X)$ of all open dense subsets of a topological space X is an example of a pseudo-complemented lattice. Except for some rather trivial cases, $\mathfrak{G}_0(X)$ does not have the zero element. We shall see in Chapter IV, § 3, § 9, that the examples just quoted are typical for relatively pseudo-complemented lattices.

12.1. *Every relatively pseudo-complemented lattice is distributive.*

This follows immediately from 11.1 (see also the remark at the beginning of § 9, p. 48).

We collect fundamental properties of the operations $\Rightarrow$ and $-$ in relatively pseudo-complemented lattices and pseudo-Boolean algebras in the form of the following theorems 12.2 and 12.3:

12.2. *In every relatively pseudo-complemented lattice* [1],

$$a \Rightarrow b = V \text{ if and only if } a \leqslant b, \tag{4}$$

$$a = b \text{ if and only if } a \Rightarrow b = V = b \Rightarrow a, \tag{5}$$

$$a \Rightarrow a = V, \tag{6}$$

$$a \Rightarrow V = V, \tag{7}$$

$$V \Rightarrow b = b, \tag{8}$$

$$(a \Rightarrow a) \cap b = b, \tag{9}$$

[1] The majority of properties (4)-(38) will be used only in VII, § 3 and IX, § 4.

(10) $a \cap (a \Rightarrow b) \leqslant b$,

(11) *if* $a_1 \leqslant a_2$, *then* $a_2 \Rightarrow b \leqslant a_1 \Rightarrow b$,

(12) *if* $b_1 \leqslant b_2$, *then* $a \Rightarrow b_1 \leqslant a \Rightarrow b_2$,

(13) $b \leqslant a \Rightarrow b$,

(14) $a \cap (a \Rightarrow b) = a \cap b$,

(15) $(a \Rightarrow b) \cap b = b$,

(16) $(a \Rightarrow b) \cap (a \Rightarrow c) = a \Rightarrow (b \cap c)$,

(17) $(a \Rightarrow c) \cap (b \Rightarrow c) = (a \cup b) \Rightarrow c$,

(18) $a \Rightarrow (b \Rightarrow c) = (a \cap b) \Rightarrow c = b \Rightarrow (a \Rightarrow c)$,

(19) $c \Rightarrow a \leqslant \big(c \Rightarrow (a \Rightarrow b)\big) \Rightarrow (c \Rightarrow b)$,

(20) $(a \Rightarrow b) \cap (b \Rightarrow c) \leqslant a \Rightarrow c$,

(21) $(a \Rightarrow b) \leqslant (b \Rightarrow c) \Rightarrow (a \Rightarrow c)$,

(22) $a \leqslant b \Rightarrow (a \cap b)$,

(23) $a \Rightarrow (b \Rightarrow c) \leqslant (a \Rightarrow b) \Rightarrow (a \Rightarrow c)$,

(24) $c \cap \big((c \cap a) \Rightarrow (c \cap b)\big) = c \cap (a \Rightarrow b)$.

(4) (6), (8) were proved in § 11, (4), (3), (5).

(5) follows from (4).

(7) follows from (4).

(9) is another formulation of (6).

(10) follows from (1) where $c = a \Rightarrow b$.

(11) and (12) can be deduced immediately from the definition of $\Rightarrow$ or in the following way. By (10), $a_2 \cap (a_2 \Rightarrow b) \leqslant b$. If $a_1 \leqslant a_2$, then $a_1 \cap (a_2 \Rightarrow b) \leqslant b$, i.e. $a_2 \Rightarrow b \leqslant a_1 \Rightarrow b$ by (1). By (10), $a \cap (a \Rightarrow b_1) \leqslant b_1$. If $b_1 \leqslant b_2$, then $a \cap (a \Rightarrow b_1) \leqslant b_2$, i.e. $a \Rightarrow b_1 \leqslant a \Rightarrow b_2$ by (1).

(13) follows from (1).

(14) follows from (10) and (13).

(15) follows from (13).

Proof of (16). By (12), $a \Rightarrow (b \cap c) \leqslant a \Rightarrow b$ and $a \Rightarrow (b \cap c) \leqslant a \Rightarrow c$. Hence $a \Rightarrow (b \cap c) \leqslant d$ where $d = (a \Rightarrow b) \cap (a \Rightarrow c)$. Since $d \leqslant a \Rightarrow b$ and $d \leqslant a \Rightarrow c$, we have $a \cap d \leqslant b$ and $a \cap d \leqslant c$ by (1). Hence $a \cap d \leqslant b \cap c$, i.e. $d \leqslant a \Rightarrow (b \cap c)$ by (1). This proves that $d = a \Rightarrow (b \cap c)$.

Proof of (17). By (11), $(a \cup b) \Rightarrow c \leqslant a \Rightarrow c$ and $(a \cup b) \Rightarrow c \leqslant b \Rightarrow c$. Hence $(a \cup b) \Rightarrow c \leqslant d$ where $d = (a \Rightarrow c) \cap (b \Rightarrow c)$. On the other hand, by 12.1 and (14) we have

$$(a \cup b) \cap d = (a \cap d) \cup (b \cap d) = \big(a \cap c \cap (b \Rightarrow c)\big) \cup \big((a \Rightarrow c) \cap b \cap c\big) \leqslant c.$$

Hence, by (1), $d \leqslant (a \cup b) \Rightarrow c$, which completes the proof of the equation $d = (a \cup b) \Rightarrow c$.

Proof of (18). For every element x, each of the following inequalities is equivalent to the next, by (1):

$$x \leqslant a \Rightarrow (b \Rightarrow c), \quad a \cap x \leqslant b \Rightarrow c, \quad (a \cap b) \cap x \leqslant c, \quad x \leqslant (a \cap b) \Rightarrow c .$$

It follows from the equivalence of the first and the last inequality that (18) holds.

Proof of (19). Since $c \Rightarrow (a \cap b) \leqslant c \Rightarrow b$ by (12), we have $c \Rightarrow \big(a \cap (a \Rightarrow b)\big) \leqslant c \Rightarrow b$ by (14) and consequently

$$(c \Rightarrow a) \cap \big(c \Rightarrow (a \Rightarrow b)\big) \leqslant c \Rightarrow b$$

by (16). This implies (19).

(20) holds by (1) since $\big((a \Rightarrow b) \cap (b \Rightarrow c)\big) \cap a = a \cap b \cap (b \Rightarrow c) = a \cap b \cap c \leqslant c$ as a result of (14).

(21) follows from (20) and (1).

Proof of (22). $b \cap a \leqslant a \cap b$. Hence, by (1), $a \leqslant b \Rightarrow (a \cap b)$.

Proof of (23). Since $a \cap (a \Rightarrow b) \cap \big(a \Rightarrow (b \Rightarrow c)\big) = a \cap b \cap (b \Rightarrow c) = a \cap b \cap c \leqslant c$ by (14); by (1) we have

$$(a \Rightarrow b) \cap \big(a \Rightarrow (b \Rightarrow c)\big) \leqslant a \Rightarrow c, \quad \text{i.e.} \quad a \Rightarrow (b \Rightarrow c) \leqslant (a \Rightarrow b) \Rightarrow (a \Rightarrow c) .$$

Proof of (24). By (16), (4) and (11) we have

$$(c \cap a) \Rightarrow (c \cap b) = \big((c \cap a) \Rightarrow c\big) \cap \big((c \cap a) \Rightarrow b\big) = (c \cap a) \Rightarrow b \geqslant a \Rightarrow b$$

which proves that

$$c \cap \big((c \cap a) \Rightarrow (c \cap b)\big) \geqslant c \cap (a \Rightarrow b) .$$

On the other hand, by (16),

$$a \cap c \cap \big((c \cap a) \Rightarrow (c \cap b)\big) = (c \cap a) \cap (c \cap b) \leqslant b$$

which implies by (1) that $c \cap \big((c \cap a) \Rightarrow (c \cap b)\big) \leqslant a \Rightarrow b$. Consequently

$$c \cap \big((c \cap a) \Rightarrow (c \cap b)\big) \leqslant c \cap (a \Rightarrow b)$$

which completes the proof of (24).

12.3. *In every pseudo-Boolean algebra,*

(25) *if* $a \leqslant b$, *then* $-b \leqslant -a$,

(26) $-\Lambda = V$, $\quad -V = \Lambda$,

(27) $a \cap -a = \Lambda$,

(28) $-(a \cap -a) = V$,

(29) $a \leqslant --a$,

(30) $---a = -a$,

(31) $-(a \cup b) = -a \cap -b$,

(32) $-a \cup -b \leqslant -(a \cap b)$,

(33) $-a \cup b \leqslant a \Rightarrow b$,

(34) $a \Rightarrow b \leqslant -b \Rightarrow -a$,

(35) $a \Rightarrow -b = -(a \cap b) = b \Rightarrow -a$,

(36) $a \Rightarrow -b = --(a \Rightarrow -b)$,

(37) $--(a \Rightarrow b) \leqslant a \Rightarrow --b$,

(38) $\Lambda \Rightarrow a = V$.

(25), (26), (27) were proved in § 10, (5), (4), (3).

(28) follows from (26) and (27).

Proof of (29). Since $a \cap -a \leqslant \Lambda$ (see (27)), we have $a \leqslant -a \Rightarrow \Lambda = --a$ by (1) and (2).

Proof of (30). Since $a \leqslant --a$ by (29), we have $---a \leqslant -a$ by (2). The converse inequality $-a \leqslant ---a$ follows from (29) where a is replaced by $-a$.

Proof of (31). By (2) and (17), $-(a \cup b) = (a \cup b) \Rightarrow \Lambda = (a \Rightarrow \Lambda) \cap (b \Rightarrow \Lambda) = -a \cap -b$.

Proof of (32). Since $(-a \cup -b) \cap (a \cap b) \leqslant \Lambda$ by distributivity and (27), we have $-a \cup -b \leqslant (a \cap b) \Rightarrow \Lambda = -(a \cap b)$ by (1) and (2).

Proof of (33). We have $(-a \cup b) \cap a \leqslant b$ by distributivity and (27) which proves (33) because of (1).

(34) follows from (1) and (20) where $c = \Lambda$.

(35) follows from (2) and (18) where $c = \Lambda$.

Proof of (36). By (35) and (30) we have

$$--(a \Rightarrow -b) = ---(a \cap b) = -(a \cap b) = a \Rightarrow -b.$$

Proof of (37). By (29) and (12), $a \Rightarrow b \leqslant a \Rightarrow --b$. Hence, by (25) and (36),

$$--(a \Rightarrow b) \leqslant --(a \Rightarrow --b) = a \Rightarrow --b.$$

(38) follows from (4).

§ 13. Filters in relatively pseudo-complemented lattices. First we shall prove the following characterization of filters:

13.1. *A subset F of a relatively pseudo-complemented lattice A is a filter if and only if it contains the unit element and satisfies the following condition:*

(1) $$\text{if} \quad a \in F \quad \text{and} \quad a \Rightarrow b \in F, \quad \text{then} \quad b \in F.$$

Suppose that (1) holds, and $1 \in F$. By § 12, (22), we have $a \Rightarrow (b \Rightarrow (a \cap b)) = 1$. If $a, b \in F$, then $b \Rightarrow (a \cap b) \in F$ and $a \cap b \in F$ by (1). If $a \in F$ and $a \leqslant b$, i.e. $a \Rightarrow b = 1 \in F$ (see § 12, (4)), then $b \in F$ by (1). This proves that F is a filter.

Conversely, if F is a filter, then $1 \in F$. If $a \in F$ and $a \Rightarrow b \in F$, then $a \cap b = a \cap (a \Rightarrow b) \in F$ (see § 12, (14)). Hence $b \in F$, i.e. F has property (1). Theorem 13.1 is proved.

Let F be a filter in a relatively pseudo-complemented lattice A. For any elements $a, b \in A$ we shall write

$$a \leqslant_F b$$

if and only if $a \Rightarrow b \in F$.

The relation $\leqslant_F$ is a quasi-ordering in A. In fact, $\leqslant_F$ is reflexive since $a \leqslant_F a$ by § 12, (6). To prove that it is transitive, let us recall that, by § 12, (20),

$$(a \Rightarrow b) \cap (b \Rightarrow c) \leqslant a \Rightarrow c \quad \text{for all} \quad a, b, c \in A .$$

If $a \leqslant_F b$ and $b \leqslant_F c$, i.e. $a \Rightarrow b \in F$ and $b \Rightarrow c \in F$, then $a \Rightarrow c \in F$, i.e. $a \leqslant_F c$.

Write

(2) $$a \sim b$$

if and only if $a \Rightarrow b \in F$ and $b \Rightarrow a \in F$.

By 5.2, relation (2) is an equivalence relation. We shall call $\sim$ the *equivalence relation determined by the filter* F, and we shall write A/F instead of $A/\sim$ (see § 3). Elements in A/F will be denoted by $|a|$ $(a \in A)$ according to § 3. By 5.2, the relation $\leqslant_F$ in A determines uniquely an ordering relation $\leqslant$ in A/F, namely

(3) $$|a| \leqslant |b| \quad \text{if and only if} \quad a \Rightarrow b \in F .$$

13.2. *For every filter F in a relatively pseudo-complemented lattice A, the relation $\leqslant$ defined by* (3) *in A/F is a lattice ordering. The lattice A/F is relatively pseudo-complemented. Moreover, the equivalence relation* (2) *is a congruence in A with respect to $\cup$, $\cap$, $\Rightarrow$, i.e.*

(4) $$|a| \cup |b| = |a \cup b|, \quad |a| \cap |b| = |a \cap b| ,$$

(5) $$|a| \Rightarrow |b| = |a \Rightarrow b| .$$

An element $|b|$ is the unit of A/F if and only if $b \in F$.

If A is a pseudo-Boolean algebra, A/∇ is also a pseudo-Boolean algebra, and the element $|\wedge|$ is the zero element of A/∇. The equivalence relation (2) *is a congruence with respect to* $-$, *i.e.*

$$-|a| = |-a| . \tag{6}$$

Since $a \leqslant a \cup b$, we have $a \Rightarrow (a \cup b) = \vee \in \nabla$ by § 12, (4). Hence

$$|a| \leqslant |a \cup b| .$$

By the same argument,

$$|b| \leqslant |a \cup b| .$$

Conversely, if $|a| \leqslant |c|$ and $|b| \leqslant |c|$, i.e. if $a \Rightarrow c \in \nabla$ and $b \Rightarrow c \in \nabla$, then, by § 12, (17),

$$(a \cup b) \Rightarrow c = (a \Rightarrow c) \cap (b \Rightarrow c) \in \nabla ,$$

i.e.

$$|a \cup b| \leqslant |c| .$$

This proves that $|a \cup b|$ is the join of $|a|$ and $|b|$.

By the same argument we prove the second of the identities (4) using § 12, (4) and (16).

If $|a| \cap |x| \leqslant |b|$, i.e. $|a \cap x| \leqslant b$, then $(a \cap x) \Rightarrow b \in \nabla$. By § 12, (18), $(a \cap x) \Rightarrow b = x \Rightarrow (a \Rightarrow b)$. Thus $|x| \leqslant |a \Rightarrow b|$. By the same argument we prove that $|x| \leqslant |a \Rightarrow b|$ implies $|a| \cap |x| \leqslant |b|$. This proves (5).

Since $a \Rightarrow \vee = \vee \in \nabla$, we have $|a| \leqslant |\vee|$ for every $a \in A$, i.e. $|\vee|$ is the unit of A/∇. On the other hand, $|b|$ is the unit of A/∇ if and only if $|\vee| \leqslant |b|$, i.e. $b = \vee \Rightarrow b \in \nabla$ (see (3) and § 12, (8)). This completes the proof of the first part of 13.2.

If A has the zero element then, for every $b \in A$, we have $\wedge \Rightarrow b = \vee \in \nabla$ (see § 12, (4)) and consequently $|\wedge| \leqslant |b|$, i.e. $|\wedge|$ is the zero element of A/∇. Hence

$$-|a| = |a| \Rightarrow |\wedge| = |a \Rightarrow \wedge| = |-a| .$$

This completes the proof of 13.2.

If ∇ is the unit filter, then the equivalence relation determined by ∇ is the relation of equality. For if $a \Rightarrow b = \vee$ and $b \Rightarrow a = \vee$, i.e. $a \leqslant b$ and $b \leqslant a$, then $a = b$. Thus if ∇ is the unit filter, then for every $a \in A$ the equivalence class $|a|$ contains only one element a. It is natural to identify $|a|$ with a. By this identification,

$$A = A/\nabla, \quad \text{provided } \nabla \text{ is the unit filter in } A . \tag{7}$$

In theorem 13.2 we have proved that every filter in a pseudo-complemented lattice determines an equivalence relation which is a congruence with respect to $\cup$, $\cap$ and $\Rightarrow$. We shall now prove that, conversely, every congruence of this kind is determined by a filter.

13.3. *If an equivalence relation $\sim$ in a relatively pseudo-complemented lattice A is a congruence with respect to $\cup$, $\cap$ and $\Rightarrow$, then the set ∇ of all $a \in A$ such that $a \sim V$ is a filter and $\sim$ is the equivalence relation determined by the filter ∇.*

By definition,

$$a \in \nabla \quad \text{if and only if} \quad a \sim V .$$

Moreover, $V \in \nabla$.

If $a \in \nabla$ and $a \leqslant b$, then $b = a \cup b \sim V \cup b = V$, i.e. $b \in \nabla$. If $a \in \nabla$ and $b \in \nabla$, i.e. $a \sim V$ and $b \sim V$, then $a \cap b \sim V \cap V = V$, i.e. $a \cap b \in \nabla$. This proves that ∇ is a filter.

We shall prove that

(8) $$\text{if } a \sim b, \text{ then } a \Rightarrow b \in \nabla \text{ and } b \Rightarrow a \in \nabla .$$

In fact, $a \Rightarrow b \sim a \Rightarrow a = V$, i.e. $a \Rightarrow b \in \nabla$. Similarly, $b \Rightarrow a \sim a \Rightarrow a = V$, i.e. $b \Rightarrow a \in \nabla$.

Conversely,

(9) $$\text{if } a \Rightarrow b \in \nabla \text{ and } b \Rightarrow a \in \nabla, \text{ then } a \sim b .$$

By hypothesis, $a \Rightarrow b \sim V$ and $b \Rightarrow a \sim V$. Thus

$$a = a \cap V \sim a \cap (a \Rightarrow b) = a \cap b$$

because of 12.2, (14). Similarly,

$$b = b \cap V \sim b \cap (b \Rightarrow a) = b \cap a = a \cap b .$$

Hence $a \sim b$ by the transitivity of $\sim$.

It follows from (8) and (9) that $\sim$ is the equivalence relation determined by ∇. Theorem 13.3 is proved.

13.4. *Let A and B be relatively pseudo-complemented lattices (or: pseudo-Boolean algebras). Let h be a homomorphism from A onto B, i.e. a mapping which preserves the operations $\cup$, $\cap$, $\Rightarrow$ (or: $\cup$, $\cap$, $\Rightarrow$, $-$). Let ∇ be the filter of all $a \in A$ such that $h(a) = V_B$. Then the mapping h_0:*

$$h_0(|a|) = h(a) \qquad (a \in A)$$

is an isomorphism from A/∇ onto B.

By 4.7 the relation $\sim$ defined as follows

(10) $$a_1 \sim a_2 \quad \text{if and only if} \quad h(a_1) = h(a_2)$$

is a congruence in A. By the definition,

$$a \in \nabla \quad \text{if and only if} \quad a \sim V ,$$

i.e. the filter ∇ defined in 13.4 coincides with the filter ∇ defined in 13.3 for congruence (10). Thus $A/\sim = A/\nabla$ and theorem 13.4 is a particular case of 4.7 (see also 13.2).

13.5. *If ∇ is a filter in a relatively pseudo-complemented lattice A, and ∇' is a filter in the relatively pseudo-complemented lattice A/∇, then the set ∇'' of all $a \in A$ such that $|a| \in \nabla'$ is a filter in A and $\nabla \subset \nabla''$.*

The mapping

$$h_0(||a||) = \big|\,|a|\,\big| \qquad (a \in A) \tag{11}$$

(where $||a||$ is the element in A/∇'' determined by $a \in A$, and $\big|\,|a|\,\big|$ is the element in $(A/\nabla)/\nabla'$ determined by $|a| \in A/\nabla$) is an isomorphism from A/∇'' onto $(A/\nabla)/\nabla'$.

The proof of the first part is easy. The second part of 13.5 follows immediately from 13.4 where $h(a) = \big|\,|a|\,\big|$, i.e. h is the superposition of the natural homomorphism from A onto A/∇, and the natural homomorphism from A/∇ onto $(A/\nabla)/\nabla'$.

13.6. *If ∇ and ∇'' are filters in a relatively pseudo-complemented lattice A, $\nabla \subset \nabla''$, then the set ∇' of all $|a| \in A/\nabla$ such that $a \in \nabla''$ is a filter in A/∇.*

The proof is by an easy verification.

13.7. *A filter ∇ in a relatively pseudo-complemented lattice A is proper if and only if the lattice A/∇ contains at least two elements.*

This follows directly from the statement that $|a|$ is the unit of A/∇ if and only if $a \in \nabla$ (see 13.2).

13.8. *A filter ∇ in a relatively pseudo-complemented lattice A is prime if and only if the unit filter in A/∇ is prime.*

The easy proof is left to the reader.

13.9. *Let ∇ be a filter in a pseudo-Boolean algebra A and let $a_0 \in A$. The filter generated by ∇ and a_0 is proper if and only if $-a_0 \notin \nabla$.*

By 8.2 and the definition of pseudo-complement, each of the following conditions is equivalent to the next:

the filter generated by ∇ and a_0 is not proper;

there exists an element $c \in \nabla$ such that $a_0 \cap c = \Lambda$;

there exists an element $c \in \nabla$ such that $c \leqslant -a_0$;

$$-a_0 \in \nabla\,.$$

The equivalence of the first and the last condition is another formulation of 13.7.

13.10. *The following conditions are equivalent for every filter ∇ in a pseudo-complemented lattice A:*

(i) *∇ is maximal;*

(ii) *∇ is proper and, for every $a \in A$, either a or $-a$ is in ∇;*

(iii) *for every* $a \in A$, *exactly one of the elements* $a, -a$ *is in* V;

(iv) A/V *has exactly two elements.*

(i) implies (ii). For if $-a \notin V$, then the filter generated by V and a is proper, by 13.9. Since it contains the maximal filter V, it coincides with V. Thus $a \in V$.

(ii) implies (iii). For if simultaneously $a \in V$ and $-a \in V$, then $\Lambda = a \cap -a \in V$ and, consequently, V is not proper.

(iii) implies (iv). In fact, (iii) implies that, for every $a \in A$, either $|a| = V$ or $-|a| = V$, i.e. $|a| = \Lambda$, and only one of these possibilities holds. Thus A/V has only two elements: Λ and V, and $\Lambda \neq V$.

(iv) implies (i). Suppose that V'' is a filter in A, $V \subset V''$, $V \neq V''$. The set V' of all $|a| \in A/V$, where $a \in V''$, is a filter in A/V. Since $V \neq V''$, V' contains an element different from the unit element of A/V. Since A/V has only two elements, the zero of A/V belongs to V', i.e. $|a| = \Lambda_{A/V}$ for an element $a \in V''$. Thus $a \Rightarrow \Lambda \in V$ since $|a| = |\Lambda|$. Since $a \in V''$ and $a \Rightarrow \Lambda \in V''$, we have $\Lambda \in V''$ by 13.1. This implies that V'' is not proper. Thus V is maximal.

Observe that conditions (i) and (iv) are also equivalent for any pseudo-complemented lattice.

CHAPTER II

BOOLEAN ALGEBRAS

§ 1. Definition and elementary properties. A lattice A is said to be *Boolean*, or: to be a *Boolean algebra* [1], if it is distributive and every element $a \in A$ has the complement $-a$. It follows from the second hypothesis that every Boolean algebra has the zero element and the unit element.

In other words, a Boolean algebra is a distributive lattice A in which with every $a \in A$ there is associated an element $-a$ such that

(l_5) $$(a \cap -a) \cup b = b\,, \quad (a \cup -a) \cap b = b\,.$$

Indeed, (l_5) means that

$$a \cap -a = \Lambda \quad \text{and} \quad a \cup -a = V\,,$$

and this implies, by I, 10.1, that $-a$ is the complement of a.

Thus Boolean algebras can also be defined as abstract algebras $\{A, \cup, \cap, -\}$ with two binary operations $\cup, \cap$ and one unary operation $-$ satisfying axioms (l_1)-(l_5) [2].

The duality principle can be applied to Boolean algebras since axioms (l_1)-(l_5) are invariant under substitution of $\cap$ for $\cup$ and $\cup$ for $\cap$.

A Boolean algebra is said to be *complete* if it is a complete lattice.

As the most natural example of Boolean algebras we introduce the notion of fields of sets. By a *field of subsets* of a space X we shall mean a set-lattice A (of subsets of X) closed with respect to set-theoretical complementation, i.e. a non-empty class A of subsets of X such that

if $A_1, A_2 \in A$, then $A_1 \cup A_2 \in A$, $A_1 \cap A_2 \in A$ and $-A_1 = X - A_1 \in A$.

By the De Morgan formulas for sets, it suffices to require here only one of the first two conditions and the third condition since the remaining condition follows from them.

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[1] For a systematic exposition of the theory of Boolean algebras see Sikorski [9]. Cf. also Birkhoff [3], Hermes [1].

[2] For a simpler set of axioms for Boolean algebras see e. g. Huntington [1].

We shall see in § 8 that this example is typical for Boolean algebras.

The class of all subsets of a fixed space X is the simplest example of a field of sets. It will always be denoted by $\mathfrak{B}(X)$. It is a complete Boolean algebra: the infinite join (meet) of any class of elements of $\mathfrak{B}(X)$, i.e. subsets of X, is the set-theoretical union (intersection) of these subsets.

The following theorem follows immediately from I, § 11, (7), (6) and (7′):

1.1. *Every Boolean algebra A is a pseudo-Boolean algebra. For all elements $a, b \in A$, the element $a \Rightarrow b$ is the complement of a relative to b, and*

$$a \Rightarrow b = -a \cup b , \tag{l_6}$$

$$-a = a \Rightarrow \wedge . \tag{1}$$

For all elements $a, b \in A$ the difference $b - a$ exists and

$$b - a = b \cap -a . \tag{2}$$

The following theorem follows from I, 12.1:

1.2. *If A is a pseudo-Boolean algebra and, for every $a \in A$, the pseudo-complement $-a$ is the complement of a, then A is a Boolean algebra.*

By 1.1 all identities and inequalities proved in I, 12.2, and I, 12.3, for pseudo-Boolean algebras also hold for every Boolean algebra. Of course, there are identities and inequalities which hold in Boolean algebras but do not hold in pseudo-Boolean algebras. Some inequalities proved for pseudo-Boolean algebras in 12.2 and 12.3 take the form of an identity in the case of Boolean algebras. As an example we quote here the identity (l_6) which, in the case of arbitrary pseudo-Boolean algebras, has only the form of the inequality $-a \cup b \leqslant a \Rightarrow b$ (I, 12.3, (33)). Other examples are collected in the following theorem:

1.3. *In every Boolean algebra,*

(3) $a \leqslant b$ *if and only if* $-b \leqslant -a$,

(4) $a \cup -a = \vee$,

(5) $--a = a$,

(6) $a \Rightarrow b = -b \Rightarrow -a$,

(7) $-a \Rightarrow b = -b \Rightarrow a$,

(8) $-(a \cap b) = -a \cup -b$,

(9) $a \cup b = -(-a \cap -b)$,

(10) $a \cap b = -(-a \cup -b)$,

(11) $-(a \Rightarrow b) = a \cap -b$,

(12) $(a \Rightarrow b) \Rightarrow a = a$.

(3) was proved in I, 10.2, (6). Observe that in the case of pseudo-Boolean algebras equivalence (3) is replaced only by the implication I, 12.3, (25).

(4) is another formulation of the second of the laws (l_5) as has been stated before. Identity (4) is dual to the identity

$$a \cap -a = \Lambda \tag{13}$$

which holds in every pseudo-Boolean algebra (I, 12.3, (27)).

(5) was proved in I, 10.2, (7). In the case of pseudo-Boolean algebras the sign $=$ must be replaced here by $\geqslant$ (see I, 12.3, (29)).

(6) and (7) can be deduced, by means of (5) and suitable substitutions, from the identity

$$a \Rightarrow -b = b \Rightarrow -a \tag{14}$$

proved for arbitrary pseudo-Boolean algebras (I, 12.3, (35)). The identities (6), (7), (14) are called the *laws of contraposition*. Note that for pseudo-Boolean algebras (6) has the form of inequality I, 12.3, (34).

(8) follows from the identity

$$-(a \cup b) = -a \cap -b\,, \tag{15}$$

true for pseudo-Boolean algebras (see I, 12.3, (31)). It suffices to substitute $-a$ and $-b$ for a and b and to apply (5). In the same way we can obtain (9) and (10). All the identities (15), (8), (9), (10) are called the *De Morgan laws*. They are Boolean analogues of the De Morgan laws for sets.

(11) follows directly from (l_6), (9) and (5).

(12) can be deduced from (l_6), (11) and § 6, (2), (5) as follows:

$$(a \Rightarrow b) \Rightarrow a = -(a \Rightarrow b) \cup a = (a \cap -b) \cup a = a\,.$$

Theorem 1.3, which is a supplement to theorems I, 12.2, and I, 12.3, does not contain any statement on the difference $b-a$. Since $b-a$ is the dual operation to $a \Rightarrow b$, such statements on $b-a$ can easily be obtained from 1.3 and I, 12.2, I, 12.3, by means of the duality principle. Another method of deducing statements on $b-a$ from statements on $a \Rightarrow b$ is based on the identities

$$b-a = -(b \Rightarrow a)\,, \tag{16}$$

$$a \Rightarrow b = -(a-b) \tag{17}$$

which follow easily from (l_6), (2) and the De Morgan laws.

From the point of view of the general theory of Boolean algebras, the two operations $a \Rightarrow b$ and $b-a$ have the same significance because they are dual. Consequently there is no reason to select one of them.

If Boolean algebras are examined from the point of view of set theory, as a generalization of the notion of field of sets, then the operation $b-a$ is usually selected because it is a natural generalization of the known set-theoretical operation of difference of sets, and no operation analogous to $a \Rightarrow b$ is examined in the set theory. However, from the point of view of applications to mathematical logic, the operation $a \Rightarrow b$ is more important because it is a Boolean analogue of the notion of logical implication, and $b-a$ has no analogue in mathematical logic. Therefore, in this book, we shall always conceive of Boolean algebras as abstract algebras

$$\{A, \cup, \cap, \Rightarrow, -\} \tag{18}$$

with three binary operations $\cup, \cap, \Rightarrow$ and one unary operation $-$. The identities (l_1)-(l_6) form a set of axioms for Boolean algebras considered as abstract algebras of the form (18). Of course, by conceiving of Boolean algebras as algebras of the form (18), we introduce a rather artificial aspect of non-symmetry to the theory but this non-symmetry is necessary for the applications in the second part of this book.

The set of axioms (l_1)-(l_6) for Boolean algebras is neither the shortest nor it is independent. Many shorter systems of axioms are known but the problem of axiomatization of the theory of Boolean algebras is out of our interest.

We shall now examine infinite operations in Boolean algebras.

1.4. *In every Boolean algebra,*

$$-\bigcup_{t\in T} a_t = \bigcap_{t\in T} -a_t , \tag{19}$$

$$-\bigcap_{t\in T} a_t = \bigcup_{t\in T} -a_t , \tag{20}$$

$$\bigcup_{t\in T} a_t = -\bigcap_{t\in T} -a_t , \tag{21}$$

$$\bigcap_{t\in T} a_t = -\bigcup_{t\in T} -a_t , \tag{22}$$

$$a \cap \bigcup_{t\in T} a_t = \bigcup_{t\in T} (a \cap a_t) , \tag{23}$$

$$a \cup \bigcap_{t\in T} a_t = \bigcap_{t\in T} (a \cup a_t) , \tag{24}$$

$$(\bigcup_{t\in T} a_t) \Rightarrow b = \bigcap_{t\in T} (a_t \Rightarrow b) , \tag{25}$$

$$(\bigcap_{t\in T} a_t) \Rightarrow b = \bigcup_{t\in T} (a_t \Rightarrow b) , \tag{26}$$

$$a \Rightarrow \bigcup_{t\in T} b_t = \bigcup_{t\in T} (a \Rightarrow b_t) , \tag{27}$$

$$a \Rightarrow \bigcap_{t\in T} b_t = \bigcap_{t\in T} (a \Rightarrow b_t) , \tag{28}$$

$$\bigcap_{t\in T} (a_t \Rightarrow b_t) \leqslant (\bigcap_{t\in T} a_t) \Rightarrow (\bigcap_{t\in T} b_t) , \tag{29}$$

$$\bigcap_{t\in T} (a_t \Rightarrow b_t) \leqslant (\bigcup_{t\in T} a_t) \Rightarrow (\bigcup_{t\in T} b_t) . \tag{30}$$

(19), (20), (21), (22) are called the *infinite De Morgan laws*. They are Boolean analogues of the infinite De Morgan laws for sets. They should

be read as follows: if the infinite join (meet) on one side of the equation exists, then the meet (join) on the another side also exists, and they are equal.

(23) and (24) are the *infinite distributive laws.*

(23), (24), (25), (26), (27), (28) should be read as follows: if the infinite join or meet on the left-hand side exists, then the infinite join or meet on the right-hand side also exists, and they are equal.

(29) and (30) hold under the hypothesis that all the infinite joins and meets in question exist.

To prove (19), suppose that the join $a = \bigcup_{t \in T} a_t$ exists. Since $a_t \leqslant a$, we have $-a \leqslant -a_t$ for every $t \in T$. On the other hand, if $b \leqslant -a_t$ for every $t \in T$, then $a_t \leqslant -b$ for every $t \in T$, i.e. $a \leqslant -b$. Hence $b \leqslant -a$. This proves that $-a = \bigcap_{t \in T} -a_t$. The proof of (19), under the hypothesis that $\bigcap_{t \in T} -a_t$ exists, is similar. In the same way we prove (20). Note that (20) can also be obtained from (19) by replacing a_t by $-a_t$. Identities (21) and (22) follow from (19) and (20) because of (5).

(23) was proved in I, 11.2. (24) follows from (23), (5) and the finite and infinite De Morgan laws:

$$a \cup \bigcap_{t \in T} a_t = -(-a \cap \bigcup_{t \in T} -a_t)$$
$$= -\bigcup_{t \in T} (-a \cap -a_t) = \bigcap_{t \in T} (a \cup a_t) .$$

(25) and (28) follow from (l_6), (24) or (23), and the infinite De Morgan laws. As an example we shall prove (25):

$$(\bigcup_{t \in T} a_t) \Rightarrow b = -(\bigcup_{t \in T} a_t) \cup b = (\bigcap_{t \in T} -a_t) \cup b$$
$$= \bigcap_{t \in T} (-a_t \cup b) = \bigcap_{t \in T} (a_t \Rightarrow b) .$$

Similarly we prove (26) and (27) using identities I, § 7, (9)-(10), instead of (23) and (24).

By I, § 7, (15), and I, § 12, (14),

$$\bigcap_{t \in T} (a_t \Rightarrow b_t) \cap \bigcap_{t \in T} a_t = \bigcap_{t \in T} (a_t \cap b_t) \leqslant \bigcap_{t \in T} b_t$$

which proves (29) because of I, § 12, (1).

Denoting $a = \bigcup_{t \in T} a_t$, $b = \bigcup_{t \in T} b_t$ and $c = \bigcap_{t \in T} (a_t \Rightarrow b_t)$ we have for every $t \in T$:

$$a_t \cap c \leqslant a_t \cap (a_t \Rightarrow b_t) \leqslant b_t \leqslant b ,$$

by I, 12.2, (10). Consequently, by (23),

$$a \cap c = \bigcup_{t \in T} (a_t \cap c) \leqslant b ,$$

i.e. $c \leqslant a \Rightarrow b$. This proves (30).

If infinite operations are taken into consideration, then every Boolean algebra A may be considered as a generalized algebra

$$\{A, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\} \tag{31}$$

with four finite operations $\cup, \cap, \Rightarrow, -$ and two infinite operations $\bigcup$ and $\bigcap$. Thus every complete Boolean algebra, i.e. complete Boolean lattice, can be considered as a complete algebra $\{A, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ in the sense defined in I, § 4, p. 28.

According to the convention at the end of I, § 7, if we write (31), we mean that the class of all sets $S \subset A$ such that $\bigcup S$ ($\bigcap S$) exists is the domain of the infinite operation $\bigcup$ ($\bigcap$) in (31). Sometimes it is necessary to restrict the domains of $\bigcup$ and $\bigcap$ to less classes $\mathfrak{Q}_1$ and $\mathfrak{Q}_2$ of sets $S \subset A$. Then we write

$$\{A, \cup, \cap, \Rightarrow, -, \bigcup|\mathfrak{Q}_1, \bigcap|\mathfrak{Q}_2\} \tag{32}$$

instead of (31) (see the similar notation for lattices in I, § 7, (25)).

§ 2. Subalgebras. From the point of view of the general notion of subalgebra (see I, § 4, p. 23) it is irrelevant whether Boolean algebras are interpreted as abstract algebras $\{A, \cup, \cap, \Rightarrow, -\}$ or as $\{A, \cup, \cap, -\}$ or even as $\{A, \cup, -\}$ or $\{A, \cap, -\}$. This follows from the fact that all the Boolean operations $\cup, \cap, \Rightarrow$ are definable by means of $\cup$ and $-$ or by means of $\cap$ and $-$ only (see § 1, (9), (10), and (1_6)).

Thus if a non-empty set A' of elements of a Boolean algebra A satisfies the condition

$$\text{if } a \in A', \text{ then } -a \in A' \tag{1}$$

and one of the two following conditions:

$$\text{if } a, b \in A', \text{ then } a \cup b \in A', \tag{2}$$

$$\text{if } a, b \in A', \text{ then } a \cap b \in A', \tag{3}$$

then it also satisfies the remaining condition (2) or (3) and each of the following conditions:

$$\text{if } a, b \in A', \text{ then } a \Rightarrow b \in A', \tag{4}$$

$$\text{if } a, b \in A', \text{ then } b - a \in A' \tag{5}$$

(see § 1, (2)). Then A' is a subalgebra of A in each of the possible interpretations of A as an abstract algebra.

Indeed, suppose e.g. that A' satisfies (1) and (2). If $a, b \in A'$, then $-a \in A'$ and $-b \in A'$ by (1). Hence it follows, by (2), that $-a \cup -b \in A'$. Consequently $a \cap b = -(-a \cup -b) \in A'$ by (1). This proves that A' satisfies (3). By the same method we verify that A' satisfies (4) and (5), etc.

Each subalgebra A' of a Boolean algebra is a Boolean algebra since axioms (l_1)-(l_6) also hold in A'.

Each subalgebra A' of a Boolean algebra A contains the unit element $\vee$ and the zero element $\wedge$ of A. In fact, if $a \in A'$, then also $-a \in A'$ and consequently $\vee = a \cup -a \in A'$, $\wedge = a \cap -a \in A'$. Of course, $\vee$ and $\wedge$ are respectively the unit element and the zero element of A'.

The set A_0 composed only of the unit element $\vee$ and the zero element $\wedge$ of A is a subalgebra of A (it is the least subalgebra of A).

2.1. *The subalgebra A' generated by a non-empty subset A_0 of a Boolean algebra is the set of all elements a of the form*

$$a = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} a_{ij} \tag{6}$$

where, for every i and j, either $a_{ij} \in A_0$ or $-a_{ij} \in A_0$.

The subalgebra A' generated by A_0 is also the set of all elements a of the form

$$a = \bigcap_{i=1}^{m} \bigcup_{j=1}^{n_i} a_{ij} \tag{7}$$

where, for every i and j, either $a_{ij} \in A_0$ or $-a_{ij} \in A_0$.

Let A_1 be the class of all elements a of the form (6). It is clear that every element $a \in A_1$ belongs to A'. Therefore it suffices to prove that A_1 is a subalgebra of A. Of course, the join of two elements of A_1 also belongs to A_1. If $a \in A_1$, then also $-a \in A_1$. In fact, by the De Morgan law, if a is given by (6), then

$$-a = \bigcap_{i=1}^{m} \bigcup_{j=1}^{n_i} -a_{ij} .$$

Applying the distributive law, we infer that $-a$ can also be represented as a finite join of finite intersections of elements which belong to A_0 or the complements of which belong to A_0. Thus $-a \in A_1$.

Similarly we prove the second part of 2.1.

2.2. *If A_0 is a sublattice of a Boolean algebra A and $\vee, \wedge \in A_0$, then the algebra A' generated by A_0 is the set of all elements a of the form*

$$a = (a_1 \Rightarrow b_1) \cap \ldots \cap (a_m \Rightarrow b_m) , \tag{8}$$

i.e. of the form

$$a = (-a_1 \cup b_1) \cap \ldots \cap (-a_m \cup b_m) , \tag{9}$$

where $a_1, \ldots, a_m, b_1, \ldots, b_m \in A_0$.

By 2.1, (7), it suffices to prove that if $a_1, \ldots, a_r, b_1, \ldots, b_s \in A_0$ $(r, s \geqslant 0$, $r+s > 0)$, then there are elements $a \in A_0$ and $b \in A_0$ such that

$$-a_1 \cup -a_2 \cup \ldots \cup -a_r \cup b_1 \cup b_2 \cup \ldots \cup b_s = -a \cup b .$$

The elements

$$a = \begin{cases} a_1 \cap a_2 \cap \ldots \cap a_r & \text{if} \quad r > 0\,, \\ V & \text{if} \quad r = 0; \end{cases}$$

$$b = \begin{cases} b_1 \cup b_2 \cup \ldots \cup b_s & \text{if} \quad s > 0\,, \\ \Lambda & \text{if} \quad s = 0 \end{cases}$$

satisfy this equation, by the De Morgan law § 1, (18). (See also § 1, (l_6)).

2.3. *The subalgebra generated by a set composed of r elements contains at most 2^{2^r} elements.*

Suppose that the set A_0 of generators is composed of elements $a_1, \ldots, a_r$. It follows easily from 2.1, (6), that every element a in the subalgebra A' generated by A_0 is a finite join of elements of the form

$$\bigcap_{j=1}^{r} \varepsilon_j \cdot a_j \tag{10}$$

where $\varepsilon_j = +1$ or -1 and where, for any element $b \in A$, the symbol $+1 \cdot b$ denotes the element b, and the symbol $-1 \cdot b$ denotes the element $-b$. The set of all elements of the form (10) contains at most 2^r elements since the number of all sequences $\{\varepsilon_1, \ldots, \varepsilon_r\}$ where $\varepsilon_j = \pm 1$ is 2^r. Since the number of all sets of elements (10) is $\leqslant 2^{2^r}$, the same estimation is true for the number of all joins of elements (10), i.e. for the number of all elements in A'.

§ 3. Boolean homomorphisms. From the point of view of the general notion of homomorphism (see I, § 4, p. 23) it is irrelevant whether Boolean algebras are interpreted as abstract algebras $\{A, \cup, \cap, \Rightarrow, -\}$ or as $\{A, \cup, \cap, -\}$ or even as $\{A, \cup, -\}$ or $\{A, \cap, -\}$. This follows from the fact that all the operations $\cup, \cap, \Rightarrow$ are definable by means of $\cup$ and $-$ or by means of $\cap$ and $-$ only (see § 1, (9), (10), and (l_6)). Thus if a mapping h of a Boolean algebra A into a Boolean algebra B satisfies the identity

$$h(-a) = -h(a) \tag{1}$$

and one of the identities

$$h(a \cup b) = h(a) \cup h(b)\,, \tag{2}$$

$$h(a \cap b) = h(a) \cap h(b)\,, \tag{3}$$

then it also satisfies the remaining identity (2) or (3) and also each of the following identities:

$$h(a \Rightarrow b) = h(a) \Rightarrow h(b)\,, \tag{4}$$

$$h(a - b) = h(a) - h(b)\,. \tag{5}$$

Then h is a homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$ in each of the possible interpretations of Boolean algebras as abstract algebras. Sometimes we shall use the term *Boolean homomorphism* for h to distinguish this notion from lattice homomorphisms (see I, § 6, p. 37).

To verify the truth of the above remarks, let us suppose, for example, that h satisfies conditions (1) and (2). Then

$$h(a \cap b) = h\big(-(-a \cup -b)\big) = -h(-a \cup -b) = -\big(h(-a) \cup h(-b)\big)$$
$$= -\big(-h(a) \cup -h(b)\big) = h(a) \cap h(b)\,,$$

i.e., h satisfies (3). In the same way we verify that h has properties (4) and (5), etc.

Every Boolean homomorphism h of a Boolean algebra $\boldsymbol{A}$ into a Boolean algebra $\boldsymbol{B}$ maps the unit element onto the unit element, and the zero element onto the zero element; in symbols

$$h(\forall_{\boldsymbol{A}}) = \forall_{\boldsymbol{B}}\,, \qquad h(\wedge_{\boldsymbol{A}}) = \wedge_{\boldsymbol{B}}\,, \tag{6}$$

since $h(\forall_{\boldsymbol{A}}) = h(a \cup -a) = h(a) \cup -h(a) = \forall_{\boldsymbol{B}}$ and similarly $\wedge$.

The homomorphic image $h(\boldsymbol{A})$ of a Boolean algebra $\boldsymbol{A}$ is a Boolean algebra since it is a subalgebra of $\boldsymbol{B}$.

3.1. *A homomorphism h of $\boldsymbol{A}$ into $\boldsymbol{B}$ is one-to-one if and only if $h(a) = \forall_{\boldsymbol{B}}$ implies $a = \forall_{\boldsymbol{A}}$ (or: if $h(a) = \wedge_{\boldsymbol{B}}$ implies $a = \wedge_{\boldsymbol{A}}$).*

The necessity is evident. To prove the sufficiency, suppose that $h(a_1) = h(a_2)$. Since $h(a_1 \Rightarrow a_2) = h(a_1) \Rightarrow h(a_2) = \forall_{\boldsymbol{B}}$, we have $a_1 \Rightarrow a_2 = \forall_{\boldsymbol{A}}$, i.e. $a_1 \leqslant a_2$. By symmetry, $a_2 \leqslant a_1$. Consequently $a_1 = a_2$, i.e. h is one-to-one.

By the duality we get the second part of 3.1.

A one-to-one Boolean homomorphism from a Boolean algebra into another is called an isomorphism (according to I, § 4) or, more precisely, a *Boolean isomorphism*. Theorem 3.1 yields necessary and sufficient conditions for a Boolean homomorphism to be a Boolean isomorphism.

Observe that every lattice isomorphism from a Boolean algebra onto another Boolean algebra is a Boolean isomorphism. In other words, if a mapping h from a Boolean algebra $\boldsymbol{A}$ onto a Boolean algebra $\boldsymbol{B}$ has properties (2) and (3), then it also has all the properties (1), (4), (5). It suffices to show that it has property (1). Since h preserves the lattice ordering and h is a mapping *onto*, equations (6) hold. Thus

$$h(a) \cap h(-a) = h(a \cap -a) = h(\wedge_{\boldsymbol{A}}) = \wedge_{\boldsymbol{B}}\,,$$
$$h(a) \cup h(-a) = h(a \cup -a) = h(\forall_{\boldsymbol{A}}) = \forall_{\boldsymbol{B}}\,.$$

Therefore the element $h(-a)$ is the complement of $h(a)$ in $\boldsymbol{B}$ (see I, 10.1), i.e. (1) holds.

If $\boldsymbol{A}$ and $\boldsymbol{B}$ are fields of subsets of spaces Y and X respectively, and φ is a mapping of X into Y such that

$$\varphi^{-1}(A) \in \boldsymbol{B} \quad \text{for every} \quad A \in \boldsymbol{A},$$

then the mapping h defined by the formula

$$h(A) = \varphi^{-1}(A) \quad \text{for} \quad A \in \boldsymbol{A}$$

is a homomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. We say then that the homomorphism h is *induced* by the point-mapping. Obviously, if φ maps X onto Y, then h is an isomorphism of $\boldsymbol{A}$ into $\boldsymbol{B}$. If φ is a one-to-one mapping of X onto Y and

$$\varphi^{-1}(A) \in \boldsymbol{B} \quad \text{if and only if} \quad A \in \boldsymbol{A},$$

then the homomorphism h induced by φ is an isomorphism of $\boldsymbol{A}$ onto $\boldsymbol{B}$. In the last case, the fields $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *equivalent* [1].

§ 4. The two-element Boolean algebra. The simplest lattice is a *one-element lattice*, i.e. set containing one element a only, with the obvious ordering relation: $a \leqslant a$. The one-element lattice is a Boolean algebra. It will also be called the *degenerate lattice* or *degenerate Boolean algebra* (according to the general definition concerning abstract algebras in I, § 4). Of course, all degenerate Boolean algebras are isomorphic.

A lattice (in particular, a Boolean algebra) is said to be *non-degenerate* if it contains at least two elements. The simplest non-degenerate lattice is the two-element lattice, i.e. a lattice $\boldsymbol{A}_0$ containing exactly two distinct elements. One of those elements is the zero element Λ of $\boldsymbol{A}_0$, the other is the unit element V of $\boldsymbol{A}_0$. The ordering relation in $\boldsymbol{A}_0$ is defined as follows:

$$\Lambda \leqslant \Lambda, \quad \Lambda \leqslant V, \quad V \leqslant V.$$

It is easy to verify that the two-element lattice $\boldsymbol{A}_0$ is Boolean, therefore it will be called the *two-element Boolean algebra*. Observe that the Boolean operations in $\boldsymbol{A}_0$ satisfy the following identities:

$$\begin{gathered} \Lambda \cup \Lambda = \Lambda, \quad \Lambda \cup V = V \cup \Lambda = V \cup V = V, \\ V \cap V = V, \quad V \cap \Lambda = \Lambda \cap V = \Lambda \cap \Lambda = \Lambda, \\ V \Rightarrow \Lambda = \Lambda, \quad \Lambda \Rightarrow \Lambda = V \Rightarrow V = \Lambda \Rightarrow V = V, \\ -\Lambda = V, \quad -V = \Lambda, \end{gathered} \tag{1}$$

[1] Marczewski [1], [2].

valid in all Boolean algebras. Thus

$$a \leqslant b \quad \text{if and only if} \quad a \Rightarrow b = V\,, \tag{2}$$

as in all relatively pseudo-complemented lattices.

For every $t \in T$ (where T is a non-empty set, finite or infinite) let a_t be an element of A_0. Then, it follows from the definition of infinite joins and meets in lattices (see I, § 7, and I, § 5) and from (1) that

$$\bigcup_{t \in T} a_t = \begin{cases} V & \text{if there exists a } t \in T \text{ such that } a_t = V\,, \\ \Lambda & \text{if } a_t = \Lambda \text{ for every } t \in T; \end{cases}$$

$$\bigcap_{t \in T} a_t = \begin{cases} V & \text{if } a_t = V \text{ for every } t \in T\,, \\ \Lambda & \text{if there exists a } t \in T \text{ such that } a_t = \Lambda\,. \end{cases} \tag{3}$$

Note that every two-element (more generally: every finite) Boolean algebra is complete.

Any two-element Boolean algebras $\boldsymbol{A}, \boldsymbol{B}$ are isomorphic, the isomorphism being defined by § 3, (6).

The field of all subsets of a one-element space is a two-element Boolean algebra. By isomorphism, every two-element Boolean algebra can be conceived, if convenient, as the field of all subsets of a one-element space.

Two-element Boolean algebras are very simple and rather uninteresting from the point of view of the general theory of Boolean algebras. However, they will play an essential role in the second part of this book (Chapters V, VII, VIII.)

§ 5. Filters and ideals. It follows immediately from the De Morgan laws and § 1, (3), that if ∇ (if Δ) is a filter (an ideal) in a Boolean algebra $\boldsymbol{A}$, then the set of all elements $-a$ where $a \in \nabla$ (where $a \in \Delta$) is an ideal (a filter) called the *adjoint* of ∇ (of Δ). By § 1, (5), the adjoint of the adjoint of ∇ (of Δ) is the filter ∇ (the ideal Δ) itself.

Thus, assigning to every filter ∇ its adjoint Δ, we establish a natural one-to-one correspondence between the class of all filters in $\boldsymbol{A}$ and the class of all ideals in $\boldsymbol{A}$. By this correspondence, it makes no difference from the practical point of view whether we talk about filters or ideals. From the point of view of applications to mathematical logic in the second part of this book we shall prefer to use the notion of filter.

The natural one-to-one correspondence between filters and ideals enables us to translate notions and constructions introduced for filters into their analogues for ideals.

For instance, in I, § 13, we have defined the quotient algebra $\boldsymbol{A}/\nabla$ for any relatively pseudo-complemented lattice $\boldsymbol{A}$ (and consequently for every Boolean algebra $\boldsymbol{A}$) and for every filter ∇ in $\boldsymbol{A}$. If $\boldsymbol{A}$ is a Boolean algebra

and Δ is an ideal in A, we can also form the quotient algebra A/Δ: by definition, A/Δ is the quotient algebra A/∇ where ∇ is the filter adjoint to Δ. Note that A/Δ is not defined for pseudo-Boolean algebras A and ideals Δ in A. There is no natural correspondence between filters and ideals in pseudo-Boolean algebras.

5.1. *If A is a Boolean algebra, and ∇ is a filter (Δ is an ideal) in A, then A/∇ (A/Δ) is also a Boolean algebra.*

In fact, it follows immediately from the equations

$$(1)\qquad \begin{aligned} |a \cup b| &= |a| \cup |b|\,, \quad |a \cap b| = |a| \cap |b|\,, \\ |a \Rightarrow b| &= |a| \Rightarrow |b|\,, \quad |-a| = -|a|\,, \end{aligned}$$

proved in I, 13.2, (4), (5), (6), that all the identities (l_1)-(l_6) hold in A/∇.

Note that, by I, 13.2,

$$(2)\qquad |\vee_A| = \vee_{A/\nabla}\,, \quad |\wedge_A| = \wedge_{A/\nabla}\,.$$

We have proved (see I, 9.1) that in every distributive lattice, in particular in every Boolean algebra, every maximal filter (ideal) is prime. In Boolean algebras the converse statement is also true. More exactly:

5.2. *The following conditions are equivalent for every filter ∇ (ideal Δ) of a Boolean algebra A:*

(a) *the filter ∇ (the ideal Δ) is maximal*;

(b) *the filter ∇ (the ideal Δ) is prime*;

(c) *the filter ∇ (the ideal Δ) is proper and, for every $x \in A$, either x or $-x$ belongs to ∇ (to Δ)*;

(d) *for every $x \in A$, exactly one of the elements x, $-x$ belongs to ∇ (to Δ)*;

(e) *the quotient algebra A/∇ (A/Δ) is a two-element Boolean algebra.*

We shall prove 5.2 only for filters. By duality we can obtain the analogous statements for ideals.

The equivalence of (a), (c), (d) and (e) was proved in I, 13.10, for any filters in pseudo-Boolean algebras. (a) implies (b) by I, 9.1. (b) implies (c) since $x \cup -x = \vee \in \nabla$.

5.3. *Let ∇ be a filter in a Boolean algebra A, and let $a \in A$. The filter generated by ∇ and $-a$ is proper if and only if $a \notin \nabla$.*

This follows immediately from I, 13.9 (where $a_0 = -a$) and the identity $--a = a$ (1.3, (5)).

§ 6. Relativization. Let g be a fixed element in a Boolean algebra A. According to I, § 11, p. 57, the symbol A_g will denote the set of all $a \in A$ such that $a \leqslant g$ (i.e. the principal ideal generated by g). The set A_g is

a distributive lattice with the same operations $\cup$ and $\cap$ restricted to A_g, that is A_g is a sublattice of A. The lattice ordering $\leqslant$ in A_g is that from A, restricted to A_g. The zero element of A is also the zero element of A_g; the element g is the unit element of A_g.

6.1. *The lattice A_g is Boolean. For every element $a \in A_g$, the element $g-a$ (i.e. the element $g \cap -a$) is the complement of a in A_g. For all elements $a, b \in A_g$, the element $g \cap (a \Rightarrow b)$ is the complement of a relative to b in A_g.*

Of course, $-$ and $\Rightarrow$ denote here operations in A. By a simple calculation we obtain

$$a \cap (g-a) = \wedge \quad \text{and} \quad a \cup (g-a) = g$$

which proves that $g-a$ is the complement of a in the lattice A_g (see I, 10.1). Since A_g is distributive, A_g is Boolean. For every element $x \in A_g$ and fixed elements $a, b \in A_g$,

$$x \cap a \leqslant b \quad \text{if and only if} \quad x \leqslant g \cap (a \Rightarrow b).$$

This proves that $g \cap (a \Rightarrow b)$ is the pseudo-complement of a relative to b in A_g. Since A_g is Boolean, it is the complement of a relative to b in A_g.

Note that A_g is not a Boolean subalgebra of A, except in the trivial case where $g = \vee$.

If A is the field $\mathfrak{B}(X)$ of all subsets of a set X, and $g \in A$, i.e. g is a subset of X, then A_g is the field $\mathfrak{B}(g)$ of all subsets of the set g. Note that if g is a one-element set, then A_g is a two-element Boolean algebra.

6.2. *If A is a Boolean algebra and $g \in A$, then the mapping*

$$h_0(a) = g \cap a \tag{1}$$

is a Boolean homomorphism from A onto A_g. The homomorphism (1) *preserves all infinite joins and meets in A.*

The proof of the first part of 6.2 is by an easy verification. To prove the second part, let us first observe that for every indexed set $\{a_t\}_{t \in T}$ of elements of A_g,

$$\bigcup_{t \in T}^{A} a_t = \bigcup_{t \in T}^{A_g} a_t, \quad \bigcap_{t \in T}^{A} a_t = \bigcap_{t \in T}^{A_g} a_t, \tag{2}$$

which justifies the omission of the superscripts at $\bigcup$ and $\bigcap$. More precisely, (2) should be understood as follows: if the join (meet) on one side exists, then the remaining join (meet) also exists and the equation holds.

To prove that h_0 preserves all infinite joins and meets it suffices to show that

$$g \cap \bigcup_{t \in T} a_t = \bigcup_{t \in T} (g \cap a_t), \quad g \cap \bigcap_{t \in T} a_t = \bigcap_{t \in T} (g \cap a_t),$$

provided the join or meet on the left-hand side exists. These equations hold because they are the distributive law § 1, (23), and the identity I, 7.1, (10).

6.3. *Let X be a non-empty set, $x_0 \in X$ and let A_0 be a two-element Boolean algebra. The equation*

$$h_0(A) = \begin{cases} V \in A_0 & \text{if} \quad x_0 \in A, \\ \Lambda \in A_0 & \text{if} \quad x_0 \notin A \end{cases} \qquad \text{for } A \subset X \tag{3}$$

defines a homomorphism from the field $\mathfrak{B}(X)$ of all subsets of X onto the two-element Boolean algebra A_0. The homomorphism h_0 preserves all infinite joins and meets in $\mathfrak{B}(X)$.

Theorem 6.3 can be obtained from 6.2 where $A = \mathfrak{B}(X)$, $g = (x_0)$ and where A_0 is identified with the two-element Boolean algebra A_g. Of course, 6.3 can also be proved directly, by a simple verification.

We come back to the case of an arbitrary Boolean algebra A. It is not difficult to verify that the mapping

$$h(a) = |a| \qquad (a \in A_g) \tag{4}$$

is an isomorphism from A_g onto A/V where V is the principal filter generated by the element g.

By duality, we can consider every principal filter as a Boolean algebra, with the same operations $\cup$ and $\cap$.

§ 7. Products of Boolean algebras. Let $\{A_n\}_{n \in N}$ be an indexed set of Boolean algebras conceived as abstract algebras

$$\{A_n, \cup, \cap, \Rightarrow, -\} \tag{1}$$

according to § 1, p. 71. By the general definition in I, § 4, p. 27, the product A of all A_n is the algebra

$$\{A, \cup, \cap, \Rightarrow, -\} \tag{2}$$

where $A = P_{n \in N} A_n$ and the operations in A are defined as follows:

$$\begin{aligned} \{a_n\}_{n \in N} \cup \{b_n\}_{n \in N} &= \{a_n \cup b_n\}_{n \in N}, \\ \{a_n\}_{n \in N} \cap \{b_n\}_{n \in N} &= \{a_n \cap b_n\}_{n \in N}, \\ \{a_n\}_{n \in N} \Rightarrow \{b_n\}_{n \in N} &= \{a_n \Rightarrow b_n\}_{n \in N}, \\ -\{a_n\}_{n \in N} &= \{-a_n\}_{n \in N}. \end{aligned} \tag{3}$$

7.1. *The product (2) of Boolean algebras (1) is a Boolean algebra.*

The proof is similar to the proof of the analogous theorem I, 6.3, for lattices.

Note that the lattice $\{A, \cup, \cap\}$ is then the product of the lattices $\{A_n, \cup, \cap\}$ (see I, § 6, p. 38 and I, § 7, p. 43). Consequently the statement in I, § 6, (14) and theorem I, 7.2 are also valid for products of Boolean algebras. In particular, if all A_n are complete Boolean algebras, so is their product A.

7.2. *If, for every $n \in N$, A_n is a field of subsets of a set X_n, and all the sets X_n are disjoint, then the mapping*

$$h(\{A_n\}_{n \in N}) = \bigcup_{n \in N} A_n \tag{4}$$

(where $A_n \in A_n$ for every $n \in N$ and $\bigcup$ denotes the set-theoretical union) is an isomorphism from the product $A = P_{n \in N} A_n$ onto the field B of all the subsets B of the union $X = \bigcup_{n \in N} X_n$ such that

$$X_n \cap B \in A_n \quad \text{for every} \quad n \in N .$$

If, for every $n \in N$, A_n is the field $\mathfrak{B}(X_n)$ of all subsets of X_n, then B is the field $\mathfrak{B}(X)$ of all subsets of X.

The proof is by an easy verification.

Isomorphism (4) yields a natural set-theoretical representation for the Boolean algebra $A = P_{n \in N} A_n$. Consequently we shall often identify the product A with the field B mentioned in 7.2 by identification of $\{A_n\}_{n \in N}$ with the union $\bigcup_{n \in N} A_n$.

7.3. *Suppose that all A_n are two-element Boolean algebras. For every element $a = \{a_n\}_{n \in N}$ in the product $A = P_{n \in N} A_n$, let $h(a)$ be the set of all $n \in N$ such that $a_n = 1_{A_n}$. The mapping h is an isomorphism of A onto the field $\mathfrak{B}(N)$ of all subsets of N.*

Theorem 7.3 can be deduced from 7.2 by assuming that A_n is the field of all subsets of the one-element set $X_n = (n)$.

The isomorphism h mentioned in 7.3 yields a natural set-theoretical representation of the product A of the two-element Boolean algebras A_n. Consequently we shall sometimes identify the product A with the field $\mathfrak{B}(N)$ by identification of $a \in A$ with the set $h(a)$.

We shall now examine the product of Boolean algebras conceived as generalized algebras with infinite operations (see the remarks at the end of § 1, p. 73).

Let

$$\{A_n, \cup, \cap, \Rightarrow, -, \bigcup | \mathfrak{Q}_{1,n}, \bigcap | \mathfrak{Q}_{2,n}\} \quad (n \in N) \tag{5}$$

be an indexed set of Boolean algebras. By the general definition in I, § 4, p. 31, we can form the product of all generalized algebras (5).

7.4. *The product of all the algebras* (5) *is the Boolean algebra*

$$\{A, \cup, \cap, \Rightarrow, -, \bigcup | \overset{*}{P}_{n \in N} \mathfrak{Q}_{1,n}, \bigcap | \overset{*}{P}_{n \in N} \mathfrak{Q}_{2,n}\}, \tag{6}$$

where $A = P_{n \in N} A_n$ and $\{A, \cup, \cap, \Rightarrow, -\}$ is the product of algebras (1).

This follows immediately from 7.1 and I, 7.3.

Consider now the case in which all A_n are two-element Boolean algebras conceived as complete Boolean algebras

$$\{A_n, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}. \tag{7}$$

By 7.4, their product is the Boolean algebra with infinite operations

$$\{A, \cup, \cap, \Rightarrow, -, \bigcup|\mathfrak{Q}_0, \bigcap|\mathfrak{Q}_0\} \tag{8}$$

where $A = \boldsymbol{P}_{n\in N} A_n$ and $\mathfrak{Q}_0$ is the set of all sets $S = \boldsymbol{P}_{n\in N} S_n$, $0 \neq S_n \subset A_n$. The algebra

$$\{A, \cup, \cap, \Rightarrow, -\} \tag{9}$$

is the product of the algebras $\{A_n, \cup, \cap, \Rightarrow, -\}$.

Under the above notation the following theorem holds:

7.5. *Let* $\boldsymbol{B}$ *be a complete* (1) *Boolean algebra. Every homomorphism from the algebra* (9) *into the algebra* $\{\boldsymbol{B}, \cup, \cap, \Rightarrow, -\}$ *is a homomorphism from the generalized algebra* (8) *into the generalized algebra* $\{\boldsymbol{B}, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$.

In other words: *every Boolean homomorphism* h: $A \to \boldsymbol{B}$ *preserves all the infinite joins and meets*

$$a' = \bigcup_{a\in S} a, \tag{10}$$

$$a'' = \bigcap_{a\in S} a, \tag{11}$$

where $S \in \mathfrak{Q}_0$.

By definition, $S = \boldsymbol{P}_{n\in N} S_n$, $0 \neq S_n \subset A_n$. Suppose equation (10) holds. By I, 7.2, $a' = \{a_n\}_{n\in N}$ where $a_n = \bigcup S_n$. Since A_n is a two-element algebra, we have $a_n \in S_n$ for every $n \in N$, and consequently $a' \in S$. Hence $h(a') \leqslant \bigcup_{a\in S} h(a)$. The converse inequality $h(a') \geqslant \bigcup_{a\in S} h(a)$ always holds (see I, § 7, (21)). Thus $h(a') = \bigcup_{a\in S} h(a)$, i.e. h preserves the join (10).

The proof that h preserves the meet (11) is by duality.

§ 8. The Stone spaces of Boolean algebras. Let A be a Boolean algebra. We recall the notation from I, § 9 (p. 50): $\mathfrak{S}(A)$ is the Stone space of all prime (= maximal) filters in A, $h(a)$ is the set of all $V \in \mathfrak{S}(A)$ such that $a \in V$, and $\boldsymbol{P}(A)$ is the class of all sets $h(a)$, $a \in A$.

8.1. *Every Boolean algebra is isomorphic to a field of sets.*

More exactly:

If A *is a Boolean algebra, then the Stone lattice* $\boldsymbol{P}(A)$ *is a field of subsets of the Stone space* $\mathfrak{S}(A)$ *and the Stone isomorphism* h *is a Boolean isomorphism of* A *onto* $\boldsymbol{P}(A)$ (2).

(1) This hypothesis is superfluous. It is assumed to simplify the proof only.

(2) Stone [2].

On account of I, 9.3, it suffices to prove that

$$h(-a) = -h(a) = \mathfrak{F}(\boldsymbol{A}) - h(a) \quad \text{for every } a \in \boldsymbol{A},$$

where the sign $-$ on the right-hand side denotes the set-theoretical difference.

If $V \in \mathfrak{F}(\boldsymbol{A}) - h(a)$, i.e. $V \notin h(a)$, then $a \notin V$ by the definition of h. By theorem 5.2 (c), $-a \in V$, i.e. $V \in h(-a)$. Conversely, if $V \in h(-a)$, i.e. $-a \in V$, then $a \notin V$ because of 5.2 (d). Hence $\Lambda \notin h(a)$, i.e. $V \in \mathfrak{F}(\boldsymbol{A}) - h(a)$.

If $\boldsymbol{A}$ is a Boolean algebra, then $\boldsymbol{P}(\boldsymbol{A})$ is said to be the *Stone field* of $\boldsymbol{A}$.

8.2. *The Stone space $\mathfrak{F}(\boldsymbol{A})$ of a Boolean algebra $\boldsymbol{A}$ is a compact totally disconnected Hausdorff space. The Stone field $\boldsymbol{P}(\boldsymbol{A})$ is the class of all both open and closed subsets of $\mathfrak{F}(\boldsymbol{A})$* [1].

$\mathfrak{F}(\boldsymbol{A})$ is compact by I, 9.4, since $\boldsymbol{A}$ has the unit. If V_1 and V_2 are different points of $\mathfrak{F}(\boldsymbol{A})$, i.e. different prime filters, then there exists an element $a \in \boldsymbol{A}$ belonging to only one of the filters, say $a \in V_1$ and $a \notin V_2$. Hence $V_1 \in h(a)$ and $V_2 \notin h(a)$, i.e. $V_2 \in -h(a) = h(-a)$. The sets $h(a)$ and $h(-a)$ are two disjoint open sets whose union is the whole space $\mathfrak{F}(\boldsymbol{A})$. Thus $\mathfrak{F}(\boldsymbol{A})$ is a totally disconnected Hausdorff space. The open set $h(a)$ is simultaneously closed since it is the complement of the open set $h(-a)$. On the other hand, if a set $G \subset \mathfrak{F}(\boldsymbol{A})$ is simultaneously open and closed, then $G = h(a_0)$ for some $a_0 \in \boldsymbol{A}$. In fact, G is the union of open sets $h(a_t)$ $(t \in T)$ by the definition of the topology in $\mathfrak{F}(\boldsymbol{A})$. Since G is a closed subset of a compact space, there exist $t_1, \ldots, t_n \in T$ such that $G = h(a_{t_1}) \cup \ldots \cup h(a_{t_n}) = h(a_0)$ where $a_0 = a_{t_1} \cup \ldots \cup a_{t_n}$.

The field $\boldsymbol{P}(\boldsymbol{A})$ has the following important property:

8.3 [2]. *Every homomorphism g from $\boldsymbol{P}(\boldsymbol{A})$ into any field $\boldsymbol{B}$ of subsets of a set X is induced by a mapping ψ, i.e. there is a mapping $\psi\colon X \to \mathfrak{F}(\boldsymbol{A})$ such that*

$$g(A) = \psi^{-1}(A) \quad \textit{for every} \quad A \in \boldsymbol{P}(\boldsymbol{A}).$$

. The superposition gh of the Stone isomorphism $h\colon \boldsymbol{A} \to \boldsymbol{P}(\boldsymbol{A})$ and the homomorphism $g\colon \boldsymbol{P}(\boldsymbol{A}) \to \boldsymbol{B}$ is a homomorphism $gh\colon \boldsymbol{A} \to \boldsymbol{B}$. Consequently, for every point $x \in X$, the set V_x of all $a \in \boldsymbol{A}$ such that $x \in g\big(h(a)\big)$ is a prime filter in $\boldsymbol{A}$, i.e. $V_x \in \mathfrak{F}(\boldsymbol{A})$. The mapping

$$\psi(x) = V_x$$

has the required properties. In fact, by definition

$$x \in g\big(h(a)\big) \quad \text{if and only if} \quad a \in V_x, \quad \text{i.e. if} \quad V_x \in h(a).$$

---

[1] Stone [3].

[2] Sikorski [1].

This proves that

$$\psi^{-1}\big(h(a)\big) = g\big(h(a)\big) \quad \text{for every} \quad a \in A .$$

Consequently replacing $h(a)$ by A, we infer that $\psi^{-1}(A) = g(A)$ for every $A \in \boldsymbol{P}(A)$.

Let T be an infinite set. If the equation

$$a = \bigcup_{t \in T}^{A} a_t \qquad \left(a = \bigcap_{t \in T}^{A} a_t\right) \tag{1}$$

holds, then $h(a)$ is the infinite join (meet) of all elements $h(a_t)$ in the Boolean algebra $\boldsymbol{P}(A)$, i.e. $h(a)$ is the least (greatest) set belonging to $\boldsymbol{P}(A)$ and containing (contained in) all the sets $h(a_t)$ However the set $h(a)$ is not, in general, the set-theoretical union (intersection) of all the sets $h(a_t)$ By I, § 7, (21), (23), we have only

$$h(a) \supset \bigcup_{t \in T} h(a_t) \qquad \left(h(a) \subset \bigcap_{t \in T} h(a_t)\right), \tag{2}$$

where, on the right-hand side (and in the sequel of this section) $\bigcup$ and $\bigcap$ denote the set-theoretical union and intersection.

The sign $\subset$ can be replaced by $=$ in (2) if and only if $a = a_{t_1} \cup \ldots \cup a_{t_n}$ for a finite sequence $t_1, \ldots, t_n \in T$. The sufficiency of this condition is obvious since h transforms finite joins (meets) into set-theoretical unions (intersections). To prove the necessity, suppose that $h(a)$ is the union of all $h(a_t)$. Since $h(a)$ is a closed subset of the compact space $\mathfrak{F}(A)$, there exists a finite sequence $t_1, \ldots, t_n \in T$ such that $h(a) = h(a_{t_1}) \cup \ldots \cup h(a_{t_n}) = h(a_{t_1} \cup \ldots \cup a_{t_n})$. Hence $a = a_{t_1} \cup \ldots \cup a_{t_n}$. The proof for intersection is similar.

8.4. *If equation* (1) *holds, then the set*

$$h(a) - \bigcup_{t \in T} h(a_t) \qquad \left(\bigcap_{t \in T} h(a_t) - h(a)\right) \tag{3}$$

is a closed, nowhere dense subset of the Stone space $\mathfrak{F}(A)$.

We shall prove the first part of 8.4. The proof of the second is similar.

The set $H = h(a) - \bigcup_{t \in T} h(a_t)$ is closed since it is the difference of a closed and an open set. Suppose that H is not nowhere dense, i.e. H contains a non-empty open set. Then there is an element $a_0 \in A$ such that

$$h(a_0) \subset H, \qquad a_0 \neq \Lambda .$$

Since $h(a_0) \subset h(a)$, we have $a_0 \leqslant a$ on account of I, § 6, (9′). Thus $a \neq a - a_0 \leqslant a$. On the other hand,

$$h(a_0) \subset h(a) - h(a_t), \quad \text{i.e.} \quad h(a_t) \subset h(a) - h(a_0) = h(a - a_0) .$$

Hence, by I, § 6, (9′), $a_t \leqslant a - a_0$ for every $t \in T$. This contradicts (1).

§ 9. Representations preserving some infinite joins and meets. Let A be a Boolean algebra and let (Q) be a set of infinite joins and meets in A:

$$\text{(Q)} \qquad \begin{aligned} a_s &= \bigcup_{t \in T'_s}^{A} a_{s,t} \qquad (s \in S') , \\ b_s &= \bigcap_{t \in T''_s}^{A} b_{s,t} \qquad (s \in S'') . \end{aligned}$$

According to I, § 7, p. 43, we shall say that a Boolean homomorphism (isomorphism) h from A into a Boolean algebra B is a Q-homomorphism (Q-isomorphism) provided it preserves all the infinite joins and meets in (Q).

It follows from a general remark in I, § 7, p. 43, that the Stone isomorphism h considered as a mapping from A into $P(A)$ is a Q-isomorphism, because it is an isomorphism onto $P(A)$. It follows from the consideration in § 8 that the Stone isomorphism h considered as an isomorphism from A into the field $\mathfrak{B}(\mathfrak{S}(A))$ of all subsets of the Stone space $\mathfrak{S}(A)$, is not a Q-homomorphism, except for some trivial cases. For, in general, $h(a_s)$ is not the set-theoretical union of all sets $h(a_{s,t})$, i.e. it is not the join of $h(a_{s,t})$ in $\mathfrak{B}(\mathfrak{S}(A))$. The analogous statement is true for meets.

For application to the second part of this book it is important to know under what conditions there exists a Q-isomorphism h_0 from A into the field $\mathfrak{B}(X)$ of all subsets of a space X, i.e. an isomorphism h_0 from A onto a field of sets such that h_0 transforms all the joins and meets in (Q) onto the corresponding set-theoretical unions and intersections. In general, such an isomorphism h_0 does not exist. We shall formulate necessary and sufficient conditions for the existence of h_0.

For this purpose we shall assume the following definition: a maximal filter V (a maximal ideal Δ) in A is said to be a Q-*filter* (a Q-*ideal*) provided the natural homomorphism from A onto A/V (onto A/Δ) is a Q-homomorphism.

In other words, a maximal (i.e. prime) filter V is a Q-filter provided that:

(a) for every $s \in S'$, if $a_s \in V$, then there is an index $t \in T'_s$ such that $a_{s,t} \in V$;

(b) for every $s \in S''$, if $b_s \notin V$, then there is an index $t \in T''_s$ such that $b_{s,t} \notin V$.

The following notation is a modification of the notation from § 8 and I, § 9: $\mathfrak{S}_Q(A)$ will denote the set of all Q-filters in A. For every $a \in A$, the symbol $h_Q(a)$ will denote the set of all Q-filters V such that $a \in V$. The symbol $P_Q(A)$ will denote the class of all sets $h_Q(a)$, $a \in A$.

By definition, $\mathfrak{S}_Q(A)$ is a subset of the Stone space $\mathfrak{S}(A)$, and

$$\text{(1)} \qquad h_Q(a) = h(a) \cap \mathfrak{S}_Q(A) .$$

Thus h_Q is a homomorphism from A into the field $\mathfrak{B}(\mathfrak{F}_Q(A))$ of all subsets of $\mathfrak{F}_Q(A)$ (viz. onto the field $P_Q(A)$). Moreover,

9.1. *The mapping h_Q is a* Q*-homomorphism from A into $\mathfrak{B}(\mathfrak{F}_Q(A))$, i.e. h_Q transforms all the infinite joins and meets in* (Q) *onto the corresponding set-theoretical unions and intersections.*

By I, § 7, (21), in order to prove that $h_Q(a_s)$ is the union of the sets $h_Q(a_{s,t})$, where $t \in T'_s$, it suffices to show that if $V \in h_Q(a_s)$, then $V \in h_Q(a_{s,t})$ for an index t. By definition of h_Q, $V \in h_Q(a_s)$ is equivalent to $a_s \in V$, and $V \in h_Q(a_{s,t})$ is equivalent to $a_{s,t} \in V$. Thus the statement follows directly from (a). This proves that h_Q preserves all the joins in (Q). Using (b) and I, § 7, (23), we can prove similarly that h_Q preserves all the meets in (Q).

9.2. *The following conditions are equivalent*:

(i) *every non-zero element $a \in A$ belongs to a* Q*-filter*;

(ii) *h_Q is a* Q*-isomorphism from A into $\mathfrak{B}(\mathfrak{F}^*_Q(A))$*;

(iii) *there exists a* Q*-isomorphism h_0 from A into the field $\mathfrak{B}(X)$ of all subsets of a space X.*

(i) means that $h_Q(a) \neq 0$ for $a \neq \Lambda$, i.e. that the homomorphism h_Q is an isomorphism (see 3.1). Thus (i) implies (ii) by 9.1. (ii) implies (iii) trivially. If (iii) holds and $a \neq \Lambda$, then the set $h_0(a)$ is not empty, i.e. there is a point $x_0 \in h_0(a)$. The set V of all $b \in A$ such that $x_0 \in h_0(b)$ is a Q-filter and $a \in V$. Thus (iii) implies (i).

9.3. *If the set* (Q) *is at most enumerable* (*i.e. if the sets S' and S'' are at most enumerable*), *then the set of all filters which are not* Q*-filters is of the first category in the Stone space $\mathfrak{S}(A)$ and condition* (i) *holds* [1].

By the definition of Q-filters, a maximal filter Λ is not a Q-filter if and only if it belongs to one of the sets

$$h(a_s) - \bigcup_{t \in T'_s} h(a_{s,t}) \qquad (s \in S'),$$

$$\bigcap_{t \in T''_s} h(b_{s,t}) - h(b_s) \qquad (s \in S'')$$

where h is the Stone isomorphism and $\bigcup$ and $\bigcap$ denote the set-theoretical union and intersection. Consequently the set M of all maximal filters which are not Q-filters is of the first category in $\mathfrak{S}(A)$ as the union of an enumerable set of nowhere dense sets (see 8.4). If $a \neq \Lambda$, then $h(a)$ is an open non-empty set of the compact Hausdorff space $\mathfrak{S}(A)$. By I, 2.2, the set $\mathfrak{S}(A) - M$ is dense, thus $h(a) - M$ is not empty, i.e. there exists a Q-filter V such that $a \in V$.

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[1] Rasiowa and Sikorski [1]. Another proof of (i) under the hypothesis of 9.3 is due to A. Tarski (see Feferman [1]).

9.4. *If the set* (Q) *is at most enumerable, then there exists a* Q*-isomorphism* h_0 *from* $\boldsymbol{A}$ *into the field* $\mathfrak{B}(X)$ *of all subsets of a space* X. *Moreover, we may assume that the cardinal of* X *is* $\leqslant 2^{\mathfrak{m}}$ *where* $\mathfrak{m}$ *is the cardinal of* $\boldsymbol{A}$ [1].

The first part of 9.4 follows immediately from 9.2 and 9.3. Namely, we may assume $X = \mathcal{S}_{\mathrm{Q}}(\boldsymbol{A})$ and $h_0 = h_{\mathrm{Q}}$. The estimation of the cardinal of $\mathcal{S}_{\mathrm{Q}}(\boldsymbol{A})$ follows directly from I, 9.5.

Observe that if $\boldsymbol{A}$ is a complete Boolean algebra, then $\boldsymbol{P}_{\mathrm{Q}}(\boldsymbol{A})$ is the field $\mathfrak{B}(\mathcal{S}_{\mathrm{Q}}(\boldsymbol{A}))$ of all subsets of $\mathcal{S}_{\mathrm{Q}}(\boldsymbol{A})$. In fact, every one-point set (V) where $V \in \mathcal{S}_{\mathrm{Q}}(\boldsymbol{A})$ is then the image of an element of $\boldsymbol{A}$, viz. $(V) = h_{\mathrm{Q}}(a_V)$ where a_V is the meet of all elements $a \in V$. Consequently, for every set $P \subset \mathcal{S}_{\mathrm{Q}}(\boldsymbol{A})$, we have $P = h_{\mathrm{Q}}(a)$ where a is the join of all a_V such that $V \in P$.

In § 10 we shall quote an example of a complete Boolean algebra which is not isomorphic to the field of all subsets of any space. For this Boolean algebra, none of the conditions (i), (ii), (iii) is satisfied.

§ 10. Minimal extensions of Boolean algebras. Every Boolean algebra $\boldsymbol{A}$ is isomorphic to a subalgebra of a complete Boolean algebra. For instance, the Stone isomorphism maps $\boldsymbol{A}$ into the complete field of all subsets of the Stone space $\mathcal{S}(\boldsymbol{A})$. This isomorphism does not, in general, preserve infinite joins and meets. In this section we shall define another isomorphism of $\boldsymbol{A}$ into a complete Boolean algebra, which preserves all the infinite joins and meets.

Let X be a topological space. A set $A \subset X$ is said to have the *Baire property* provided there exists an open set G such that

(1) $\qquad A - G$ and $G - A$ are of the first category.

The union of two sets A, A' having the Baire property also has the Baire property. In fact, suppose that (1) is satisfied and G' is an open set such that $A' - G'$ and $G' - A'$ are of the first category. Then $G \cup G'$ is open and

$$(A \cup A') - (G \cup G') \subset (A - G) \cup (A' - G'),$$

$$(G \cup G') - (A \cup A') \subset (G - A) \cup (G' - A').$$

Since the sets on the right-hand side are of the first category, so are the sets on the left-hand side.

The complement $-A = X - A$ of a set A with the Baire property also has this property. Indeed, suppose that (1) holds. The set $G_0 = -\mathrm{C}G$

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[1] Rasiowa and Sikorski [1].

is open and the set $\mathbf{C}G-G$ is nowhere dense. We have

$$(-A)-G_0 = (-G_0)-A = \mathbf{C}G-A \subset (\mathbf{C}G-G) \cup (G-A)\,,$$
$$G_0-(-A) = G_0 \cap A = A \cap -\mathbf{C}G = A-\mathbf{C}G \subset A-G\,,$$

which proves that the sets on the left-hand side are of the first category.

Thus the class of all sets, having the Baire property, is a field of subsets of the space X. Of course, this field contains all open subsets.

Consider now the Stone space $\mathfrak{S}(A)$ of a Boolean algebra A. Let B be the field of all sets $A \subset \mathfrak{S}(A)$ having the Baire property and let Δ be the ideal of all sets $A \subset \mathfrak{S}(A)$ of the first category. The Boolean algebra $A^* = B/\Delta$ is called the *minimal extension* [1] of A. The mapping

$$h^*(a) = |h(a)| \in A^* \quad \text{for} \quad a \in A\,,$$

where h is the Stone isomorphism of A onto the Stone field $P(A)$ of all both open and closed subsets of $\mathfrak{S}(A)$, is a homomorphism. If $h^*(a)$ is the zero of A^*, then $h(a) \in \Delta$, i.e. the open set $h(a)$ is of the first category. By I, 2.2, the set $h(a)$ is empty. Since h is an isomorphism, we have $a = \Lambda_A$. This proves that h^* is an isomorphism. We call h^* the *canonical isomorphism* of A into A^*.

10.1 [2]. *The minimal extension A^* of an arbitrary Boolean algebra A is complete. The canonical isomorphism h^* preserves all the infinite joins and meets, i.e.*

(2) *if* $a = \bigcup^A_{t\in T} a_t$, *then* $h^*(a) = \bigcup^{A^*}_{t\in T} h^*(a_t)$,

(3) *if* $b = \bigcap^A_{t\in T} b_t$, *then* $h^*(b) = \bigcap^{A^*}_{t\in T} h^*(b_t)$.

In this section the sign $\bigcup$ without any superscript will denote the set-theoretical union only.

If (1) holds, then $|A| = |G|$. Therefore every element of A^* can be represented in the form $|G|$ where G is open.

First we shall prove that

$$|\bigcup_{t\in T} G_t| = \bigcup^{A^*}_{t\in T} |G_t| \tag{4}$$

for every indexed set $\{G_t\}_{t\in T}$ of open subsets of $\mathfrak{S}(A)$.

Let G_0 be the union of all sets G_t. Since $G_t \subset G_0$, we have

$$|G_t| \leqslant |G_0| \quad \text{for every} \quad t \in T\,.$$

On the other hand, suppose that $|G| \in A^*$ (G open) is an element in A^* such that

$$|G_t| \leqslant |G| \quad \text{for every} \quad t \in T\,.$$

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[1] The notion of minimal extension was defined by MacNeille [1] in another way. This equivalent definition is due to Sikorski [3].

[2] MacNeille [1].

Since $|G_t - G| = |G_t| - |G| = \wedge_{A^*}$, the set $G_t - G$ is of the first category. The set $G_t - \mathbf{C}G$ is open and of the first category (since it is a subset of $G_t - G$); thus it is empty on account of I, 2.2, i.e.

$$G_t \subset \mathbf{C}G \quad \text{for every} \quad t \in T .$$

Hence $G_0 \subset \mathbf{C}G = G \cup (\mathbf{C}G - G)$ and consequently

$$|G_0| \leqslant |\mathbf{C}G| = |G| \cup |\mathbf{C}G - G| .$$

Since the set $\mathbf{C}G - G$ is nowhere dense, $|\mathbf{C}G - G|$ is the zero element of A^* and consequently

$$|G_0| \leqslant |G|$$

which completes the proof of (4). Since every indexed set of elements of A^* can be represented in the form $\{|G_t|\}_{t \in T}$, where the sets G_t are open, it follows from (4) that its join exists, i.e. A^* is complete.

Suppose that the hypothesis of (2) holds. We have (see § 8, (2))

$$h(a) = \big(h(a) - \bigcup_{t \in T} h(a)\big) \cup \bigcup_{t \in T} h(a_t) .$$

By 8.4, the element $|h(a) - \bigcup_{t \in T} h(a_t)|$ is the zero of A^*. Hence

$$h^*(a) = |h(a)| = |\bigcup_{t \in T} h(a_t)| = \bigcup_{t \in T}^{A^*} |h(a_t)| = \bigcup_{t \in T}^{A^*} h^*(a_t)$$

on account of (4).

(3) follows from (2) and the De Morgan laws (see § 1, (19)-(22)).

An element a_0 of a Boolean algebra A is said to be an *atom* provided $a_0 \neq \wedge$ and there is no element $a \leqslant a_0$ such that $\wedge \neq a \neq a_0$. For instance, if A is a field of sets, every one-point set belonging to A is an atom of A.

A Boolean algebra A is said to be *atomic* provided, for every element $a \neq \wedge$, there is an atom $a_0 \leqslant a$. For instance, the field of all subsets of a fixed space is atomic. More generally, every *complete field of sets* (i.e. a field A such that the union and intersection of every indexed set of sets belonging to A belong to A) is atomic. Indeed, if a set $A \in A$ is not empty, let $x_0 \in A$. The intersection A_0 of all sets in A which contain x_0 is an atom of A and $A_0 \subset A$.

A Boolean algebra A is said to be *atomless* if it contains no atom, i.e. for every element $a \neq \wedge$ there is an element $a' \leqslant a$, $\wedge \neq a' \neq a$. For instance, if A_0 is the field of all subsets of an infinite space and Δ_0 is the ideal of all finite subsets, then $A = A_0/\Delta_0$ is atomless. In fact, if $a = |A| \in A$ ($A \in A_0$) is not the zero of A, then the set A is infinite. If A' is a subset of A such that both A' and $A - A'$ are infinite, then $a' = |A'| \leqslant a$ and $\wedge \neq a' \neq a$.

The property of being atomless (or: atomic) is invariant under isomorphisms. Therefore no atomless Boolean algebra is isomorphic to a complete field of sets.

If $\boldsymbol{A}$ is atomless, then the minimal extension $\boldsymbol{A}^*$ of $\boldsymbol{A}$ is also atomless. In fact, if $|G| \in \boldsymbol{A}^* = \boldsymbol{B}/\Delta$ (see p. 89) is not the zero of $\boldsymbol{A}^*$, then we can suppose that G is a non-empty open set. By definition of the topology in the Stone space $\mathfrak{S}(\boldsymbol{A})$, there is an element $a \neq \wedge_{\boldsymbol{A}}$ such that $h(a) \subset G$. Let $a' \leqslant a$, $\wedge_{\boldsymbol{A}} \neq a' \neq a$. The element $|h(a')|$ differs from zero and $|G|$ and $|h(a')| \leqslant |G|$.

Consequently we infer that if $\boldsymbol{A}$ is atomless, then the complete Boolean algebra $\boldsymbol{A}^*$ is not isomorphic to any complete field of sets. Thus the imbedding of a Boolean algebra $\boldsymbol{A}$ into a complete one, given by the canonical homomorphism h^*, is of a quite different character from that given by the Stone isomorphism (see the beginning of this section).

§ 11. The Cantor discontinuum. Let E be a non-empty set and let $\mathfrak{D}$ be the Cartesian product

$$\mathfrak{D} = U^E,$$

where U is the set consisting only of the integers 0 and 1. By definition, $\mathfrak{D}$ is the set of all $u = \{u_\alpha\}_{\alpha \in E}$ such that $u_\alpha = 0$ or $u_\alpha = 1$ for every $\alpha \in E$.

For every $\alpha \in E$, let D_α be the set of all $u \in \mathfrak{D}$ such that $u_\alpha = 1$. Let $\boldsymbol{D}_0$ be the class of all sets D_α ($\alpha \in E$) and let $\boldsymbol{D}$ be the field (of subsets of $\mathfrak{D}$) generated by $\boldsymbol{D}_0$. We shall consider $\mathfrak{D}$ as a topological space, the class $\boldsymbol{D}_0$ being assumed as a subbasis (see I, 2.1). The topological space thus obtained is called the *Cantor discontinuum over* E. By definition, the field $\boldsymbol{D}$ is a basis for $\mathfrak{D}$.

11.1. *Every mapping g from $\boldsymbol{D}_0$ into a Boolean algebra $\boldsymbol{B}$ can be extended to a Boolean homomorphism h of $\boldsymbol{D}$ into $\boldsymbol{B}$* [1].

According to the general definition of free algebras (see I, § 4), theorem 11.1 can also be formulated as follows:

$\boldsymbol{D}$ is a free Boolean algebra and $\boldsymbol{D}_0$ is a set of free generators.

Since every Boolean algebra is isomorphic to a field of sets, it suffices to prove 11.1 for the case in which $\boldsymbol{B}$ is a field of subsets of a space X.

For every $\alpha \in E$, let

$$\varphi_\alpha(x) = \begin{cases} 1 & \text{if} \quad x \in g(D_\alpha), \\ 0 & \text{if} \quad x \notin g(D_\alpha), \end{cases}$$

and let

$$\varphi(x) = \{\varphi_\alpha(x)\} \in \mathfrak{D}.$$

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[1] See e. g. Sikorski [4].

By definition

$$\varphi^{-1}(D_\alpha) = g(D_\alpha) .$$

Therefore the formula

$$h(A) = \varphi^{-1}(A) \quad \text{for} \quad A \in \boldsymbol{D}$$

defines a homomorphism (of $\boldsymbol{D}$ into $\boldsymbol{B}$) which is an extension of g.

11.2. *The Cantor discontinuum $\mathfrak{D}$ is a totally disconnected compact space homeomorphic to the Stone space $\mathfrak{S}(\boldsymbol{D})$ of the Boolean algebra $\boldsymbol{D}$. The field $\boldsymbol{D}$ is composed of all both open and closed subsets of $\mathfrak{D}$ and is equivalent to Stone field $\boldsymbol{P}(\boldsymbol{D})$ of $\boldsymbol{D}$.*

From the remarks at the end of § 3, it follows that it suffices to prove that there exists a one-to-one mapping φ of $\mathfrak{S}(\boldsymbol{D})$ onto $\mathfrak{D}$ such that

(1) $$\varphi^{-1}(A) \in \boldsymbol{P}(\boldsymbol{D}) \quad \text{if and only if} \quad A \in \boldsymbol{D} .$$

Let h be the Stone isomorphism of $\boldsymbol{D}$ onto $\boldsymbol{P}(\boldsymbol{D})$. Setting $X = \mathfrak{S}(\boldsymbol{D})$ and $g =$ the mapping h restricted to $\boldsymbol{D}_0$ in the proof of 11.1 we define, by the same method, a mapping φ: $\mathfrak{S}(\boldsymbol{D}) \to \mathfrak{D}$ such that

(2) $$h(A) = \varphi^{-1}(A) \quad \text{for} \quad A \in \boldsymbol{D} .$$

h^{-1} is an isomorphism of $\boldsymbol{P}(\boldsymbol{D})$ onto the field $\boldsymbol{D}$ of subsets of $\mathfrak{D}$. By theorem 8.3, there exists a mapping ψ: $\mathfrak{D} \to \mathfrak{S}(\boldsymbol{D})$ such that

(3) $$h^{-1}(B) = \psi^{-1}(B) \quad \text{for every} \quad B \in \boldsymbol{P}(\boldsymbol{D}) .$$

Consequently

$$\varphi^{-1}\psi^{-1}(B) = B \quad \text{for every} \quad B \in \boldsymbol{P}(\boldsymbol{D})$$

and

$$\psi^{-1}\varphi^{-1}(A) = A \quad \text{for every} \quad A \in \boldsymbol{D} .$$

Since any two distinct points in $\mathfrak{D}$ are separated by a set $A \in \boldsymbol{D}$ (i.e. exactly one of them belongs to A) and since an analogous statement holds for points in $\mathfrak{S}(\boldsymbol{D})$, we have $\psi = \varphi^{-1}$. Thus φ is a one-to-one mapping of $\mathfrak{S}(\boldsymbol{D})$ onto $\mathfrak{D}$. Property (1) follows from (2) and (3).

φ is a homeomorphism of $\mathfrak{S}(\boldsymbol{D})$ onto $\mathfrak{D}$ which realizes the equivalence between $\boldsymbol{P}(\boldsymbol{D})$ and $\boldsymbol{D}$.

CHAPTER III

TOPOLOGICAL BOOLEAN ALGEBRAS

§ 1. Definition and elementary properties. By a *topological Boolean algebra* [1] we shall understand a Boolean algebra $\mathfrak{A}$ with an operation $\mathbf{I}$ which, to every element $a \in \mathfrak{A}$, associates an element $\mathbf{I}a \in \mathfrak{A}$ in such a way that the following axioms are satisfied:

$$(i_1) \qquad \mathbf{I}(a \cap b) = \mathbf{I}a \cap \mathbf{I}b\,,$$

$$(i_2) \qquad \mathbf{I}a \leqslant a\,,$$

$$(i_3) \qquad \mathbf{II}a = \mathbf{I}a\,,$$

$$(i_4) \qquad \mathbf{I}V = V\,,$$

for all $a, b \in \mathfrak{A}$. In other words: a topological Boolean algebra is an abstract algebra $\{\mathfrak{A}, \cup, \cap, \Rightarrow, -, \mathbf{I}\}$ such that $\{\mathfrak{A}, \cup, \cap, \Rightarrow, -\}$ is a Boolean algebra, and $\mathbf{I}$ satisfies (i_1)-(i_4).

Every operation $\mathbf{I}$ (in a Boolean algebra) satisfying (i_1)-(i_4) is called an *interior operation*.

For instance, if X is a topological space and $\mathfrak{A}$ is a field of subsets of X such that

$$\mathbf{I}A \in \mathfrak{A} \quad \text{for every} \quad A \in \mathfrak{A}$$

(where $\mathbf{I}A$ denotes the interior of A—see I, § 2), then $\mathfrak{A}$ is a topological Boolean algebra. Every topological Boolean algebra of this type will be called a *topological field of sets* (more exactly: a *topological field of subsets of* X). In particular, the class $\mathfrak{B}(X)$ of all subsets of a topological space X is a topological field.

The axioms (i_1)-(i_4) mentioned above are identical with the axioms (i_1)-(i_4) mentioned in I, § 2. Consequently, the interior operation $\mathbf{I}$ in a topological Boolean algebra has, roughly speaking, the same properties as the interior of sets in topological spaces. Therefore we shall use the same terminology as in the case of topological spaces.

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[1] For a systematic exposition of topological Boolean algebras see McKinsey and Tarski [1], Sikorski [2], Nöbeling [1].

For instance, the element $\mathbf{I}a$ will be called the *interior* of the element a in the topological Boolean algebra $\mathcal{A}$. The element $-\mathbf{I}-a$ will be called the *closure* of $a \in \mathcal{A}$ and be denoted by $\mathbf{C}a$. By definition

$$\mathbf{C}a = -\mathbf{I}-a \quad \text{and} \quad \mathbf{I}a = -\mathbf{C}-a\,. \tag{1}$$

Similarly as in I, § 2, we prove that

$$\mathbf{C}(a \cup b) = \mathbf{C}a \cup \mathbf{C}b\,, \tag{c_1}$$

$$a \leqslant \mathbf{C}a\,, \tag{c_2}$$

$$\mathbf{C}\mathbf{C}a = \mathbf{C}a\,, \tag{c_3}$$

$$\mathbf{C}\Lambda = \Lambda \tag{c_4}$$

for any $a, b \in \mathcal{A}$. Conversely (see the analogous remark on p. 13), if $\mathcal{A}$ is a Boolean algebra with an operation $\mathbf{C}$ which, to every $a \in \mathcal{A}$, associates an element $\mathbf{C}a \in \mathcal{A}$ in such a way that (c_1)-(c_4) are satisfied, then the operation $\mathbf{I}$ defined by the second of equations (1) satisfies (i_1)-(i_4), i.e. $\mathcal{A}$ is a topological Boolean algebra [1].

Every operation $\mathbf{C}$ (in a Boolean algebra) satisfying (c_1)-(c_4) is called a *closure operation*.

It follows immediately from (i_1) and (c_1) that, for arbitrary elements a, b in a topological Boolean algebra $\mathcal{A}$,

$$a \leqslant b \quad \text{implies} \quad \mathbf{I}a \leqslant \mathbf{I}b \text{ and } \mathbf{C}a \leqslant \mathbf{C}b\,. \tag{2}$$

An element $a \in \mathcal{A}$ is said to be *open (closed)* if $a = \mathbf{I}a$ (if $a = \mathbf{C}a$). For instance, the interior (closure) of any element is open (closed) by (i_3) (by (c_3)). Analogously to the case of topological spaces we can prove that a is open (closed) if and only if its complement $-a$ is closed (open). The meet (join) of a finite number of open (closed) elements is open (closed). The join (meet) of any number of open (closed) elements is open (closed) whenever it exists in $\mathcal{A}$. In particular, the join (meet) of a finite number of open (closed) elements is always open (closed). If a is open (closed) and b is closed (open), then $a-b$ is open (closed).

Analogously to the case of topological spaces we observe that, if b is open, then, for every a,

$$b \leqslant a \quad \text{if and only if} \quad b \leqslant \mathbf{I}a \tag{3}$$

because of (i_2), (i_3) and (2). This implies that $\mathbf{I}a$ is the greatest open element contained in a. Similarly, if b is closed, then for every a

$$a \leqslant b \quad \text{if and only if} \quad \mathbf{C}a \leqslant b \tag{4}$$

[1] Therefore topological Boolean algebras are also called *closure algebras* by many authors.

on account of (c_2), (c_3) and (2). This implies that $\mathbf{C}a$ is the least closed element containing a.

An element a of a topological Boolean algebra $\boldsymbol{A}$ is said to be *dense* (more exactly: *dense in* $\boldsymbol{A}$) if $\mathbf{C}a = \forall$. The notions of boundary element and nowhere dense element can also be introduced for arbitrary topological Boolean algebras in the same way as for topological spaces; however, these notions will not play any essential part in our investigation.

Generalizing definitions from p. 14, we say that a class $\boldsymbol{B}$ of open elements of a topological Boolean algebra $\boldsymbol{A}$ is a *basis* of $\boldsymbol{A}$ if every open element in $\boldsymbol{A}$ is a (finite or infinite) join of some elements in $\boldsymbol{B}$. A class $\boldsymbol{B}_0$ of open elements is said to be a *subbasis* of $\boldsymbol{A}$ if the class $\boldsymbol{B}$, composed of the zero element $\wedge$, the unit element $\forall$ and of all finite meets $a_1 \cap \ldots \cap a_n$ where $a_1, \ldots, a_n \in \boldsymbol{B}_0$, is a basis of $\boldsymbol{A}$. Of course, if $\boldsymbol{B}_0$ is a subbasis containing the zero and the unit elements, and if $b_1 \cap b_2 \in \boldsymbol{B}_0$ for all $b_1, b_2 \in \boldsymbol{B}_0$, then $\boldsymbol{B}_0$ is a basis.

The following theorem is a generalization of I, 2.1:

1.1. *For every class* $\boldsymbol{B}_0$ *of elements of a complete Boolean algebra* $\boldsymbol{A}$ *there exists exactly one interior operation* $\mathbf{I}$ *in* $\boldsymbol{A}$ *such that* $\boldsymbol{B}_0$ *is a subbasis of the topological Boolean algebra* $\boldsymbol{A}$ *with the interior operation* $\mathbf{I}$.

The proof of 1.1 is the same as the proof of I, 2.1.

According to the general definition in I, § 4, by a *topological subalgebra* of a topological Boolean algebra $\boldsymbol{A}$ we understand every Boolean subalgebra $\boldsymbol{A}'$ (of the Boolean algebra $\boldsymbol{A}$) which is closed under the interior operation, i.e. such that

$$\mathbf{I}a \in \boldsymbol{A}' \quad \text{for every} \quad a \in \boldsymbol{A}'.$$

Of course, a Boolean subalgebra $\boldsymbol{A}'$ of $\boldsymbol{A}$ is a topological subalgebra of $\boldsymbol{A}$ if and only if it is closed under the closure operation, i.e. if

$$\mathbf{C}a \in \boldsymbol{A}' \quad \text{for every} \quad a \in \boldsymbol{A}'.$$

We shall always assume that the signs $\mathbf{I}$ and $\mathbf{C}$ bind more strongly than $\cup$, $\cap$ and $\Rightarrow$, i.e. expressions like $\mathbf{I}a \Rightarrow b$, $\mathbf{C}a \cup b$ should be understood as $(\mathbf{I}a) \Rightarrow b$, $(\mathbf{C}a) \cup b$, etc.

§ 2. Relativization to principal ideals. As we observed in II, 6.1, the principal ideal $\boldsymbol{A}_g$ generated by an element g in a Boolean algebra $\boldsymbol{A}$ can also be considered as a Boolean algebra, the join and meet in $\boldsymbol{A}_g$ being the same as in $\boldsymbol{A}$ and the complement of $a \in \boldsymbol{A}_g$ (i.e. $a \leqslant g$) being defined as $g - a$.

When $\boldsymbol{A}$ is a topological Boolean algebra (with an interior operation $\mathbf{I}$ and a closure operation $\mathbf{C}$), $\boldsymbol{A}_g$ will also be interpreted as a topo-

logical Boolean algebra, the closure operation $\mathbf{C}_g$ in $\boldsymbol{A}_g$ being defined by the equation

$$\mathbf{C}_g a = g \cap \mathbf{C}a \quad \text{for} \quad a \in \boldsymbol{A}_g . \tag{1}$$

It is evident that $\mathbf{C}_g$ satisfies axioms (c_1), (c_2), (c_4). Since

$$\mathbf{C}_g\mathbf{C}_g a = g \cap \mathbf{C}(g \cap \mathbf{C}a) \leqslant g \cap \mathbf{C}\mathbf{C}a = g \cap \mathbf{C}a = \mathbf{C}_g a$$
$$= g \cap \mathbf{C}a \leqslant g \cap \mathbf{C}(g \cap \mathbf{C}a) = \mathbf{C}_g\mathbf{C}_g a$$

for $a \leqslant g$, axiom (c_3) also holds.

The interior operation $\mathbf{I}_g$, determined in $\boldsymbol{A}_g$ by $\mathbf{C}_g$ on account of the general rule § 1, (1), is given by the identity

$$\mathbf{I}_g a = g \cap \mathbf{I}(a \cup -g) = g \cap \mathbf{I}(g \Rightarrow a) \quad \text{for} \quad a \in \boldsymbol{A}_g . \tag{2}$$

Observe that an element $a \in \boldsymbol{A}_g$ is closed in $\boldsymbol{A}_g$ (i.e. $a = \mathbf{C}_g a$) if and only if it is of the form

$$a = g \cap b \tag{3}$$

where b is a closed element in $\boldsymbol{A}$ (i. e. $\mathbf{C}b = b$, $b \in \boldsymbol{A}$). In fact, if $a = \mathbf{C}_g a$, i.e. $a = g \cap \mathbf{C}a$, it suffices to assume $b = \mathbf{C}a$. On the other hand, if $a = g \cap b$ and $b = \mathbf{C}b$, then

$$\mathbf{C}_g a = g \cap \mathbf{C}(g \cap b) \leqslant g \cap \mathbf{C}b = g \cap b = a \leqslant \mathbf{C}_g a$$

which proves $a = \mathbf{C}_g a$.

Consequently, if g is closed in $\boldsymbol{A}$, then an element $a \leqslant g$ is closed in $\boldsymbol{A}_g$ if and only if it is closed in $\boldsymbol{A}$. If g is closed, then formula (1) can be simplified:

$$\mathbf{C}_g a = \mathbf{C}a \quad \text{for every} \quad a \in \boldsymbol{A}_g . \tag{1'}$$

Passing to complements, we assert that an element $a \in \boldsymbol{A}_g$ is open in $\boldsymbol{A}_g$ (i.e. $a = \mathbf{I}_g a$) if and only if it is of the form (3) where b is open in $\boldsymbol{A}$ (i.e. $b = \mathbf{I}b$, $b \in \boldsymbol{A}$).

Consequently, if g is open in $\boldsymbol{A}$, then an element $a \in \boldsymbol{A}_g$ is open in $\boldsymbol{A}_g$ if and only if it is open in $\boldsymbol{A}$. If g is open, then formula (2) can be simplified as follows:

$$\mathbf{I}_g a = \mathbf{I}a \quad \text{for every} \quad a \in \boldsymbol{A}_g . \tag{2'}$$

In fact, if $a \leqslant g$ and $g = \mathbf{I}g$, then

$$\mathbf{I}_g a = g \cap \mathbf{I}(a \cup -g) = \mathbf{I}g \cap \mathbf{I}(a \cup -g) = \mathbf{I}\big(g \cap (a \cup -g)\big) = \mathbf{I}a .$$

2.1. *If $\boldsymbol{B}$ is a basis of a topological algebra $\boldsymbol{A}$, and $g \in \boldsymbol{A}$, then the set of all elements $g \cap b$, where $b \in \boldsymbol{B}$, is a basis for $\boldsymbol{A}_g$. If $\boldsymbol{B}_0$ is a subbasis for $\boldsymbol{A}$, then the set of all elements $g \cap b$, where $b \in \boldsymbol{B}_0$, is a subbasis for $\boldsymbol{A}_g$.*

If $a \in A_g$ is open in A_g, then $a = g \cap a_0$ where a_0 is open in A. Since B is a basis in A, there exists an indexed set $\{b_t\}_{t \in T}$ of elements in B such that $a_0 = \bigcup_{t \in T} b_t$. By the infinite distributive law II, 1.4, (23), $a = \bigcup_{t \in T} g \cap b_t$. This proves that elements $g \cap b$, where $b \in B$, form a basis for A_g.

If B_0 is a subbasis of A, then the class of all elements $b_1 \cap \ldots \cap b_n$, where $b_1, \ldots, b_n \in B_0$, form a basis for A together with the zero and unit element of A. By the part of 2.1, just proved, the elements $g \cap b_1 \cap \ldots \cap b_n$ $(b_1, \ldots, b_n \in B_0)$ form a basis for A_g together with the zero and unit element of A_g. This proves the second part of 2.1, since

$$g \cap b_1 \cap \ldots \cap b_n = (g \cap b_1) \cap \ldots \cap (g \cap b_n) .$$

When A is the topological Boolean algebra $\mathfrak{B}(X)$ of all subsets of a topological space X and G is a subset of X, then $\mathfrak{B}(X)_G = \mathfrak{B}(G)$, i.e. G becomes a topological space with the interior operation $\mathbf{I}_G$ and the closure operation $\mathbf{C}_G$ defined by the equations

$$\begin{gathered} \mathbf{I}_G A = G \cap \mathbf{I}\big(A \cup (X - G)\big) = G \cap \mathbf{I}(G \Rightarrow A), \\ \mathbf{C}_G A = G \cap \mathbf{C}A \quad \text{for} \quad A \subset G . \end{gathered} \tag{4}$$

If G is open, then

$$\mathbf{I}_G A = \mathbf{I}A \quad \text{for} \quad A \subset G . \tag{4'}$$

It follows from (3) (or from 2.1 and I, § 2, (11)) that the topological space G with the interior operation $\mathbf{I}_G$ just defined is a topological subspace of the topological space X, in the sense defined in I, § 2, p. 19. Thus equations (4) are definitions of the interior operation and the closure operation in topological subspaces of topological spaces.

§ 3. Topological homomorphisms and isomorphisms. Interior mappings. According to the general definition in I, § 4, a mapping h of a topological Boolean algebra A into a topological Boolean algebra B is said to be a *homomorphism* if it preserves all Boolean operations and the interior operation $\mathbf{I}$, i.e. if

$$h(a \cup b) = h(a) \cup h(b) , \quad h(a \cap b) = h(a) \cap h(b) , \tag{1}$$

$$h(a \Rightarrow b) = h(a) \Rightarrow h(b) , \quad h(-a) = -h(a) , \tag{2}$$

$$h(\mathbf{I}a) = \mathbf{I}h(a) \tag{3}$$

for all $a, b \in A$. If, in addition, h is one-to-one, it is called an *isomorphism*.

Of course, (3) can be replaced by the condition

$$h(\mathbf{C}a) = \mathbf{C}h(a) \tag{4}$$

which, because of the second of the identities (2), is equivalent to (3) (see § 1, (1)).

Sometimes it is necessary to consider topological Boolean algebras as Boolean algebras only, and to consider mappings h which satisfy only (1) and (2), i.e. which preserve only Boolean operations. To avoid mistakes, a mapping h: $\boldsymbol{A} \to \boldsymbol{B}$ which satisfies only (1) and (2) will be called a *Boolean homomorphism* (according to I, § 5), and a mapping h: $\boldsymbol{A} \to \boldsymbol{B}$ which satisfies (1), (2) and (3) will be called a *topological homomorphism*. The same terminology is used for isomorphisms.

If there exists a topological isomorphism h of $\boldsymbol{A}$ onto $\boldsymbol{B}$, then $\boldsymbol{A}$ and $\boldsymbol{B}$ are said to be *isomorphic* or *homeomorphic*. Of course, h^{-1} is then a topological isomorphism of $\boldsymbol{B}$ onto $\boldsymbol{A}$.

For instance, if X and Y are topological spaces and φ is a homeomorphism of X onto Y, then the mapping

$$h(B) = \varphi^{-1}(B) \quad \text{for} \quad B \subset Y$$

is a topological isomorphism of $\mathfrak{B}(Y)$ onto $\mathfrak{B}(X)$. One can prove that every topological isomorphism h of $\mathfrak{B}(Y)$ onto $\mathfrak{B}(X)$ is of the above form, i.e. it is induced by a homeomorphism of X onto Y (in fact, h transforms atoms of $\mathfrak{B}(Y)$ onto atoms of $\mathfrak{B}(X)$, i.e. it establishes a one-to-one correspondence φ between points in X and Y, and φ is a homeomorphism).

Thus the notion of topological isomorphisms is a generalization of the notion of homeomorphisms between topological spaces.

The following theorem yields an important example of a topological homomorphism:

3.1. *If g is an open element of a topological algebra $\boldsymbol{A}$, then the mapping*

$$h(a) = g \cap a \quad (a \in \boldsymbol{A}) \tag{5}$$

is a topological homomorphism of $\boldsymbol{A}$ onto $\boldsymbol{A}_g$ (see § 2).

Indeed, it follows from II, 6.2, that h is a Boolean homomorphism. It also preserves the interior operation since, by § 2, (2′), and (i_1),

$$h(\mathbf{I}a) = g \cap \mathbf{I}a = \mathbf{I}g \cap \mathbf{I}a = \mathbf{I}(g \cap a) = \mathbf{I}h(a) = \mathbf{I}_g h(a) .$$

It is worth while to remark that the statement converse to 3.1 is also true: if (5) is a topological homomorphism, then g is open. Indeed, then $g = \mathbf{I}_g g = \mathbf{I}_g h(g) = h(\mathbf{I}g) = g \cap \mathbf{I}g = \mathbf{I}g$.

Let φ be a mapping of a topological space X into a topological space Y. We shall now discuss under what conditions the Boolean homomorphism h induced by φ, i.e. the Boolean homomorphism

$$h(B) = \varphi^{-1}(B) \quad (B \subset Y) \tag{6}$$

of $\mathfrak{B}(Y)$ into $\mathfrak{B}(X)$, is topological, i.e.

$$\varphi^{-1}(\mathbf{I}B) = \mathbf{I}\varphi^{-1}(B) \quad \text{for every set} \quad B \subset Y . \tag{7}$$

To explain this, we introduce the following definitions.

The mapping φ is said to be *continuous* if, for every open set $B \subset Y$, the set $\varphi^{-1}(B)$ is open in X (or—by passing to complements—if $\varphi^{-1}(B)$ is closed in X for every closed set $B \subset Y$).

It follows immediately from the definition that a mapping φ is continuous if and only if

(8) $\varphi^{-1}(B)$ is open in X for every set B in a subbasis of Y.

The mapping φ is continuous if and only if

$$\varphi^{-1}(\mathbf{I}B) \subset \mathbf{I}\varphi^{-1}(B) \quad \text{for every set} \quad B \subset Y. \tag{9}$$

Indeed, if (9) is true and $B = \mathbf{I}B$, then $\mathbf{I}\varphi^{-1}(B) \subset \varphi^{-1}(B) \subset \mathbf{I}\varphi^{-1}(B)$ by (i_2), which proves that $\varphi^{-1}(B)$ is open. Conversely, if the counter-image of every open set is open, then $\varphi^{-1}(\mathbf{I}B) = \mathbf{I}\varphi^{-1}(\mathbf{I}B) \subset \mathbf{I}\varphi^{-1}(B)$ (see (i_2)), i.e. (9) holds.

The mapping φ is said to be *open* if $\varphi(A)$ is open in Y for every open subset A of X.

It follows immediately from the definition that φ is open if and only if

(10) $\varphi(A)$ is open in Y for every set A in a basis of X.

The mapping φ is open if and only if

$$\varphi(\mathbf{I}A) \subset \mathbf{I}\varphi(A) \quad \text{for every set} \quad A \subset X. \tag{11}$$

In fact, if (11) is true and $A = \mathbf{I}A$, then $\varphi(A) = \varphi(\mathbf{I}A) \subset \mathbf{I}\varphi(A) \subset \varphi(A)$, i.e. $\varphi(A) = \mathbf{I}\varphi(A)$ is open. Conversely, if φ is open, then $\varphi(\mathbf{I}A) = \mathbf{I}\varphi(\mathbf{I}A) \subset \mathbf{I}\varphi(A)$ (see (i_2)) which proves (11).

The mapping φ is open if and only if

$$\mathbf{I}\varphi^{-1}(B) \subset \varphi^{-1}(\mathbf{I}B) \quad \text{for every set} \quad B \subset Y. \tag{12}$$

It suffices to prove the equivalence of (11) and (12). If (11) holds for $A = \varphi^{-1}(B)$, then $\varphi\big(\mathbf{I}\varphi^{-1}(B)\big) \subset \mathbf{I}\varphi\big(\varphi^{-1}(B)\big) \subset \mathbf{I}B$, which implies (12). Conversely, if (12) holds for $B = \varphi(A)$, then $\mathbf{I}A \subset \mathbf{I}\varphi^{-1}\big(\varphi(A)\big) \subset \varphi^{-1}\big(\mathbf{I}\varphi(A)\big)$, which implies (11).

A mapping $\varphi\colon X \to Y$ is said to be an *interior mapping* from X into Y if it is continuous and open.

It follows immediately from (9) and (12) that

3.2. *The Boolean homomorphism h defined by* (6) *is topological if and only if φ is an interior mapping* [1].

[1] Wallace [1]; see also Sikorski [5], [7].

Observe that, by the definition of continuity (see also the end of I, § 2), a mapping φ of a topological space X onto a topological space Y is a homeomorphism if and only if both φ and φ^{-1} are continuous.

One can prove that each of the following conditions is necessary and sufficient for a mapping φ of a topological space X into a topological space Y to be continuous:

$$\mathbf{C}\varphi^{-1}(B) \subset \varphi^{-1}(\mathbf{C}B) \quad \text{for every} \quad B \subset Y;$$
$$\varphi(\mathbf{C}A) \subset \mathbf{C}\varphi(A) \quad \text{for every} \quad A \subset X;$$

φ is open if and only if $\varphi^{-1}(\mathbf{C}B) \subset \mathbf{C}\varphi^{-1}(B)$ for every $B \subset Y$.

§ 4. Extensions and imbeddings of topological Boolean algebras. The following lemma will play a fundamental part in this section:

4.1. *Let $\boldsymbol{B}_0$ be a subalgebra of a complete Boolean algebra $\boldsymbol{B}$. Every interior operation $\mathbf{I}_0$ in $\boldsymbol{B}_0$ can be extended to an interior operation $\mathbf{I}$ in $\boldsymbol{B}$ in such a way that open elements of $\boldsymbol{B}_0$ constitute a basis for $\boldsymbol{B}$* (1).

By 1.1 there exists an interior operation $\mathbf{I}$ in $\boldsymbol{B}$ such that the class $\boldsymbol{G}_0$ of all open elements in $\boldsymbol{B}_0$ is a subbasis of the topological Boolean algebra $\boldsymbol{B}$. Since the class $\boldsymbol{G}_0$ contains the unit and the zero element of $\boldsymbol{B}$, and $b_1 \cap b_2 \in \boldsymbol{G}_0$ for $b_1, b_2 \in \boldsymbol{G}_0$, the class $\boldsymbol{G}_0$ is a basis of the topological algebra $\boldsymbol{B}$ (see p. 95). We shall prove that $\mathbf{I}$ is an extension of $\mathbf{I}_0$, i.e. that $\mathbf{I}a = \mathbf{I}_0 a$ for $a \in \boldsymbol{B}_0$. By definition, $\mathbf{I}a$ is the union of all $b \in \boldsymbol{G}_0$ such that $b \leqslant a$. On the other hand, $\mathbf{I}_0 a$ is the greatest element $b \in \boldsymbol{G}_0$ such that $b \leqslant a$ (see the remark after statement (3) on p. 94). Hence it follows that $\mathbf{I}a = \mathbf{I}_0 a$.

4.2. *For every topological Boolean algebra $\boldsymbol{A}$ there exists a complete topological Boolean algebra $\boldsymbol{A}^*$ and a topological isomorphism h of $\boldsymbol{A}$ into $\boldsymbol{A}^*$ such that h preserves all infinite joins and meets in $\boldsymbol{A}$, and the elements $h(a)$, where a is open in $\boldsymbol{A}$, constitute a basis for $\boldsymbol{A}^*$* (2).

Let h be the canonical Boolean isomorphism of $\boldsymbol{A}$ into its minimal extension $\boldsymbol{A}^*$ (see II, § 10). It follows from II, 10.1, that h preserves all infinite joins and meets in $\boldsymbol{A}$. Substitute $\boldsymbol{B}_0 = h(\boldsymbol{A})$ and $\boldsymbol{B} = \boldsymbol{A}^*$ in lemma 4.1. By the isomorphism h the interior operation $\mathbf{I}$ in $\boldsymbol{A}$ induces an interior operation $\mathbf{I}$ in $\boldsymbol{B}_0$, namely the operation defined by the formula

$$\mathbf{I}h(a) = h(\mathbf{I}a) \quad (a \in \boldsymbol{A}). \tag{1}$$

Now we extend this interior operation to the whole algebra $\boldsymbol{A}^*$ by means of 4.1. Thus $\boldsymbol{A}^*$ becomes a complete topological Boolean algebra and h is the required topological isomorphism.

(1) McKinsey and Tarski [1].

(2) Rasiowa [2].

4.3. *Every topological Boolean algebra A is isomorphic to a topological field of sets, i.e. there exists a topological space X and a topological isomorphism h of A into $\mathfrak{B}(X)$. In addition, we can assume that the cardinal of X is $\leqslant 2^{\mathfrak{m}}$ where $\mathfrak{m}$ is the cardinal of A, and that the class of all sets $h(a)$, where a is open in A, is a basis for X. Moreover, given any enumerable set (Q) of infinite joins and meets in A, we can always assume that h is a (Q)-isomorphism from A into $\mathfrak{B}(X)$* (1).

By II, 9.4, there exists a Boolean (Q)-isomorphism h of A onto a field B_0 of subsets of a space X with cardinal $\leqslant 2^{\mathfrak{m}}$. We define the interior operation $\mathbf{I}$ in B_0 by (1) and extend it to the whole algebra $\mathfrak{B}(X)$ by 4.1. Thus X becomes a topological space, and the sets $h(a)$, where a is open, constitute an open basis for X.

The following lemma will be useful in IV, § 10, and XI, § 9:

4.4. *Let A be a topological Boolean algebra, let A_0 be a finite subset of A, and let A' be the Boolean subalgebra generated by A_0. Then there exists an interior operation $\mathbf{I}'$ in A' such that, for every $a \in A'$,*

$$\text{if} \quad \mathbf{I}a \in A_0, \quad \text{then} \quad \mathbf{I}'a = \mathbf{I}a .$$

The algebra A' contains at most 2^{2^r} elements where r is the number of elements in A_0 (2).

The estimation of the number of elements of A' follows immediately from II, 2.3.

Let B_0 be the class of all open elements in A which belong to A_0. By 1.1 there exists an interior operation $\mathbf{I}'$ in A' such that B_0 is a subbasis of A' considered as a topological Boolean algebra with the interior operation $\mathbf{I}'$. Namely $\mathbf{I}'V = V$ and, for $a \neq V$ $(a \in A')$, $\mathbf{I}'a$ is the union of all finite intersections $b_1 \cap \ldots \cap b_n \leqslant a$ where $b_1, \ldots, b_n \in B_0$, or $\mathbf{I}'a = \Lambda$ if such the intersections do not exist. Thus

$$\mathbf{I}'a \leqslant \mathbf{I}a \quad \text{for} \quad \text{every} \quad a \in A'.$$

If $\mathbf{I}a \in A_0$, then $\mathbf{I}a \in B_0$ and $\mathbf{I}a \leqslant a$. Hence it follows that $\mathbf{I}a \leqslant \mathbf{I}'a$, which proves that $\mathbf{I}a = \mathbf{I}'a$.

§ 5. Strongly compact spaces. A topological space is said to be *strongly compact* provided, for every indexed set $\{G_t\}_{t \in T}$ of open subsets of X, if X is the union of all G_t, then there exists an index t_0 such that $X = G_{t_0}$. By passing to complements, we see that X is strongly compact if the intersection of any class of non-empty closed sets is not empty.

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(1) The first part of theorem 4.3 was proved by McKinsey and Tarski [1]; the second is due to Rasiowa and Sikorski [3].

(2) McKinsey and Tarski [1].

Thus, if X is strongly compact, then the intersection F_0 of all non-empty closed subsets of X is not empty. F_0 is the least non-empty closed subset of X, i.e. $F_0 \subset F$ for every non-empty closed set $F \subset X$. The complement $G_0 = X - F_0$ is the greatest open proper subset of X, i.e. $G \subset G_0$ for every open subset $G \neq X$. Conversely, if the class of all non-empty closed subsets contains the least set (or equivalently: if the class of all proper open subsets contains the greatest set), then X is strongly compact.

5.1. *Every topological space X is an open subspace of a strongly compact space X_0 such that $X_0 - X$ is a one-point set, and the class of all open subsets of X_0 is composed of X_0 and all open subsets of X.*

Let x_0 be any element which does not belong to X, and let $X_0 = X \cup (x_0)$. By I, 2.1, there exists an interior operation in X_0 such that the class $\boldsymbol{B}_0$ of all open subsets in X is a subbasis in X_0. By definition, the class of all open subsets of the space X_0 is composed of the set X_0 and all open subsets of X. In particular, X is an open subset of X_0. It is easy to verify that X is a topological subspace of X_0 (see I, § 2, (10), and (11)). Since X_0 is the only open set (in X_0) which contains the point x_0, the space X_0 is strongly compact. Namely, X is the greatest open proper subset of X_0, and (x_0) is the least non-empty closed subset of X_0.

The space X_0 constructed in the proof of 5.1 will be called the *one-point strong compactification* of the space X.

§ 6. Metric spaces. By a *metric space* we understand a set X with a real function $\varrho(x, y)$ defined for all $x, y \in X$ and such that

(m_1) $\quad 0 \leqslant \varrho(x, y) < \infty; \quad \varrho(x, y) = 0$ if and only if $x = y$;

(m_2) $\quad \varrho(x, y) = \varrho(y, x)$;

(m_3) $\quad \varrho(x, y) \leqslant \varrho(x, z) + \varrho(z, y)$.

More exactly, a metric space is a pair $\{X, \varrho\}$ where X is a set and ϱ is a function (called *metric*) satisfying the above conditions. The number $\varrho(x, y)$ is called the *distance* between x and y.

For every subset A of a metric space X, let

$$\varrho(x, A) = \infty \quad \text{if} \quad A = 0$$

and let

$$\varrho(x, A) = \text{l.u.b. } \varrho(x, y) \quad \text{where} \quad y \in A$$

provided $A \neq 0$. By definition, $0 \leqslant \varrho(x, A) < \infty$ for $A \neq 0$, and $\varrho(x, A) = 0$ if $x \in A$.

It follows immediately from the definition that

$$\varrho(x, A \cup B) = \min\big(\varrho(x, A), \varrho(x, B)\big), \tag{1}$$

and

$$\text{(2)} \qquad \text{if} \quad A \subset B, \quad \text{then} \quad \varrho(x, A) \geqslant \varrho(x, B) .$$

If $A \neq 0$, then for every $y \in A$ and $x, x' \in X$

$$\varrho(x, A) \leqslant \varrho(x, y) \leqslant \varrho(x, x') + \varrho(x', y) .$$

Hence

$$\varrho(x, A) \leqslant \varrho(x, x') + \varrho(x', A) .$$

Substituting x', x for x, x', we obtain

$$\varrho(x', A) \leqslant \varrho(x', x) + \varrho(x, A) .$$

Hence, by (m_2),

$$\text{(3)} \qquad |\varrho(x, A) - \varrho(x', A)| \leqslant \varrho(x, x') \qquad (A \neq 0) .$$

A sequence of points $x_n \in X$ $(n = 1, 2, \ldots)$ is said to *converge* to a point $x \in X$ (or: x is said to be the *limit* of $\{x_n\}$) provided

$$\lim_{n \to \infty} \varrho(x_n, x) = 0 .$$

We then write

$$x = \lim_{n \to \infty} x_n \qquad \text{or} \qquad x_n \to x .$$

For instance, if $x_n = x$ for every n, then $x_n \to x$.

Observe that every sequence $\{x_n\}$ has at most one limit. Indeed, if $x_n \to x$ and $x_n \to y$, then by (m_2) and (m_3)

$$0 \leqslant \varrho(x, y) \leqslant \varrho(x_n, x) + \varrho(x_n, y) \to 0$$

which proves $\varrho(x, y) = 0$, i.e. $x = y$ on account of (m_1).

If $\{x_{m_n}\}$ is a subsequence of $\{x_n\}$, and $x_n \to x$, then also $x_{m_n} \to x$.

The inequality (3) implies that

$$\text{(4)} \qquad \text{if} \quad x_n \to x, \quad \text{then} \quad \varrho(x_n, A) \to \varrho(x, A) .$$

Every metric space X will be considered to be a topological space with the closure operation defined as follows:

$\mathbf{C}A$ is the set of all x such that $\varrho(x, A) = 0$, i.e. such that there exists a sequence $x_n \to x$, $x_n \in A$ for $n = 1, 2, \ldots$ In other words, $\mathbf{C}A$ is the set of all limits of sequences of points in A.

It follows directly from the definition that $\mathbf{C}$ satisfies the axioms (c_2) and (c_4). The axiom (c_1) follows from (1). The axiom (c_3) follows from the equation

$$\varrho(x, A) = \varrho(x, \mathbf{C}A)$$

which is obvious for $A = 0$ and can be proved as follows for $A \neq 0$. Let $\varepsilon > 0$ be arbitrary, let $z \in \mathbf{C}A$ be such that

$$\varrho(x, z) \leqslant \varrho(x, \mathbf{C}A) + \varepsilon$$

and let $y \in A$ be such that $\varrho(z, y) < \varepsilon$. Then, by (m_3),

$$\varrho(x, A) \leqslant \varrho(x, y) \leqslant \varrho(x, \mathbf{C}A) + 2\varepsilon ,$$

which, for $\varepsilon \to 0$, gives $\varrho(x, A) \leqslant \varrho(x, \mathbf{C}A)$. The inverse inequality follows from (2).

Passing to complements in the definition of the closure operation, we infer that, for every set $A \subset X$, $\mathbf{I}A$ is the set of all $x \in X$ such that $\varrho(x, X-A) > 0$, i.e. such that $\inf_n \varrho(x_n, x) > 0$ for every sequence $\{x_n\}$ of points in $X-A$. Hence, if A is open, $x \in A$ and $x_n \to x$, then $x_n \in A$ for n sufficiently great. If A is closed and $x \notin A$, then $\varrho(x, A) > 0$.

Denote by $\mathbf{K}(x_0, \varepsilon)$ (by $\overline{\mathbf{K}}(x_0, \varepsilon)$) the set of all $x \in X$ such that $\varrho(x, x_0) < \varepsilon$ (such that $\varrho(x, x_0) \leqslant \varepsilon$).

It follows from the above consideration that $x_0 \in \mathbf{I}A$ if and only if there exists an $\varepsilon > 0$ such that $\mathbf{K}(x_0, \varepsilon) \subset A$. Thus a set A is open if and only if, for every $x_0 \in A$, there exists an $\varepsilon > 0$ such that $\mathbf{K}(x_0, \varepsilon) \subset A$. In the last two statements $\mathbf{K}(x_0, \varepsilon)$ can be replaced by $\overline{\mathbf{K}}(x_0, \varepsilon)$.

Assuming a one-point set (x_0) in (4) as the set A, we infer that

$$\text{if} \quad x_n \to x, \quad \text{then} \quad \varrho(x_n, x_0) \to \varrho(x, x_0) . \tag{5}$$

Hence it follows that the set $\mathbf{K}(x_0, \varepsilon)$ is open and the set $\overline{\mathbf{K}}(x_0, \varepsilon)$ is closed. Since $\mathbf{K}(x_0, \varepsilon) \subset \overline{\mathbf{K}}(x_0, \varepsilon)$ we have (see § 1, (4))

$$\mathbf{C}\mathbf{K}(x_0, \varepsilon) \subset \overline{\mathbf{K}}(x_0, \varepsilon) . \tag{6}$$

It follows immediately from the definition of $\overline{\mathbf{K}}(x_0, \varepsilon)$ and (m_2), (m_3) that

$$\varrho(x, y) \leqslant 2\varepsilon \quad \text{for all point} \quad x, y \in \overline{\mathbf{K}}(x_0, \varepsilon) . \tag{7}$$

The simplest example of a metric space is given by the set X of all real numbers, the distance being defined as usual: $\varrho(x, y) = |x-y|$. The sets $\mathbf{K}(x_0, \varepsilon)$ are then open intervals. It follows easily from the above considerations that the interior operation determined in X by the metric coincides with the interior operation defined in X in I, § 2, p. 15.

Every metric space is a T_1-space, i.e. every one-point set (y) (and consequently every finite subset) is closed. In fact, if $x \neq y$, then $\varrho(x, (y)) = \varrho(x, y) > 0$ by (m_1), which proves $\mathbf{C}(y) = (y)$.

Every metric space is normal (and, consequently, Hausdorff and regular—see I, § 2). This follows immediately from the following theorem:

6.1. *If $F_1, \ldots, F_m$ are disjoint closed subsets of a metric space X, then there exist open sets $G_1, \ldots, G_m$ such that the sets $\mathbf{C}G_1, \ldots, \mathbf{C}G_m$ are disjoint, and $F_i \subset \mathbf{C}G_i$ for $i = 1, \ldots, m$.*

It suffices to prove 6.1 in the case for which all the sets F_i are non-empty, i.e. $\varrho(x, F_i) < \infty$. The set of all $x \in X$ such that

$$\varrho(x, F_i) \geqslant \tfrac{1}{2} \min_{j \neq i} \big(\varrho(x, F_j)\big)$$

is closed since if $x_n \to x$ and all x_n satisfy this inequality, so does x by (4). Consequently its complement G_i, i.e. the set G_i of all $x \in X$ such that

$$\varrho(x, F_i) < \tfrac{1}{2} \min_{j \neq i} \big(\varrho(x, F_j)\big)$$

is open. F_i is a subset of G_i since if $x \in F_i$, then the left-hand side of this inequality equals 0, and the right-hand side is positive because $x \notin F_j$ for $j \neq i$. It follows from (4) that the closure $\mathbf{C}G_i$ of G_i is contained in the set A_i of all $x \in X$ such that

$$\varrho(x, F_i) \leqslant \tfrac{1}{2} \min_{j \neq i} \big(\varrho(x, F_j)\big) .$$

The sets $A_1, \ldots, A_m$ are disjoint. In fact, if $x \in A_i$ and $x \in A_j$ for $j \neq i$, then

$$\varrho(x, F_i) \leqslant \tfrac{1}{2} \varrho(x, F_j)$$

and, by symmetry,

$$\varrho(x, F_j) \leqslant \tfrac{1}{2} \varrho(x, F_i) .$$

This implies $\varrho(x, F_i) = 0 = \varrho(x, F_j)$, i.e. $x \in \mathbf{C}F_i \cap \mathbf{C}F_j = F_i \cap F_j$. Contradiction!

Every subset G of a metric space X is also a metric space with the same metric. The closure operation $\mathbf{C}_G$ determined in G by the metric in the manner described in this section satisfies the equation

$$\mathbf{C}_G A = G \cap \mathbf{C}A \quad \text{for} \quad A \subset G ,$$

i.e. it coincides with the closure operation $\mathbf{C}_G$ defined in § 2, (4).

§ 7. A fundamental lemma on metric spaces. We start with remarks on topological spaces which supplement I, § 2.

If G is an open subset of a topological space X, and A is a boundary subset of X, then

$$\mathbf{C}(G - A) = \mathbf{C}G . \tag{1}$$

Indeed, $\mathbf{I}(G \cap A) = \mathbf{I}G \cap \mathbf{I}A = 0$. Passing to the complements we infer that $\mathbf{C} - (G \cap A) = X$, i.e. $\mathbf{C}\big(-G \cup (G - A)\big) = X$. Since $-G$ is closed, we obtain $-G \cup \mathbf{C}(G - A) = X$, which implies $G \subset \mathbf{C}(G - A)$ and con-

sequently $\mathbf{C}G \subset \mathbf{C}(G-A)$. The converse inclusion $\mathbf{C}(G-A) \subset \mathbf{C}G$ being also true (see I, § 2, (2)), we obtain (1).

For all subsets A, B of a topological space X,

$$\mathbf{C}A - \mathbf{C}B \subset \mathbf{C}(A-\mathbf{C}B)\,. \tag{2}$$

Indeed, $\mathbf{C}A \subset \mathbf{C}\big((A-\mathbf{C}B) \cup \mathbf{C}B\big) = \mathbf{C}(A-\mathbf{C}B) \cup \mathbf{C}\mathbf{C}B = \mathbf{C}(A-\mathbf{C}B) \cup \mathbf{C}B$, which implies (2).

A subset H of a topological space X is said to be *dense-in-itself* provided

$$x \in \mathbf{C}\big(H-(x)\big) \quad \text{for every} \quad x \in H\,. \tag{3}$$

In particular, the empty set is dense-in-itself. The whole space X is dense-in-itself if and only if no one-point set is open, i.e. if every one-point set is a boundary set. Thus every one-point subset (and, consequently, every finite subset) of a dense-in-itself T_1-space (in particular, of a dense-in-itself metric space) is nowhere dense. Every dense-in-itself non-empty T_1-space is infinite (since the complement of every finite subset is dense and therefore is not empty).

Every open subset G of a dense-in-itself topological space X is dense-in-itself. Indeed, for every $x \in G$ we have $x \in \mathbf{C}\big(X-(x)\big) = \mathbf{C}(X-G) \cup \mathbf{C}\big(G-(x)\big) = (X-G) \cup \mathbf{C}\big(G-(x)\big)$. Since $x \notin X-G$, we infer that $x \in \mathbf{C}\big(G-(x)\big)$.

A subset H of a metric space is dense-in-itself if and only if every $x \in H$ is the limit of a sequence $x_n \in H$, $x_n \neq x$ for $n = 1, 2, \ldots$

Let $\varepsilon > 0$. A subset A of a metric space X is said to be an *ε-set* provided $\varrho(x, y) \geqslant \varepsilon$ for any pair of distinct points $x, y \in A$.

Every ε-set A is closed. Indeed, suppose that $x_n \to x$, $x_n \in A$ for $n = 1, 2, \ldots$ Then, for $n \geqslant n_0$,

$$\varrho(x_n, x) < \tfrac{1}{2}\varepsilon\,,$$

and consequently

$$\varrho(x_n, x_{n_0}) \leqslant \varrho(x_n, x) + \varrho(x_{n^0}, x) < \varepsilon\,,$$

which proves that $x_n = x_{n_0}$ for $n \geqslant n_0$. Hence $x = \lim_{n\to\infty} x_n = x_{n_0} \in A$.

Every ε-set A in a dense-in-itself metric space X is nowhere dense. Indeed, let $x \in A$. There exists a sequence $x_n \in X$ such that $x_n \to x$ and $x_n \neq x$ for all n. For $n \geqslant n_0$ we have $\varrho(x_n, x) < \varepsilon$. Thus $x_n \in X-A$ for $n \geqslant n_0$, which implies that $x \in \mathbf{C}(X-A)$. This proves $A \subset \mathbf{C}(X-A)$. Since also $X-A \subset \mathbf{C}(X-A)$ (see (c_2)), we have $X = \mathbf{C}(X-A)$, i.e. the complement of A is dense. Since A is closed, A is nowhere dense.

A subset A of a set B (in a metric space X) is said to be a *maximal ε-subset* of B provided A is an ε-set and A is not a proper subset of any ε-set contained in B; in other words: if A is a maximal element of the ordered (with respect to the set-theoretical inclusion) set $\boldsymbol{K}$ of all ε-subsets of B. The set $\boldsymbol{K}$ satisfies the condition of I, 5.1 (the union of any chain of ε-sets is an ε-set!). Therefore $\boldsymbol{K}$ contains a maximal element, i.e. every set B contains a maximal ε-subset A.

If A is a maximal ε-subset of B, then

$$\varrho(x, A) \leqslant \varepsilon \quad \text{for every} \quad x \in \mathbf{C}B\,. \tag{4}$$

For if $\varrho(x, A) > \varepsilon$ for an $x \in B$, then $A \subset A \cup (x) \neq A$ and $A \cup (x)$ is an ε-subset of B, which contradicts the hypothesis that A is a maximal ε-subset. This proves inequality (4) for $x \in B$. Applying § 6, (4), we obtain (4) for every $x \in \mathbf{C}B$.

If G is an open subset of a dense-in-itself metric space X, then for every $\varepsilon > 0$ and for every positive integer m there exist disjoint closed nowhere dense sets $A_1, \ldots, A_m$ such that

$$\varrho(x, A_i) \leqslant \varepsilon \quad \text{for every} \quad x \in \mathbf{C}G \quad (i = 1, \ldots, m)\,. \tag{5}$$

Indeed, let A_1 be a maximal ε-subset of G and, by induction, let A_i be a maximal ε-subset of $G-(A_1 \cup \ldots \cup A_{i-1})$ $(1 \leqslant i \leqslant m)$. Then, by (4), the inequality (5) holds for $x \in \mathbf{C}(G-(A_1 \cup \ldots \cup A_{i-1}))$. By equation (1), $\mathbf{C}(G-(A_1 \cup \ldots \cup A_{i-1}) = \mathbf{C}G$ since the union of nowhere dense sets $A_1, \ldots, A_{i-1}$ is nowhere dense.

We shall apply the above remarks to the proof of the following lemma which will play a fundamental part in the proofs of some representation theorems for topological Boolean algebras in the next section.

7.1 [1]. *Let G be an open non-empty subset of a dense-in-itself metric space X and let r, s be non-negative integers. Then there exist disjoint non-empty sets $G_1, \ldots, G_r$, $B_0, B_1, \ldots, B_s$ such that*

$$G = G_1 \cup \ldots \cup G_r \cup B_0 \cup B_1 \cup \ldots \cup B_s\,; \tag{6}$$

$$G_1, \ldots, G_r \quad \textit{are open}\,; \tag{7}$$

$$\mathbf{C}G_i - G_i = \mathbf{C}B_j = \mathbf{C}G - (G_1 \cup \ldots \cup G_r) \quad \textit{for} \quad i = 1, \ldots, r \quad \textit{and} \quad j = 0, 1, \ldots, s\,. \tag{8}$$

By induction on n we shall define sequences of sets

$$\{G_{i,n}\}\ (i = 1, \ldots, r) \quad \text{and} \quad \{B_{j,n}\}\ (j = 0, 1, \ldots, s)$$

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[1] Tarski [8], McKinsey and Tarski [1].

such that

(9) the sets $G_{i,n}$ are open; the sets $B_{j,n}$ are closed and nowhere dense;

(10) for every fixed n, all the sets $X-G$, $\mathbf{C}G_{i,n}$ $(i=1,\dots,r)$, $B_{j,n}$ $(j=0,1,\dots,s)$ are disjoint;

$$\text{(11)} \qquad \mathbf{C}G_{i,n} \subset G_{i,n+1}, \qquad B_{j,n} \subset B_{j,n+1};$$

(12) for every $x \in \mathbf{C}G-(G_{1,n} \cup \dots \cup G_{r,n})$

$$\varrho(x, G_{i,n}) \leqslant 1/n \ \ (i=1,\dots,r) \quad \text{and} \quad \varrho(x, B_{j,n}) \leqslant 1/n \ \ (j=0,1,\dots,s);$$

$$\text{(13)} \qquad G_{i,1} \neq 0 \neq B_{j,1} \qquad (i=1,\dots,r,\ j=0,1,\dots,s).$$

Namely, let

$$\text{(14)} \qquad G_{i,0} = 0 = B_{j,0}.$$

Suppose that the sets $G_{i,n}$, $B_{j,n}$ are defined so that (9) and (10) hold $(n \geqslant 0)$. By (5), there exist disjoint closed nowhere dense subsets $A_1, \dots, A_{r+s}$ of the open set $X_n = G-(\mathbf{C}G_{1,n} \cup \dots \cup \mathbf{C}G_{r,n} \cup B_{0,n} \cup \dots \cup B_{s,n})$ such that

$$\text{(16)} \qquad \varrho(x, A_k) \leqslant 1/(n+1) \quad \text{for every} \quad x \in \mathbf{C}X_n \qquad (k=0,\dots,r+s).$$

The sets

$$X-G, \qquad B_{j,n+1} = B_{j,n} \cup A_j \quad (j=0,1,\dots,s),$$
$$\mathbf{C}G_{i,n} \cup A_{s+1} \quad (i=1,\dots,r)$$

being closed and disjoint, there exist open sets $G_{i,n+1}$ such that

$$\text{(17)} \qquad \mathbf{C}G_{i,n} \subset G_{i,n+1} \qquad (i=1,\dots,r)$$

and the sets

$$X-G, \qquad \mathbf{C}G_{1,n+1}, \quad \dots, \quad \mathbf{C}G_{r,n+1}, \qquad B_{0,n+1} \cup \dots \cup B_{s,n+1}$$

are disjoint (see 6.1). Thus (9) and (10) are satisfied for $n+1$. By (17), (1) and (2) we have

$$\mathbf{C}G-(G_{1,n+1} \cup \dots \cup G_{r,n+1}) \subset \mathbf{C}G-\mathbf{C}(G_{1,n} \cup \dots \cup G_{i,n}) \subset \mathbf{C}X_n.$$

This proves (see (16)) that (12) holds for $n+1$.

It follows immediately from the above definition (the case $n=0$) that (13) is true (see (14)). Therefore the sets

$$G_i = \bigcup_{n=1}^{\infty} G_{i,n} \qquad (i=1,\dots,r)$$
$$B_j = \bigcup_{n=1}^{\infty} B_{j,n} \qquad (j=1,\dots,s)$$
$$B_0 = G-(G_1 \cup \dots \cup G_r \cup B_1 \cup \dots \cup B_s) \supset \bigcup_{n=1}^{\infty} B_{0,n}$$

are not empty. By (10) and (11), they are disjoint. It follows immediately from the definition that they have properties (6) and (7).

It follows from (12) that

$$\varrho(x, G_i) = 0 = \varrho(x, B_j) \qquad (i = 1, \ldots, r,\ j = 0, 1, \ldots, s) \tag{18}$$

for every point x in the set

$$B = \mathbf{C}G - (G_1 \cup \ldots \cup G_r) = \bigcap_{n=1}^{\infty} \left(\mathbf{C}G - (G_{1,n} \cup \ldots \cup G_{r,n})\right).$$

It follows from (18) that $B \subset \mathbf{C}G_i$. Since B is disjoint with G_i, we obtain $B \subset \mathbf{C}G_i - G_i$. On the other hand, $G_i \subset \mathbf{C}G - (G_1 \cup \ldots \cup G_{i-1} \cup G_{i+1} \cup \ldots \cup G_r)$. The set on the right-hand side of this inclusion being closed, we infer that $\mathbf{C}G_i$ also is its subset, which proves that $\mathbf{C}G_i - G_i \subset B$. Consequently

$$\mathbf{C}G_i - G_i = B \qquad (i = 1, \ldots, r).$$

It follows from (18) that $B \subset \mathbf{C}B_j$. Since $B_j \subset B$ and B is closed, we also have $\mathbf{C}B_j \subset B$, which proves

$$\mathbf{C}B_j = B \qquad (j = 0, 1, \ldots, s)$$

and completes the proof of (8).

§ 8. Finite topological Boolean algebras. Every finite Boolean algebra $\boldsymbol{A}$ is atomic. Indeed, let $a_0 \neq \Lambda$, $a_0 \in \boldsymbol{A}$. The set of all $a \subset a_0$, $a \neq \Lambda$, is finite, therefore it contains a minimal (with respect to the Boolean ordering $\leqslant$) element b. The element b is an atom contained in a_0.

Every finite Boolean algebra $\boldsymbol{A}$ is isomorphic to the field of all subsets of a finite set X. Indeed, let X be the set of all atoms of $\boldsymbol{A}$ and, for every $a \in \boldsymbol{A}$, let $h(a)$ be the set of all atoms which are included in a. It is easy to verify that h is a Boolean isomorphism of $\boldsymbol{A}$ onto the field of all subsets of X.

Every finite topological Boolean algebra is isomorphic (i.e. homeomorphic) with the topological field $\mathfrak{B}(X)$ of all subsets of a finite topological space X. In fact, let h and X have meanings as above. We make X a topological space with the interior operation induced by the Boolean isomorphism h, i.e.

$$\mathbf{I}h(a) = h(\mathbf{I}a) \qquad \text{for} \qquad a \in \boldsymbol{A}.$$

By definition, h is a topological isomorphism of $\boldsymbol{A}$ onto $\mathfrak{B}(X)$.

On the other hand, every finite topological space yields an example of a finite topological Boolean algebra $\mathfrak{B}(X)$. Thus finite topological Boolean algebras can be identified with finite topological spaces. However, these finite topological spaces are rather irregular; in general they are neither T_1-spaces nor T_0-spaces. From the point of view of applications given later in this book, it seems more important to represent

finite topological Boolean algebras as topological fields of subsets of more natural topological spaces (e.g. metric spaces). This representation is given by theorem 8.3 below.

A finite topological Boolean algebra $\boldsymbol{A}$ is said to be *strongly compact* if, considered as a finite topological space, it is strongly compact in the sense defined in § 5, i.e. if the class of all non-zero closed elements in $\boldsymbol{A}$ has the least element.

8.1. *For every non-degenerate finite topological Boolean algebra* $\boldsymbol{A}$, *there exists a decomposition*

$$V = g_1 \cup \ldots \cup g_r \qquad (r > 0) \tag{1}$$

such that $g_1, \ldots, g_r$ *are open non-zero elements and all the topological Boolean algebras* $\boldsymbol{A}_{g_1}, \ldots, \boldsymbol{A}_{g_r}$ (see § 2) *are strongly compact.*

Let $b_1, \ldots, b_r$ be all the minimal elements of the ordered (by the Boolean relation $\leqslant$) set of all non-zero closed elements in $\boldsymbol{A}$. It follows from the definition that, for every closed element b in $\boldsymbol{A}$ and for every integer i $(1 \leqslant i \leqslant r)$,

$$\text{either} \quad b_i \leqslant b \quad \text{or} \quad b \cap b_i = \Lambda$$

for if not, $b \cap b_i$ would be a non-zero element less than $b_i \neq b \cap b_i$. Moreover, for every closed non-zero element $b \in \boldsymbol{A}$ there exists an integer i such that $b_i \leqslant b$.

Let a_i be the join of all closed elements b such that $b \cap b_i = \Lambda$, and let $g_i = -a_i$ $(1 \leqslant i \leqslant r)$. By definition, g_i is open and $b_i \subset g_i$. Every element closed in $\boldsymbol{A}_{g_i}$ is of the form $g_i \cap b$ where b is closed in $\boldsymbol{A}$ (see § 2, (3)). If $g_i \cap b \neq \Lambda$, then $b \cap b_i \neq \Lambda$ by definition of a_i and g_i. Hence $b_i \leqslant b$ and, consequently, $b_i \leqslant g_i \cap b$, which proves that b_i is the least closed element in $\boldsymbol{A}_{g_i}$. Thus $\boldsymbol{A}_{g_i}$ is strongly compact.

We shall now prove (1). Suppose that (1) is not true, i.e. the closed element $b = -(g_1 \cup \ldots \cup g_r)$ is not equal to Λ. Then there exists an integer i such that $b_i \leqslant b$. On the other hand, $b_i \leqslant g_i$. Contradiction!

8.2 [1]. *Let* G *be an open non-empty subset of a dense-in-itself metric space* X. *For every non-degenerate finite strongly compact topological Boolean algebra* $\boldsymbol{A}$ *there exists a topological isomorphism* h *of* $\boldsymbol{A}$ *into* $\mathfrak{B}(G)$ *such that*

$$\mathbf{C}G - G \subset \mathbf{C}h(a) \quad \text{for every} \quad a \neq \Lambda, \quad a \in \boldsymbol{A}. \tag{2}$$

The proof is by induction on the number of atoms of $\boldsymbol{A}$.

If $\boldsymbol{A}$ has only one atom, i.e. $\boldsymbol{A}$ is a two-element Boolean algebra, then the mapping

$$h(\Lambda) = 0, \qquad h(V) = G$$

is the required topological isomorphism.

[1] McKinsey and Tarski [1].

Suppose that A has k atoms ($k > 1$) and that the theorem is true for topological Boolean algebras with less than k atoms. Let b be the least closed non-zero element in A, and let $g = -b$ be the greatest open element distinct from V. By 8.1 there exist elements $g_1, \ldots, g_r$ ($r \geqslant 0$) open in A_g (and, consequently, in A too—see § 2) such that

$$g = g_1 \cup \ldots \cup g_r$$

and all topological Boolean algebras $A_{g_1}, \ldots, A_{g_r}$ are strongly compact (if $g = \Lambda$, we assume $r = 0$). Let $b_0, b_1, \ldots, b_s$ ($s \geqslant 0$) be all the atoms contained in b.

Let $G_1, \ldots, G_r$, $B_0, B_1, \ldots, B_s$ be sets satisfying all the conditions of lemma 7.1. By the induction hypothesis, there exists a topological isomorphism h_i of A_{g_i} into $\mathfrak{B}(G_i)$ such that

$$\mathbf{C}G_i - G_i \subset \mathbf{C}h_i(a) \quad \text{for all} \quad a \neq \Lambda, \quad a \in A_{g_i} \quad (1 \leqslant i \leqslant r). \tag{3}$$

Let

$$h_0(a) = \bigcup_{b_j \leqslant a} B_j \quad \text{for} \quad a \leqslant b. \tag{4}$$

The Boolean isomorphism h (of A into $\mathfrak{B}(G)$) defined by the equation

$$h(a) = h_0(b \cap a) \cup h_1(g_1 \cap a) \cup \ldots \cup h_r(g_r \cap a) \tag{5}$$

is a topological isomorphism satisfying condition (2).

To prove this, let us observe first that if $a \neq V$, then also $\mathbf{I}a \neq V$, and consequently $\mathbf{I}a \subset g$ since g is the greatest open element $\neq V$. Thus

$$\mathbf{I}a = g \cap \mathbf{I}a \quad \text{for every} \quad a \in A, \quad a \neq V. \tag{6}$$

Let $B = \mathbf{C}G - (G_1 \cup \ldots \cup G_r)$. If $a \neq \Lambda$, then

either $b \cap a \neq \Lambda$ or $g_i \cap a \neq \Lambda$ for an integer i $(1 \leqslant i \leqslant r)$.

In the first case, there exists an integer j $(0 \leqslant j \leqslant s)$ such that $b_j \leqslant b \cap a$ and consequently $B_j \subset h_0(b \cap a)$. Then $\mathbf{C}h_0(b \cap a) \supset \mathbf{C}B_j = B$ (see 7.1, (8)). Hence $B \subset \mathbf{C}h(a)$.

In the second case, $B = \mathbf{C}G_i - G_i \subset \mathbf{C}h_i(g_i \cap a)$ (see (3) and 7.1, (8)), which implies $B \subset \mathbf{C}h(a)$.

Thus

$$\mathbf{C}G - G \subset \mathbf{C}G - (G_1 \cup \ldots \cup G_r) \subset \mathbf{C}h(a) \quad \text{for every} \quad a \neq \Lambda, \quad a \in A. \tag{7}$$

This proves (2).

Replacing the element a by $-a$ in (7) we obtain, for every $a \neq V$ ($a \in A$),

$$\begin{aligned} \mathbf{I}h(a) = G \cap \mathbf{I}h(a) &= G \cap -\mathbf{C}h(-a) \subset G \cap -\big(G - (G_1 \cup \ldots \cup G_r)\big) \\ &= G_1 \cup \ldots \cup G_r. \end{aligned}$$

This implies that

$$\mathbf{I}h(a) = (G_1 \cup \ldots \cup G_r) \cap \mathbf{I}h(a) = \big(\mathbf{I}G_1 \cap \mathbf{I}h(a)\big) \cup \ldots \cup \big(\mathbf{I}G_r \cap \mathbf{I}h(a)\big)$$
$$= \mathbf{I}\big(G_1 \cap h(a)\big) \cup \ldots \cup \mathbf{I}\big(G_r \cap h(a)\big) = \mathbf{I}\big(h_1(g_1 \cap a)\big) \cup \ldots \cup \mathbf{I}h_r(g_r \cap a)$$

for every $a \neq V$, $a \in A$.

Hence, for every $a \in A$, $a \neq V$,

$$h(\mathbf{I}a) = h(g \cap \mathbf{I}a) = h_1(g_1 \cap \mathbf{I}a) \cup \ldots \cup h_r(g_r \cap \mathbf{I}a)$$
$$= h_1\big(\mathbf{I}_{g_1}(g_1 \cap a)\big) \cup \ldots \cup h_r\big(\mathbf{I}_{g_r}(g_r \cap a)\big)$$
$$= \mathbf{I}_{G_1}h_1(g_1 \cap a) \cup \ldots \cup \mathbf{I}_{G_r}h_r(g_r \cap a)$$
$$= \mathbf{I}h_1(g_1 \cap a) \cup \ldots \cup \mathbf{I}h_r(g_r \cap a) = \mathbf{I}h(a) = \mathbf{I}_G h(a)$$

because of (6) and § 2, (2'), (4'). Since also $h(\mathbf{I}V) = h(V) = G = \mathbf{I}_G h(V)$, h is a topological isomorphism.

8.3. *Let $X \neq 0$ be a dense-in-itself metric space. For every non-degenerate finite topological Boolean algebra A there exists an open dense set $G \subset X$ such that A is topologically isomorphic to a topological subalgebra of $\mathfrak{B}(G)$.*

If, in addition, X is totally disconnected, then we can assume that $G = X$, i.e. A is then isomorphic to a topological subalgebra of $\mathfrak{B}(X)$ (1).

By theorem 8.1,

$$V = g_1 \cup \ldots \cup g_r \qquad (r > 0)$$

where $g_1, \ldots, g_r$ are open non-zero elements and $A_{g_1}, \ldots, A_{g_r}$ are strongly compact.

Since the space X is infinite, there exist r open non-empty disjoint subsets $G_1, \ldots, G_r$ of X such that the open set

$$G = G_1 \cup \ldots \cup G_r$$

is dense in X. For instance, take r distinct points $x_1, \ldots, x_r$ in X. Applying theorem 6.1 to one-point closed sets $F_i = (x_i)$, $i = 1, \ldots, r$, we infer that there exist r disjoint open sets $H_1, \ldots, H_r$ such that $x_i \in H_i$ for $i = 1, \ldots, r$. The sets $G_1 = H_1, \ldots, G_{r-1} = H_{r-1}$, $G_r = X - \mathbf{C}(H_1 \cup \ldots \cup H_{r-1})$ have the required property.

When X is totally disconnected, we may assume $G = X$, since, for every positive integer r, any infinite totally disconnected space is the union of r disjoint, non-empty, both open and closed sets (proof by induction on r).

---

(1) McKinsey and Tarski [1].

By 8.2, there exists a topological isomorphism h_i of A_{g_i} into $\mathfrak{B}(G_i)$ $(i = 1, \ldots, r)$. The Boolean isomorphism

$$h(a) = h_1(g_1 \cap a) \cup \ldots \cup h_r(g_r \cap a)$$

of A into $\mathfrak{B}(G)$ is topological. Indeed,

$$\begin{aligned} h(\mathbf{I}a) &= h_1(g_1 \cap \mathbf{I}a) \cup \ldots \cup h_r(g_r \cap \mathbf{I}a) = h_1\big(\mathbf{I}_{g_1}(g_1 \cap a)\big) \cup \ldots \cup h_r\big(\mathbf{I}_{g_r}(g_r \cap a)\big) \\ &= \mathbf{I}_{G_1} h_1(g_1 \cap a) \cup \ldots \cup \mathbf{I}_{G_r} h_r(g_r \cap a) = \mathbf{I}\big(G_1 \cap h(a)\big) \cup \ldots \cup \mathbf{I}\big(G_r \cap h(a)\big) \\ &= \big(G_1 \cap \mathbf{I}h(a)\big) \cup \ldots \cup \big(G_r \cap \mathbf{I}h(a)\big) = G \cap \mathbf{I}h(a) = \mathbf{I}h(a) = \mathbf{I}_G h(a) \end{aligned}$$

because of § 2, (2′) and (4′).

If A is a degenerate topological algebra and X is any topological space, then A is isomorphic to $\mathfrak{B}(G)$ for an open set $G \subset X$, viz. for the empty set $G = 0$.

§ 9. Cartesian products of topological spaces. Any Cartesian product $X_0 = P_{t \in T} X_t$ of topological spaces X_t will always be considered as a topological space with the interior operation $\mathbf{I}$ determined, by means of theorem I, 2.1, by the following class B_0 of sets:

$$P_{t \in T} G_t \tag{1}$$

where $G_t = X_t$ for all $t \in T$ except a finite number of indices $t_1, \ldots, t_m$, and G_{t_i} is an open subset of X_{t_i} for $i = 1, \ldots, m$.

The class $\boldsymbol{B}_0$ is a basis of the topological space X_0 since the empty set and the whole space X_0 are of the form (1) and the product of two sets (1) is also of the form (1).

Observe that if a fixed basis $\boldsymbol{B}_t$ is chosen in every space X_t, then the class of all sets (1), where $G_{t_i} \in \boldsymbol{B}_{t_i}$, is also a basis of X_0. If we suppose only that the $\boldsymbol{B}_t$ are subbases in X_t respectively, then the class of all sets (1) with G_{t_i} in $\boldsymbol{B}_{ti}$ form a subbasis of X_0.

The mapping which, to every point $x = \{x_t\} \in X_0$, associates its t_0^{th} coordinate x_{t_0} (t_0 being fixed) will be called the *projection* onto the t_0^{th} axis and denoted by π_{t_0}. By definition

$$\pi_{t_0}(x) = x_{t_0} \quad \text{for} \quad x = \{x_t\}\,.$$

The mapping π_{t_0} is an interior mapping, i.e.

$$\pi_{t_0}^{-1}(\mathbf{I}A) = \mathbf{I}\pi_{t_0}^{-1}(A) \quad \text{for every set} \quad A \subset X_0\,.$$

Indeed, let $G = \mathbf{I}A$. The set $\pi_{t_0}^{-1}(G)$ is open in X_0 since it is of the form (1). Consequently $\pi_{t_0}^{-1}(G) \subset \mathbf{I}\pi_{t_0}^{-1}(A)$. On the other hand, if a non-empty set (1) is contained in $\pi_{t_0}^{-1}(A)$, then $G_{t_0} \subset A$. Consequently $G_{t_0} \subset G$, which implies that the set (1) is contained in $\pi_{t_0}^{-1}(G)$. This proves that $\mathbf{I}\pi_{t_0}^{-1}(A) \subset \pi_{t_0}^{-1}(G)$.

Observe that, in terms of the projections π_t, the open sets (1) can be written in the form

$$\pi_{t_1}^{-1}(G_1) \cap \ldots \cap \pi_{t_n}^{-1}(G_n) \tag{2}$$

where $t_1, \ldots, t_n \in T$ and $G_1, \ldots, G_n$ are open subsets of spaces $X_{t_1}, \ldots, X_{t_n}$, respectively. If the sets $G_1, \ldots, G_n$ range over only certain fixed bases of spaces X_t, respectively, the sets (2) also form a basis for X_0. Hence it follows that if $\boldsymbol{B}_t$ are subbases of X_t, $t \in T$, then the class of all the sets

$$\pi_t^{-1}(G) \quad \text{where} \quad G \in \boldsymbol{B}_t, \ t \in T$$

form a subbasis for X_0.

From the point of view of applications given later in this book, the most important case is that in which T is the set of all positive integers n and all the spaces X_n are equal: $X_n = Y$ for $n = 1, 2, \ldots$ Then the Cartesian product will be denoted by $Y^{\aleph_0}$. By definition, $Y^{\aleph_0}$ is the set of all sequences $y = \{y_n\}$ where $y_n \in Y$ for $n = 1, 2, \ldots$ The class of all sets

$$H = G_1 \times G_2 \times \ldots \times G_m \times Y \times Y \times \ldots \tag{3}$$

where G_n are open (or: belong to a given basis for Y) is an open basis for $Y^{\aleph_0}$.

9.1. *The space* $(Y^{\aleph_0})^{\aleph_0}$ *is homeomorphic to* $Y^{\aleph_0}$.

Let ψ be a one-to-one mapping of the set of all positive integers onto the set of all pairs of positive integers. The elements of $(Y^{\aleph_0})^{\aleph_0}$ are infinite sequences

$$x = \{x_1, x_2, \ldots\}$$

where, for every m,

$$x_m = \{y_{m,1}, y_{m,2}, \ldots\}$$

is a sequence of positive integers. The formula

$$\varphi(x) = \{y_{\psi(1)}, y_{\psi(2)}, \ldots\}$$

defines a one-to-one mapping of $(Y^{\aleph_0})^{\aleph_0}$ onto $Y^{\aleph_0}$. It is easy to verify that φ transforms sets in the basis (3) constructed for $(Y^{\aleph_0})^{\aleph_0}$ onto sets in the basis (3) for $Y^{\aleph_0}$, and so does φ^{-1}. This proves that φ is a homeomorphism.

Consider now the following special topological space Y_0 formed of four elements:

$$Y_0 = (g_1, g_2, f_1, f_2)$$

with the interior operation defined as follows:

$$\mathrm{I}A = \begin{cases} Y_0 & \text{if} \quad A = Y_0, \\ (g_1, g_2) & \text{if} \quad (g_1, g_2) \subset A \neq Y_0, \\ 0 & \text{otherwise.} \end{cases}$$

The space Y_0 has exactly three open subsets:

$$0, \quad G_0 = (g_1, g_2), \quad Y_0,$$

and exactly three closed subsets:

$$0, \quad F_0 = (f_1, f_2), \quad Y_0.$$

Hence it follows that the sets

$$\begin{aligned} H_1 &= G_0 \times Y_0 \times Y_0 \times \dots, \\ H_2 &= Y_0 \times G_0 \times Y_0 \times \dots, \\ H_3 &= Y_0 \times Y_0 \times G_0 \times \dots \\ &\dots\dots\dots\dots \end{aligned} \tag{4}$$

form a subbasis for $Y_0^{\aleph_0}$.

A topological space X is said to be *separable* if it has an at most enumerable open basis $G_1, G_2, \dots$ For instance, the space of all real numbers is separable (see the remark in I, § 2, p. 15).

9.2. *Every separable space X with cardinal $\leqslant 2^{\aleph_0}$ is homeomorphic to a subset of $Y_0^{\aleph_0}$.*

Since the cardinal of X is $\leqslant 2^{\aleph_0}$, there exists a sequence $\{A_n\}$ of sets separating X, i.e. such that for arbitrary distinct points x_1, x_2 in X there exists a set A_n containing exactly one of them. Let $\{G_n\}$ be an open basis for X. The mapping

$$\varphi(x) = \{y_n(x)\} \in Y_0^{\aleph_0} \quad \text{for} \quad x \in X,$$

where

$$y_n(x) = \begin{cases} g_1 & \text{if} \quad x \in G_n \cap A_n, \\ g_2 & \text{if} \quad x \in G_n - A_n, \\ f_1 & \text{if} \quad x \in A_n - G_n, \\ f_2 & \text{if} \quad x \notin A_n \cup G_n, \end{cases}$$

is a one-to-one transformation of X onto a subset $Y = \varphi(X) \subset Y_0^{\aleph_0}$. Since sets $H_n \cap Y$ form a subbasis for the space Y (see I, § 2, (11)) and

$$\varphi(G_n) = H_n \cap Y \quad \text{and} \quad \varphi^{-1}(H_n \cap Y) = G_n,$$

the transformation φ is a homeomorphism because of I, § 2, (8) or (8′).

The metric space $\mathcal{N}$ of all irrational numbers (with the usual metric $\varrho(x, y) = |x-y|$) is homeomorphic to the space $N^{\aleph_0}$ where N is the set of all positive integers with the trivial interior operation

$$\mathbf{I}A = A \quad \text{for every} \quad A \subset N .$$

In fact, the class of all one-point sets is an open basis for N. Consequently, the class of all sets

$$N_{k_1,\ldots,k_r} = \text{the set of all } \{m_n\} \in N^{\aleph_0} \text{ such that } m_1 = k_1, \ \ldots, \ m_r = k_r$$

is an open basis in $N^{\aleph_0}$. Every irrational number x can be uniquely represented as a continued fraction

$$x = \frac{1}{|m_1} + \frac{1}{|m_2} + \frac{1}{|m_3} + \ldots$$

The one-to-one mapping

$$\varphi(x) = \{m_1, m_2, \ldots\}$$

of $\mathcal{N}$ onto $N^{\aleph_0}$ is a homeomorphism by I, § 2, (8'), since $\varphi^{-1}(N_{k_1,\ldots,k_r})$ is an interval in $\mathcal{N}$, and the class of all such intervals forms a basis for $\mathcal{N}$.

Since $\mathcal{N}$ is homeomorphic to $N^{\aleph_0}$, the space $\mathcal{N}^{\aleph_0}$ is homeomorphic to $(N^{\aleph_0})^{\aleph_0}$, i.e. to $N^{\aleph_0}$ by 9.1. Thus the spaces $\mathcal{N}$ and $\mathcal{N}^{\aleph_0}$ are homeomorphic.

9.3. *The space $Y_0^{\aleph_0}$ is an interior image of the space $\mathcal{N}$ of all irrational numbers.*

The space $\mathcal{N}$ being dense-in-itself, by 7.1 there exists a decomposition into non-empty disjoint sets:

$$\mathcal{N} = G \cup F_1 \cup F_2$$

such that G is open and $\mathbf{C}G - G = \mathbf{C}F_1 = \mathbf{C}F_2$. Hence

$$\mathcal{N} = \mathbf{C}\mathcal{N} = (\mathbf{C}G - G) \cup G \cup \mathbf{C}F_1 \cup \mathbf{C}F_2 = (\mathbf{C}G - G) \cup G = \mathbf{C}G .$$

Applying theorem 7.1 for the second time, we infer that there exists a decomposition

$$G = G_1 \cup G_2$$

into two disjoint non-empty sets, such that $\mathbf{C}G_1 = \mathbf{C}G_2 = \mathbf{C}G = \mathcal{N}$.

The mapping

$$\varphi_0(x) = \begin{cases} g_i & \text{for} \quad x \in G_i , \\ f_i & \text{for} \quad x \in F_i \end{cases} \qquad (i = 1, 2)$$

is an interior mapping of $\mathcal{N}$ onto Y_0. In fact, if H is any open non-empty subset of $\mathcal{N}$, then

$$\varphi_0(H) = \begin{cases} (g_1, g_2) & \text{if} \quad H \subset G, \\ Y_0 & \text{if} \quad H \not\subset G, \end{cases}$$

since G_1, G_2 are dense in $\mathcal{N}$ (and, consequently, they intersect H) and F_1, F_2 are dense in the subspace $F = \mathcal{N} - G$ (and, consequently, they intersect H provided the set $F \cap H = H - G$, open in F, is not empty).

The mapping

$$\varphi(x) = \{\varphi_0(x_n)\} \quad \text{for} \quad x = \{x_n\} \in N^{\aleph_0}$$

is an interior mapping of $N^{\aleph_0}$ onto $Y_0^{\aleph_0}$ since it transforms every set of the form (3) (in the space Y_0) onto an open set in $Y^{\aleph_0}$ (see § 3, (10)).

Let ψ be a homeomorphism of $\mathcal{N}$ onto $\mathcal{N}^{\aleph_0}$ (see 9.1). The superposition $\varphi(\psi(x))$ is the required interior mapping of $\mathcal{N}$ onto $Y_0^{\aleph_0}$.

9.4. *For every separable topological space X with a cardinal $\leqslant 2^{\aleph_0}$ there exists a set X_0 of irrational numbers and an interior mapping φ of X_0 onto X* (1).

By 9.2 there exists a homeomorphism φ_0 of X onto a subset $\varphi_0(X)$ of the space $Y_0^{\aleph_0}$. By 9.3 there exists an interior mapping ψ of $\mathcal{N}$ onto $Y_0^{\aleph_0}$. Let $X_0 = \psi^{-1}(\varphi_0(X))$. The mapping

$$\varphi(x) = \varphi_0^{-1}(\psi(x))$$

has the required property.

§ 10. A representation theorem for enumerable topological Boolean algebras. We shall prove the following theorem:

10.1. *For every enumerable topological Boolean algebra A and for every given enumerable set* (Q) *of infinite joins and meets in A, there exists a set X_0 of irrational numbers and a topological* (Q)*-isomorphism of A into $\mathfrak{B}(X_0)$* (2).

By 4.3 there exists a separable topological space X with a cardinal $\leqslant 2^{\aleph_0}$, and a topological (Q)-isomorphism h of A into $\mathfrak{B}(X)$. By 9.4 there exists a set X_0 of irrational numbers and an interior mapping φ of X_0 onto X. By 3.2 the isomorphism h_1 of $\mathfrak{B}(X)$ into $\mathfrak{B}(X_0)$,

$$h_1(A) = \varphi^{-1}(A) \quad \text{for} \quad A \subset X,$$

is topological. The isomorphism h_1 preserves all set-theoretical unions and intersections. Hence it follows that the topological isomorphism of A into $\mathfrak{B}(X)$,

$$h_0(a) = h_1(h(a)) \quad \text{for} \quad a \in A,$$

is a (Q)-isomorphism.

(1) Шварц [1]. See also Sikorski [7] and Пономарев [1].

(2) Sikorski [7].

§ 11. Complete spaces. A metric space X is said to be *complete* if, for every sequence $x_n \in X$, the condition

$$\lim_{k,n\to\infty} \varrho(x_n, x_k) = 0 \tag{1}$$

implies the convergence of x_n, i.e. the existence of a point $x \in X$ such that $\lim_{n\to\infty} \varrho(x_n, x) = 0$.

For instance the metric space of all real numbers with the ordinary distance $\varrho(x, y) = |x-y|$ is complete, by a well-known theorem.

In the space $N^{\aleph_0}$ of all sequences of positive integers (see § 9, p. 116) we can introduce a metric φ such that the interior operation determined by the metric coincides with the product topology defined in § 9, p. 113. Namely for any two distinct points $x = \{m_p\}$, $y = \{m'_p\}$ in $N^{\aleph_0}$ we define

$$\varrho(x, y) = 1/p_0$$

where p_0 is the smallest integer p such that $m_p \neq m'_p$. The easy verification is left to the reader.

The metric space $N^{\aleph_0}$ is complete. For if a sequence $x_n = \{m_{n,1}, m_{n,2}, \ldots\} \in N^{\aleph_0}$ satisfies condition (1), then for every p there exist integers n_p and m_p such that

$$m_{n,p} = m_p \quad \text{for all} \quad n > n_p .$$

Consequently $x_n \to x$ where $x = \{m_1, m_2, \ldots\} \in N^{\aleph_0}$.

A topological space is said to be *topologically complete* if it is homeomorphic to a complete metric space.

For instance, the space $\mathcal{N}$ of all irrational numbers (with the ordinary distance $p(x, y) = |x-y|$) is not complete but it is topologically complete since it is homeomorphic to the complete metric space $N^{\aleph_0}$ (see § 9, p. 116).

The following theorem is an analogue of theorem I, 2.2, for topologically complete spaces:

11.1. *If X is a topologically complete space, then every set $A \subset X$ of the first category is a boundary set*, i.e. *$X-A$ is dense* [1].

Since sets of the first category, dense sets and boundary sets are invariant under homeomorphisms (see I, § 2, p. 19), it suffices to prove 11.1 only when X is a complete metric space. The proof is a slight modification of that of I, 2.2.

It suffices to prove that for every open set $B_0 \neq 0$, there exists a point $x \in B_0 - A$.

---

[1] This is the Baire theorem mentioned on p. 18.

By hypothesis, A is the union of a sequence $\{A_n\}$ of nowhere dense sets. Since the set A_1 is nowhere dense, the open set $B_0 - \mathbf{C}A_1$ is not empty, i.e. it contains a point x_1. Let ε_1 be a positive number such that $\overline{\mathrm{K}}(x_1, \varepsilon_1) \subset B_0 - \mathbf{C}A_1$ (see § 6, p. 104) and $\varepsilon_1 < 1$. The open set $\mathrm{K}(x_1, \varepsilon_1) - \mathbf{C}A_2$ contains a point x_2 by the same argument. Let ε_2 be a positive number such that $\overline{\mathrm{K}}(x_2, \varepsilon_2) \subset \mathrm{K}(x_1, \varepsilon_1) - \mathbf{C}A_2$ and $\varepsilon_2 < 1/2$.

Continuing this procedure we define a sequence of points $x_n \in X$ and positive numbers ε_n such that

$$\overline{\mathrm{K}}(x_n, \varepsilon_n) \subset \mathrm{K}(x_{n-1}, \varepsilon_{n-1}) - \mathbf{C}A_n \quad \text{for} \quad n > 1$$

and $\varepsilon_n < 1/n$. Since

$$x_n \in \overline{\mathrm{K}}(x_k, \varepsilon_k) \quad \text{for} \quad n \geqslant k\,, \tag{2}$$

we have $\varrho(x_n, x_k) \leqslant 2\varepsilon_k < 2/k$ (see § 6, (7)). Hence it follows that the sequence $\{x_n\}$ satisfies condition (1) and consequently it converges to a point $x \in X$. Since the set $\overline{\mathrm{K}}(x_k, \varepsilon_k)$ is closed, we infer from (2) that $x \in \overline{\mathrm{K}}(x_k, \varepsilon_k)$ for every k. Hence it follows that $x \in B_0$ and $x \notin A$, by the definition of the sets $\mathrm{K}(x_n, \varepsilon_n)$. This proves theorem 11.1.

In I, § 2, p. 16, we showed that in any topological space

$$A - \mathbf{I}A \quad \text{is a boundary set} \tag{3}$$

for every set $A \subset X$. We shall prove that, for every set $A \subset X$, the set

$$A - \mathbf{IC}A \quad \text{is nowhere dense}\,. \tag{4}$$

In fact, the set $\mathbf{C}A - \mathbf{IC}A$ is closed and, by (3), nowhere dense. The set (4) is a subset of $\mathbf{C}A - \mathbf{IC}A$ and therefore it is also nowhere dense.

Note that, conversely, every nowhere dense set is of the form $A - \mathbf{IC}A$. For if A is nowhere dense, then $\mathbf{IC}A = 0$ and consequently $A = A - \mathbf{IC}A$.

By (4), if $\{A_n\}$ is any sequence of subsets of a topological space X, then the union

$$\bigcup_{n=1}^{\infty} (A_n - \mathbf{IC}A_n)$$

is a set of the first category (and every set of the first category is of this form!). By 11.1, if X is topologically complete, then the complement of this set is dense in X, i.e.

$$\mathbf{C} - \bigcup_{n=1}^{\infty} (A_n - \mathbf{IC}A_n) = X\,,$$

or equivalently:

$$-\mathbf{I} \bigcup_{n=1}^{\infty} (A_n \cap -\mathbf{I} - \mathbf{I} - A_n) = X \tag{5}$$

for any sequence $\{A_n\}$ of subsets of a topologically complete space. The identity (5) is another formulation of 11.1.

Note that identity (5) does not hold in all topological spaces. For instance, if X is the metric space of all rational numbers with the ordinary distance $\varrho(x, y) = |x-y|$, all elements in X are arranged in a sequence $x_1, x_2, \ldots$, and $A_n = (x_n)$, then

$$-\mathbf{I}\bigcup_{n=1}^{\infty}(A_n \cap -\mathbf{I}-\mathbf{I}-A_n) = \mathbf{0}\,. \tag{6}$$

This also proves that the metric space of all rational numbers is not topologically complete.

By I, 2.2, identity (5) also holds in any compact Hausdorff space.

Note that in the case of metric spaces, theorem I, 2.2, is a particular case of 11.1 since it can be proved that every compact metric space is complete.

§ 12. Quotient algebras. A filter $\mathcal{F}$ in a topological Boolean algebra $\boldsymbol{A}$ is said to be an I-*filter* if, for every $a \in \boldsymbol{A}$,

$$a \in \mathcal{F} \quad \text{implies} \quad \mathbf{I}a \in \mathcal{F}\,. \tag{1}$$

12.1. *If $\mathcal{F}$ is an* I-*filter in a topological Boolean algebra $\boldsymbol{A}$, then the quotient Boolean algebra $\boldsymbol{A}/\mathcal{F}$ is topological, the interior operation* **I** *in $\boldsymbol{A}/\mathcal{F}$ being defined by the equality*

$$\mathbf{I}|a| = |\mathbf{I}a| \quad (a \in \boldsymbol{A})\,. \tag{2}$$

Moreover,

$$\mathbf{C}|a| = |\mathbf{C}a| \quad (a \in \boldsymbol{A})\,. \tag{3}$$

First observe that, for any $a, b \in \boldsymbol{A}$,

$$\mathbf{I}(a \Rightarrow b) \leqslant \mathbf{I}a \Rightarrow \mathbf{I}b\,, \tag{4}$$

since $\mathbf{I}a \cap \mathbf{I}(a \Rightarrow b) = \mathbf{I}\big(a \cap (a \Rightarrow b)\big) = \mathbf{I}(a \cap b) \leqslant \mathbf{I}b$.

If $|a| = |b|$, then $a \Rightarrow b \in \mathcal{F}$, and consequently $\mathbf{I}(a \Rightarrow b) \in \mathcal{F}$ by (1). Hence $\mathbf{I}a \Rightarrow \mathbf{I}b \in \mathcal{F}$ by (4). Since also $b \Rightarrow a \in \mathcal{F}$, we have $\mathbf{I}b \Rightarrow \mathbf{I}a \in \mathcal{F}$ by the same argument. This proves that

$$|a| = |b| \quad \text{implies} \quad |\mathbf{I}a| = |\mathbf{I}b|\,.$$

Thus (2) really defines an operation **I** in $\boldsymbol{A}/\mathcal{F}$. It follows immediately from the commutativity of **I** and $|\ |$, expressed by (2), that **I** satisfies the axioms (i_1)-(i_4). The identity (3) follows directly from (2) since the closure **C** in $\boldsymbol{A}/\mathcal{F}$ is defined by the equation

$$\mathbf{C}|a| = -\mathbf{I}-|a|$$

according to § 1, (1). Hence $\mathbf{C}|a| = |-\mathbf{I}-a| = |\mathbf{C}a|$.

Theorem 12.1 can also be formulated as follows.

We have proved in I, § 13, and II, § 5, that if ∇ is a filter in a Boolean algebra $\boldsymbol{A}$, then the equivalence relation $\sim$ defined as follows:

(5) $\quad a \sim b$ if and only if $a \Rightarrow b \in \nabla$ and $b \Rightarrow a \in \nabla$

is a congruence with respect to the Boolean operation $\cup, \cap, \Rightarrow, -$, and $\boldsymbol{A}/\nabla$ is a Boolean algebra. Theorem 12.1 states that if $\boldsymbol{A}$ is a topological Boolean algebra and ∇ is an I-filter, then the equivalence relation $\sim$ is also a congruence with respect to the operation $\mathbf{I}$, i.e. it is a congruence in the topological Boolean algebra $\{\boldsymbol{A}, \cup, \cap, \Rightarrow, -, \mathbf{I}\}$. Moreover $\boldsymbol{A}/\nabla$, i.e. the algebra $\boldsymbol{A}/\sim$ is also a topological Boolean algebra. Note that the definition (2) was formulated according to the general rule in I, § 4, p. 25.

The condition that ∇ be an I-filter is also necessary for the equivalence relation $\sim$ to be a congruence with respect to $\mathbf{I}$. For if $\sim$ is a congruence with respect to $\mathbf{I}$ and $a \in \nabla$, then $a \sim 1$ and consequently $\mathbf{I}a \sim \mathbf{I}1 = 1$, i.e. $\mathbf{I}a \in \nabla$.

Note that a principal filter ∇ generated by an element $g \in \boldsymbol{A}$ is an I-filter if and only if g is open. In this case the Boolean isomorphism $h(|a|) = a \cap g$ (II, § 6, (4)) from $\boldsymbol{A}/\nabla$ onto $\boldsymbol{A}_g$ is topological.

By duality, an ideal Δ in a topological Boolean algebra $\boldsymbol{A}$ is called a C-*ideal* if, for every $a \in \boldsymbol{A}$,

(5) $\quad a \in \Delta$ implies $\mathbf{C}a \in \Delta$.

A theorem dual to 11.1 also holds for quotient algebras $\boldsymbol{A}/\Delta$ where Δ is a C-ideal in $\boldsymbol{A}$.

§ 13. Products of topological Boolean algebras. Direct unions of topological spaces. Let $\{\boldsymbol{A}_n\}_{n \in N}$ be an indexed set of topological Boolean algebras conceived as abstract algebras

(1) $\quad \{\boldsymbol{A}_n, \cup, \cap, \Rightarrow, -, \mathbf{I}\}$

according to § 1, p. 93. By the general definition from I, § 4, p. 27, the product $\boldsymbol{A}$ of all $\boldsymbol{A}_n$ is the algebra

(2) $\quad \{\boldsymbol{A}, \cup, \cap, \Rightarrow, -, \mathbf{I}\}$

where $\boldsymbol{A} = \boldsymbol{P}_{n \in N} \boldsymbol{A}_n$ and the operations in $\boldsymbol{A}$ are defined as follows:

$$
\begin{aligned}
\{a_n\}_{n \in N} \cup \{b_n\}_{n \in N} &= \{a_n \cup b_n\}_{n \in N}, \\
\{a_n\}_{n \in N} \cap \{b_n\}_{n \in N} &= \{a_n \cap b_n\}_{n \in N}, \\
\{a_n\}_{n \in N} \Rightarrow \{b_n\}_{n \in N} &= \{a_n \Rightarrow b_n\}_{n \in N}, \\
-\{a_n\}_{n \in N} &= \{-a_n\}_{n \in N}, \\
\mathbf{I}\{a_n\}_{n \in N} &= \{\mathbf{I}a_n\}_{n \in N}.
\end{aligned}
\tag{3}
$$

13.1. *The product* (2) *of all topological Boolean algebras* (1) *is a topological Boolean algebra.*

The proof is similar to the proof of the analogous theorem I, 6.3, for lattices and theorem II, 7.1, for Boolean algebras.

Note that the Boolean algebra $\{\boldsymbol{A}, \cup, \cap, \Rightarrow, -\}$ is then the product of the Boolean algebras $\{\boldsymbol{A}_n, \cup, \cap, \Rightarrow, -\}$ (see II, § 7, (3)). Consequently all statements from II, § 7, on products of Boolean algebras can also be applied to products of topological Boolean algebras. In particular, if all $\boldsymbol{A}_n$ are complete topological Boolean algebras, their product $\boldsymbol{A}$ is also.

It follows from the last two identities (3) and from § 1, (1), that

$$\mathbf{C}\{a_n\}_{n\in N} = \{\mathbf{C}a_n\}_{n\in N} \tag{4}$$

for any $\{a_n\}_{n\in N} \in \boldsymbol{A}$. The last of the identities (3) and identity (4) immediately imply the following theorem:

13.2. *An element* $a = \{a_n\}_{n\in N}$ *in the product* $\boldsymbol{A}$ *of all topological Boolean algebras* $\boldsymbol{A}_n$ *is open* (*closed*) *if and only if all* a_n *are open* (*closed*) *in* $\boldsymbol{A}_n$.

Let $\{X_n\}_{n\in N}$ be an indexed set of disjoint topological spaces, and let X be the union of all the spaces X_n. We shall consider X as a topological space, the class of all open sets in the spaces X_n being assumed as a basis in X. The topological space X just defined is said to be the *direct union* of the topological spaces X_n. By definition, a set $G \subset X$ is open if and only if the intersection $X_n \cap G$ is an open subset of X_n for every $n \in N$. In particular, all sets X_n are open subsets of X.

The following theorem is an analogue of II 7.2:

13.3. *If, for every* $n \in N$, $\boldsymbol{A}_n$ *is a topological field of subsets of a topological space* X_n, *and all the spaces* X_n *are disjoint, then the mapping*

$$h(\{A_n\}_{n\in N}) = \bigcup_{n\in N} A_n \tag{5}$$

(*where* $A_n \in \boldsymbol{A}_n$ *for every* $n \in N$, *and* $\bigcup$ *denotes the set-theoretical union*) *is a topological isomorphism from the product* $\boldsymbol{A} = \boldsymbol{P}_{n\in N} \boldsymbol{A}_n$ *onto the topological field* $\boldsymbol{B}$ *of all subsets* B *of the direct union* X *of all* X_n, *such that*

$$X_n \cap B \in A_n \quad \text{for every} \quad n \in N.$$

If, for every $n \in N$, $\boldsymbol{A}_n$ *is the topological field* $\mathfrak{B}(X_n)$ *of all subsets of* X_n, *then* $\boldsymbol{B}$ *is the topological field* $\mathfrak{B}(X)$ *of all subsets of* X.

The proof is by an easy verification.

Isomorphism (5) yields a natural set-theoretical representation for the topological Boolean algebra $\boldsymbol{A} = \boldsymbol{P}_{n\in N} \boldsymbol{A}_n$. Consequently we can identify the product $\boldsymbol{A}$ with the field $\boldsymbol{B}$, mentioned in 13.2, by identification of $\{A_n\}_{n\in N} \in \boldsymbol{A}$ with the union $\bigcup_{n\in N} A_n$.

CHAPTER IV

PSEUDO-BOOLEAN ALGEBRAS

§ 1. Preliminaries. We recall (see I, § 12) that a lattice $\boldsymbol{A}$ is said to be relatively pseudo-complemented if, for any elements $a, b \in \boldsymbol{A}$, the set of all $x \in \boldsymbol{A}$ such that $a \cap x \leqslant b$ contains the greatest element. The greatest element is denoted by $a \Rightarrow b$ and called the pseudo-complement of a relative to b. By definition,

$$a \cap x \leqslant b \quad \text{if and only if} \quad x \leqslant a \Rightarrow b \,. \tag{1}$$

Many properties of the operation $a \Rightarrow b$ were given in theorems I, 12.2 and I, 12.3.

As we mentioned in I, § 12, p. 59, every relatively pseudo-complemented lattice $\boldsymbol{A}$ will be considered as an abstract algebra

$$\{\boldsymbol{A}, \cup, \cap, \Rightarrow\} \tag{2}$$

with three binary operations. Algebras (2) can be characterized by a simple set of axioms. Namely

1.1. *An abstract algebra* (2) *with three binary operations is a relatively pseudo-complemented lattice if and only if it satisfies the lattice axioms* (l_1)-(l_3) (see p. 35) *and also the following*:

$$a \cap (a \Rightarrow b) = a \cap b \,, \tag{l_7}$$

$$(a \Rightarrow b) \cap b = b \,, \tag{l_8}$$

$$(a \Rightarrow b) \cap (a \Rightarrow c) = a \Rightarrow (b \cap c) \,, \tag{l_9}$$

$$(a \Rightarrow a) \cap b = b \,. \tag{l_{10}}$$

If an algebra $\boldsymbol{A}$ satisfies axioms (l_1)-(l_3), then it is a lattice by I, 6.1. If, in addition, axioms (l_7)-(l_{10}) hold in $\boldsymbol{A}$, then $\boldsymbol{A}$ is a relatively pseudo-complemented lattice and the operation $\Rightarrow$ given in $\boldsymbol{A}$ coincides with the relative pseudo-complement defined in I, § 11. To prove this, it suffices to show that the binary operation $\Rightarrow$ in (2) satisfies condition (1). If $x \leqslant a \Rightarrow b$, then

$$a \cap x \leqslant a \cap (a \Rightarrow b) = a \cap b \leqslant b$$

by (l_7). Conversely, if $a \cap x \leqslant b$, i.e. $x \cap a = x \cap a \cap b$, then by (l_8), (l_{10}) and (l_9)

$$x \leqslant a \Rightarrow x = (a \Rightarrow x) \cap (a \Rightarrow a) = a \Rightarrow (x \cap a) = a \Rightarrow (x \cap a \cap b)$$
$$= \big(a \Rightarrow (x \cap a)\big) \cap (a \Rightarrow b) \leqslant a \Rightarrow b .$$

On the other hand, axioms (l_1)-(l_3) hold in any lattice by theorem I, 6.1. By theorem I, 12.2, (14), (15), (16), (9) axioms (l_7)-(l_{10}) hold in any relatively pseudo-complemented lattice.

We recall that every relatively pseudo-complemented lattice has the unit element but it does not always have the zero element. Every relatively pseudo-complemented lattice with the zero element $\wedge$ is called a pseudo-Boolean algebra. If $\boldsymbol{A}$ is a pseudo-Boolean algebra, then every element $a \in \boldsymbol{A}$ has the pseudo-complement $-a$. Namely (see I, § 11, (6))

$$-a = a \Rightarrow \wedge . \tag{3}$$

Many properties of the pseudo-complement $-a$ in pseudo-Boolean algebras were given in theorem I, 12.3.

As we mentioned in I, § 12, every pseudo-Boolean algebra $\boldsymbol{A}$ will be considered as an abstract algebra

$$\{\boldsymbol{A}, \cup, \cap, \Rightarrow, -\} \tag{4}$$

with three binary operations $\cup, \cap, \Rightarrow$ and one unary operation $-$. Algebras (4) can easily be characterized by a set of axioms, as follows:

1.2. *An abstract algebra* (4) *with three binary operations* $\cup, \cap, \Rightarrow$ *and one unary operation* $-$ *is a pseudo-Boolean algebra if and only if it satisfies axioms* [1] (l_1)-(l_3), (l_7)-(l_{10}) *and the following*:

$$-(a \Rightarrow a) \cup b = b , \tag{l_{11}}$$

$$a \Rightarrow \big(-(a \Rightarrow a)\big) = -a . \tag{l_{12}}$$

This follows immediately from 1.1 since (l_{11}) asserts that $-(a \Rightarrow a) = \wedge$ and (l_{12}) is another formulation of (3).

If a relatively pseudo-complemented lattice $\boldsymbol{A}$ has no zero element, then $\boldsymbol{A}$ can easily be extended to a pseudo-Boolean algebra $\boldsymbol{A}'$ by adding a new element which is the zero of $\boldsymbol{A}'$. This follows directly from the following more general theorem:

1.3. *Let* $\boldsymbol{A}$ *be a relatively pseudo-complemented lattice and let* $\wedge' \notin \boldsymbol{A}$. *The set* $\boldsymbol{A}'$ *consisting of* $\wedge'$ *and of all elements in* $\boldsymbol{A}$

[1] For a simpler set of axioms for pseudo-Boolean algebras see e.g. Monteiro [1].

is a lattice provided the operations $\cup$ *and* $\cap$ *in* $\boldsymbol{A}$ *are extended over* $\boldsymbol{A}'$ *by the rules*:

$$\Lambda' \cup a = a = a \cup \Lambda' \quad \text{and} \quad \Lambda' \cap a = \Lambda' = a \cap \Lambda' \quad \text{for every} \quad a \in \boldsymbol{A}'.$$

Λ' *is the zero element of the lattice* $\boldsymbol{A}'$. $\boldsymbol{A}'$ *is a pseudo-Boolean algebra. If* $a, b \in \boldsymbol{A}$, *then the pseudo-complement*

$$a \Rightarrow b$$

of a *relative to* b *in* $\boldsymbol{A}$ *coincides with the pseudo-complement of* a *relative to* b *in* $\boldsymbol{A}'$. *For every* $b \in \boldsymbol{A}'$,

$$\Lambda' \Rightarrow b = V,$$

where V *is the common unit element of* $\boldsymbol{A}$ *and* $\boldsymbol{A}'$. *For every* $a \in \boldsymbol{A}$, *the pseudo-complement* $-a$ *of* a *in* $\boldsymbol{A}'$ *is equal to* Λ':

$$-a = a \Rightarrow \Lambda' = \Lambda'.$$

Moreover, for any set of elements $a_t \in \boldsymbol{A}$ $(t \in T)$

$$\bigcup_{t \in T}^{\boldsymbol{A}} a_t = \bigcup_{t \in T}^{\boldsymbol{A}'} a_t, \qquad \bigcap_{t \in T}^{\boldsymbol{A}} a_t = \bigcap_{t \in T}^{\boldsymbol{A}'} a_t.$$

More precisely: if the join (meet) on the left-hand side exists, so does the join (meet) on the right-hand side and the equation holds; if the join (meet) on the right-hand side exists and belongs to $\boldsymbol{A}$, *so does the join (meet) on the right-hand side and the equation holds.*

The proof of 1.3 is by an easy verification.

Observe that if $\boldsymbol{A}$ has the zero element Λ, then Λ is not the zero element of $\boldsymbol{A}'$.

The main purpose of this Chapter is to prove some representation theorem for relatively pseudo-complemented lattices and pseudo-Boolean algebras. By 1.3 it suffices to investigate only the case of pseudo-Boolean algebras. Therefore the next sections deal almost exclusively with pseudo-Boolean algebras.

For any topological Boolean algebra $\boldsymbol{B}$, we shall denote by $\mathfrak{G}(\boldsymbol{B})$ the set of all open elements in $\boldsymbol{B}$. Since the join and meet of two open elements is open, $\mathfrak{G}(\boldsymbol{B})$ is a sublattice of $\boldsymbol{B}$. For the present, let us denote the relative complement in $\boldsymbol{B}$ by $\rightarrow$, and the complement in $\boldsymbol{B}$ by $-$, the usual symbols $\Rightarrow$ and $-$ being reserved to denote the relative pseudo-complement and the pseudo-complement in the lattice $\mathfrak{G}(\boldsymbol{B})$.

1.4. *The lattice* $\mathfrak{G}(\boldsymbol{B})$ *of all open elements in a topological Boolean algebra* $\boldsymbol{B}$ *is a pseudo-Boolean algebra. Moreover, for all* $a, b \in \mathfrak{G}(\boldsymbol{B})$,

$$a \Rightarrow b = \mathbf{I}(a \rightarrow b), \tag{5}$$

$$-a = \mathbf{I}-a \tag{6}$$

where $\mathbf{I}$ *is the interior operation in* $\boldsymbol{B}$.

In particular, if elements a, b are both open and closed in $\boldsymbol{B}$, then $a \Rightarrow b = a \rightarrow b = -a \cup b$ and $-a = -a$; consequently elements $a \Rightarrow b$ and $-a$ are both open and closed in $\boldsymbol{B}$.

To prove (5) it suffices to show that, for any $a, b, x \in \mathfrak{G}(\boldsymbol{B})$,

$$a \cap x \leqslant b \quad \text{if and only if} \quad x \leqslant \mathbf{I}(a \rightarrow b)$$

(see (1)). In fact, $a \cap x \leqslant b$ is equivalent to $x \leqslant a \rightarrow b$ since $\rightarrow$ is the relative complement in $\boldsymbol{B}$. Since x is open, the inequality $x \leqslant a \rightarrow b$ is equivalent to $x \leqslant \mathbf{I}(a \rightarrow b)$ (see III, § 1, (3)). Thus $a \cap x \leqslant b$ is equivalent to $x \leqslant \mathbf{I}(a \rightarrow b)$.

(6) follows immediately from (5) since

$$-a = a \Rightarrow 0 = \mathbf{I}(a \rightarrow 0) = \mathbf{I}-a .$$

Theorem 1.4 yields an important example of pseudo-Boolean algebras. This example is typical because we shall prove in § 3 that every pseudo-Boolean algebra $\boldsymbol{A}$ is of the form $\mathfrak{G}(\boldsymbol{B})$ for a topological Boolean algebra $\boldsymbol{B}$.

If $\boldsymbol{B} = \mathfrak{B}(X)$, i.e. if B is the field of all subsets of a topological space X, then we shall write simply $\mathfrak{G}(X)$ instead of $\mathfrak{G}(\mathfrak{B}(X))$. By definition, $\mathfrak{G}(X)$ is the pseudo-Boolean algebra of all open subsets of a topological space X. The algebra $\mathfrak{G}(X)$ was described earlier in I, § 12 (p. 59). The relative pseudo-complement $\Rightarrow$ and the pseudo-complement $-$ in $\mathfrak{G}(X)$ satisfy the identities

$$A \Rightarrow B = \mathbf{I}(A \rightarrow B) , \qquad -A = \mathbf{I}-A \tag{7}$$

for any open subsets A, B of X, where $\rightarrow$ and $-$ denote the relative complement and complement in $\mathfrak{B}(X)$.

According to the general definition in I, § 4, a subalgebra of a pseudo-Boolean algebra $\boldsymbol{A}$ is a non-empty subset of $\boldsymbol{A}$, closed under the operations $\cup, \cap, \Rightarrow, -$. Every subalgebra of a pseudo-Boolean algebra is also a pseudo-Boolean algebra.

Similarly, by a subalgebra of a relatively pseudo-complemented lattice $\boldsymbol{A}$ we shall understand any subset of $\boldsymbol{A}$ which is closed under the operations $\cup, \cap, \Rightarrow$. Every subalgebra of a relatively pseudo-complemented lattice is also a relatively pseudo-complemented lattice.

For instance, the set $\mathfrak{G}_0(\boldsymbol{B})$ of all open dense elements in a topological Boolean algebra $\boldsymbol{B}$ is a subalgebra of the relatively pseudo-complemented lattice $\mathfrak{G}(\boldsymbol{B})$ of all open elements in $\boldsymbol{B}$. For if $a, b \in \mathfrak{G}_0(\boldsymbol{B})$, then the open element $a \Rightarrow b$ is also dense since $b \leqslant a \Rightarrow b$ (see (1_8)) and consequently $1 = \mathbf{C}b \leqslant \mathbf{C}(a \Rightarrow b)$. Thus we infer that

1.5. *The lattice $\mathfrak{G}_0(\boldsymbol{B})$ of all open dense elements in a topological Boolean algebra $\boldsymbol{B}$ is a relatively pseudo-complemented lattice, the operation $\Rightarrow$ being defined by* (5).

If $\boldsymbol{B}$ is the topological field $\mathfrak{B}(X)$ of all subsets of a topological space X, we write $\mathfrak{G}_0(X)$ instead of $\mathfrak{G}_0(\mathfrak{B}(X))$ (see I, § 12, p. 59). Thus theorem 1.5 is a generalization of the remark on the lattice $\mathfrak{G}_0(X)$ from I, § 12, p. 59.

According to the general definition in I, § 4, a relatively pseudo-complemented lattice or a pseudo-Boolean algebra is said to be *degenerate* if it contains only one element, i.e. if its unit element is its zero element. Otherwise it is called *non-degenerate*.

§ 2. Pseudo-Boolean homomorphisms and isomorphisms. According to the general definition in I, § 4, a mapping h of a pseudo-Boolean algebra $\boldsymbol{A}$ into a pseudo-Boolean algebra $\boldsymbol{A}'$ is said to be a *homomorphism* provided it preserves all the operations $\cup, \cap, \Rightarrow, -$, i.e.

$$h(a \cup b) = h(a) \cup h(b), \quad h(a \cap b) = h(a) \cap h(b), \tag{1}$$

$$h(a \Rightarrow b) = h(a) \Rightarrow h(b), \tag{2}$$

$$h(-a) = -h(a). \tag{3}$$

According to I, § 4, every one-to-one homomorphism is called an *isomorphism*. In order that a one-to-one mapping of a pseudo-Boolean algebra $\boldsymbol{A}$ onto another pseudo-Boolean algebra $\boldsymbol{A}'$ be an isomorphism, it suffices that h be a *lattice isomorphism*, i.e. that h satisfy (1) only. This follows from the fact that the relative pseudo-complement and the pseudo-complement are uniquely determined by the join and meet (see § 1, (1) and (3)).

In order to distinguish homomorphisms (isomorphisms) in the sense just defined from the Boolean and topological homomorphisms (isomorphisms) discussed in II, § 3, and III, § 3, mappings (one-to-one mappings) satisfying (1), (2), (3) will also be called *pseudo-Boolean homomorphisms* (*isomorphisms*).

Every pseudo-Boolean homomorphism h of $\boldsymbol{A}$ into $\boldsymbol{A}'$ maps the unit element of $\boldsymbol{A}$ onto the unit element of $\boldsymbol{A}'$, and the zero element of $\boldsymbol{A}$ onto the zero element of $\boldsymbol{A}'$:

$$h(V) = V, \quad h(\Lambda) = \Lambda.$$

This follows from the following identities:

$$h(V) = h(a \Rightarrow a) = h(a) \Rightarrow h(a) = V,$$

$$h(\Lambda) = h(a \cap -a) = h(a) \cap -h(a) = \Lambda$$

(see I, § 12, (6), and (27)).

Conversely, it follows immediately from § 1, (3), that any mapping (of $\boldsymbol{A}$ into $\boldsymbol{A}'$) satisfying (1) and (2) and transforming the zero element

onto the zero element also satisfies (3), i.e. it is a pseudo-Boolean homomorphism.

As an example of a pseudo-Boolean homomorphism we mention the natural homomorphism $h(a) = |a|$ from $\boldsymbol{A}$ onto $\boldsymbol{A}/V$ where V is any filter in a pseudo-Boolean algebra $\boldsymbol{A}$ (see I, 13.2). Other examples of pseudo-Boolean homomorphisms are given by the two following theorems (see also theorem 8.2):

2.1. *If h is a topological homomorphism of a topological Boolean algebra $\boldsymbol{B}$ into a topological Boolean algebra $\boldsymbol{B}'$, then h, restricted to $\mathfrak{G}(\boldsymbol{B})$ only, is also a pseudo-Boolean homomorphism of $\mathfrak{G}(\boldsymbol{B})$ into $\mathfrak{G}(\boldsymbol{B}')$.*

This follows immediately from the definition of operations in $\mathfrak{G}(\boldsymbol{B})$ and $\mathfrak{G}(\boldsymbol{B}')$ (see § 1, (5), (6), p. 125).

The next theorem follows immediately from 2.1 and II, 3.2:

2.2. *If φ is an interior mapping of a topological space X into another topological space Y, then the mapping h*

$$h(B) = \varphi^{-1}(B) \quad \text{for} \quad B \in \mathfrak{G}(Y) \tag{4}$$

is a pseudo-Boolean homomorphism of $\mathfrak{G}(Y)$ into $\mathfrak{G}(X)$. If, in addition, φ maps X onto Y, then h is a pseudo-Boolean isomorphism [1].

One can prove that, with certain additional hypotheses about Y, if (4) is a pseudo-Boolean homomorphism, then φ is an interior mapping [1].

§ 3. Representation theorems. The following theorem explain the connection between pseudo-Boolean algebras and topological Boolean algebras:

3.1. *For every pseudo-Boolean algebra $\boldsymbol{A}$ there exists a topological Boolean algebra $\boldsymbol{B}$ such that $\boldsymbol{A} = \mathfrak{G}(\boldsymbol{B})$* [2].

If $\boldsymbol{A}$ is finite, $\boldsymbol{B}$ is also finite. If the cardinal of $\boldsymbol{A}$ is $\mathfrak{m} \geqslant \aleph_0$, then the cardinal of $\boldsymbol{B}$ is also $\mathfrak{m}$.

First observe that, for every distributive lattice $\boldsymbol{A}$ with the zero and unit element, there exists a Boolean algebra $\boldsymbol{B}$ such that

(a) $\boldsymbol{A}$ is sublattice of $\boldsymbol{B}$; the zero (unit) element of $\boldsymbol{A}$ is the zero (unit) element of $\boldsymbol{B}$;

(b) if the cardinal of $\boldsymbol{A}$ is finite, the cardinal of $\boldsymbol{B}$ is also; if the cardinal of $\boldsymbol{A}$ is $\mathfrak{m} \geqslant \aleph_0$, then the cardinal of $\boldsymbol{B}$ is $\mathfrak{m}$;

(c) every element $b \in \boldsymbol{B}$ is of the form

$$b = (a_1 \to a_1') \cap \ldots \cap (a_n \to a_n') \,, \tag{1}$$

i.e. of the form

$$b = (-a_1 \cup a_1') \cap \ldots \cap (-a_n \cup a_n') \tag{1'}$$

---

[1] Sikorski [7].

[2] McKinsey and Tarski [2].

where $a_1, a_1', \ldots, a_n, a_n' \in \boldsymbol{A}$. The symbols $\rightarrow$ and $-$ here denote the Boolean relative complement and complement in $\boldsymbol{B}$, respectively.

Since every distributive lattice is isomorphic to a set lattice (see I, 9.3), it suffices to prove this when $\boldsymbol{A}$ is a lattice of subsets of a space X, the join and the meet being the set-theoretical union and intersection respectively, $\wedge = 0$ and $\vee = X$. This follows from II, 2.2 (where $\boldsymbol{A}_0$ is replaced by $\boldsymbol{A}$, and $\boldsymbol{A}$ by the field $\mathfrak{B}(X)$ of all subsets of X). Indeed, by II, 2.2, the class of all sets (1) is the field generated by $\boldsymbol{A}$. Thus (a) and (c) hold. Also (b) can easily be verified.

Now let $\boldsymbol{A}$ be a pseudo-Boolean algebra, and let $\boldsymbol{B}$ be the Boolean algebra satisfying (a), (b), (c).

For any $a, a' \in \boldsymbol{A}$ we have

$$a \Rightarrow a' \leqslant a \rightarrow a'. \tag{2}$$

In fact, $a \cap (a \Rightarrow a') \leqslant a'$ by § 1, (l_7), what implies $a \Rightarrow a' \leqslant -a \cup a' = a \rightarrow a'$.

We shall now prove that, for arbitrary $a_1, a_1', \ldots, a_n, a_n', a, a' \in \boldsymbol{A}$, the inequality

$$(a_1 \rightarrow a_1') \cap \ldots \cap (a_n \rightarrow a_n') \leqslant a \rightarrow a' \tag{3}$$

implies

$$(a_1 \Rightarrow a_1') \cap \ldots \cap (a_n \Rightarrow a_n') \leqslant a \Rightarrow a'. \tag{3'}$$

In fact, (3) is equivalent to

$$(a_1 \rightarrow a_1') \cap \ldots \cap (a_n \rightarrow a_n') \cap a \leqslant a'.$$

Hence, by (2),

$$(a \Rightarrow a_1') \cap \ldots \cap (a_n \Rightarrow a_n') \cap a \leqslant a',$$

which implies (3′) on account of § 1, (1).

It follows immediately from (3) and (3′) that the equation

$$(a_1 \rightarrow a_1') \cap \ldots \cap (a_n \rightarrow a_n') = (a_1 \rightarrow a_1') \cap \ldots \cap (a_m \rightarrow a_m'),$$

where $a_1, a_1', \ldots, a_n, a_n', a_1, a_1', \ldots, a_m, a_m' \in \boldsymbol{A}$, implies the equation

$$(a_1 \Rightarrow a_1) \cap \ldots \cap (a_n \Rightarrow a_n') = (a_1 \Rightarrow a_1') \cap \ldots \cap (a_m \Rightarrow a_m').$$

Now we define the interior operation in $\boldsymbol{B}$ as follows: If $b \in \boldsymbol{B}$ is of the form (1), then

$$\mathbf{I}b = (a_1 \Rightarrow a_1') \cap \ldots \cap (a_n \Rightarrow a_n').$$

It follows from the above consideration that $\mathbf{I}b$ does not depend on the representation of the element b in the form (1). Obviously

$$\mathbf{I}b \in \boldsymbol{A} \quad \text{for every} \quad b \in \boldsymbol{B}, \tag{4}$$

$$\mathbf{I}a = a \quad \text{for every} \quad a \in \boldsymbol{A}. \tag{5}$$

The operation $\mathbf{I}$ satisfies all the axioms III, § 1, (i_1), (i_2), (i_3), (i_4) (see p. 93). In fact, (i_1) follows immediately from the definition of $\mathbf{I}$; (i_2) fol-

lows from (2); (i_3) follows from (i_1), (4) and (5); (i_4) follows immediately from (5).

Properties (4), (5) imply immediately the equation $\boldsymbol{A} = \mathfrak{G}(\boldsymbol{B})$. The relative pseudo-complement and pseudo-complement given in $\boldsymbol{A}$ coincide with the relative pseudo-complement and pseudo-complement induced in $\mathfrak{G}(\boldsymbol{B})$ by the interior operation $\mathbf{I}$ in the manner described in § 1, p. 125. This follows from the fact that the relative pseudo-complement and pseudo-complement, if they exist, are uniquely determined by the join and meet.

3.2. *For every relatively pseudo-complemented lattice $\boldsymbol{A}$ there exists a topological Boolean algebra $\boldsymbol{B}$ such that $\boldsymbol{A} = \mathfrak{G}_0(\boldsymbol{B})$, i.e. $\boldsymbol{A}$ is the lattice of all open dense elements in $\boldsymbol{B}$.*

If $\boldsymbol{A}$ is finite, $\boldsymbol{B}$ is also finite. If the cardinal of $\boldsymbol{A}$ is $\mathfrak{m} \geqslant \aleph_0$, then the cardinal of $\boldsymbol{B}$ is also $\mathfrak{m}$.

Let $\boldsymbol{A}'$ be the pseudo-Boolean algebra obtained from $\boldsymbol{A}$ by adding a new zero element by the method described in theorem 1.3. By 3.1 there exists a topological Boolean algebra $\boldsymbol{B}$ such that $\boldsymbol{A}' = \mathfrak{G}(\boldsymbol{B})$ and the cardinal of $\boldsymbol{B}$ satisfies the conditions mentioned in 3.2. We shall prove that $\boldsymbol{A} = \mathfrak{G}_0(\boldsymbol{B})$, i.e. that all elements in $\boldsymbol{A}$ are dense in $\boldsymbol{B}$.

By 1.3, for every element $a \in \boldsymbol{A}$, the pseudo-complement $-a$ of a in $\boldsymbol{A}' = \mathfrak{G}(\boldsymbol{B})$ is equal to the zero element of $\boldsymbol{A}'$, i.e. to the zero element of $\boldsymbol{B}$. Hence it follows from 1.4, (6), that $\mathbf{C}a = -\mathbf{I}-a = --a = 1'$ (where $-$ denotes the Boolean complementation in $\boldsymbol{B}$).

§ 4. Finite pseudo-Boolean algebras. Suppose that $\boldsymbol{A}$ is a finite pseudo-Boolean algebra. By 3.1 we may suppose that $\boldsymbol{A} = \mathfrak{G}(\boldsymbol{B})$ where $\boldsymbol{B}$ is a finite topological Boolean algebra. It follows from III, § 8, that $\boldsymbol{B}$ can be interpreted as the topological field $\mathfrak{B}(X)$ of all subsets of a finite topological space X. Hence it follows that $\boldsymbol{A}$ can be represented as the algebra $\mathfrak{G}(X)$ of all open subsets of a finite topological space X. Conversely, if X is any finite topological space, then $\mathfrak{G}(X)$ is a finite pseudo-Boolean algebra.

However, the representation of finite pseudo-Boolean algebras as algebras of all open subsets of finite topological spaces is not important from the point of view of applications later in this book because such topological spaces are often singular (in general, they are not T_1-spaces). It is more important to represent them as subalgebras of the algebras of all open subsets of more natural topological spaces, e.g. of metric spaces. This representation is given by the following theorem:

4.1. *Let $X \neq 0$ be a dense-in-itself metric space, and let $\boldsymbol{A}$ be a non-degenerate finite pseudo-Boolean algebra. Then there exists a dense open set $G \subset X$ such that $\boldsymbol{A}$ is isomorphic to a subalgebra of $\mathfrak{G}(G)$.*

If, in addition, X is totally disconnected, we may assume $G = X$, i.e. $\boldsymbol{A}$ is then isomorphic to a subalgebra of $\mathfrak{G}(X)$ (1).

This theorem follows immediately from 3.1 and III, 8.3.

A finite pseudo-Boolean algebra $\boldsymbol{A}$ is said to be *strongly compact* provided the equation $V = a_1 \cup \ldots \cup a_n$ implies that $V = a_i$ for an $i \leqslant n$. By 3.1, $\boldsymbol{A} = \mathfrak{G}(\boldsymbol{B})$ where $\boldsymbol{B}$ is a finite topological Boolean algebra. It follows immediately from the definition that $\boldsymbol{A}$ is strongly compact if and only if $\boldsymbol{B}$ is strongly compact (see III, § 8, p. 110). Hence and from III, 8.2 we obtain the following theorem:

4.2. *Let G be an open non-empty subset of a dense-in-itself topological space X. For every non-degenerate finite strongly compact pseudo-Boolean algebra $\boldsymbol{A}$ there exists a pseudo-Boolean isomorphism h of $\boldsymbol{A}$ into $\mathfrak{B}(G)$ such that, for every $a \neq V$, $a \in \boldsymbol{A}$, the set $\mathbf{Ch}(a)$ contains the set-theoretical difference $\mathbf{C}G - G$* (2).

§ 5. Dense elements. An element a of a pseudo-Boolean algebra $\boldsymbol{A}$ is said to be *dense* (3) provided $-a = \Lambda$.

To explain the name "dense", let us consider the case where $\boldsymbol{A}$ is the algebra $\mathfrak{G}(\boldsymbol{B})$ of all open elements of a topological Boolean algebra $\boldsymbol{B}$. Then an element $a \in \boldsymbol{A}$ (i.e. an open element $a \in \boldsymbol{B}$) is dense in the sense just defined if and only if it is a dense element of the topological algebra $\boldsymbol{B}$ in the sense defined in III, § 1 (p. 95), i.e. if $\mathbf{C}a = V$. In fact, denoting the complement in $\boldsymbol{B}$ by $-$, we have

$$-a = \mathbf{I} - a = - \mathbf{C}a \quad \text{for every} \quad a \in \boldsymbol{A}$$

(see § 1, (6), and III, § 1, (1)). Thus, for every $a \in \boldsymbol{A}$,

$$-a = \Lambda \quad \text{if and only if} \quad \mathbf{C}a = V .$$

5.1. *An element a of a pseudo-Boolean algebra $\boldsymbol{A}$ is dense if and only if $--a = V$.*

If a is dense, then $-a = \Lambda$ and consequently $--a = -\Lambda = V$ (see I, 12.3, (26)). Conversely, if $--a = V$, then $-a = ---a = -V = \Lambda$ on account of I, 12.3, (30), (26).

5.2. *An element a of a pseudo-Boolean algebra $\boldsymbol{A}$ is dense if and only if $a \cap c \neq \Lambda$ for every element $c \neq \Lambda$, $c \in \boldsymbol{A}$.*

Indeed, $-a = \Lambda$ if and only if the zero element is the only element c such that $a \cap c = \Lambda$ (see the definition of $-a$ in I, § 10).

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(1) McKinsey and Tarski [2]; cf. also Tarski [8].

(2) McKinsey and Tarski [2].

(3) See Tarski [8].

5.3. *An element a of a pseudo-Boolean algebra $\boldsymbol{A}$ is dense if and only if it is of the form*

$$a = b \cup -b \tag{1}$$

where b is an element of $\boldsymbol{A}$.

In fact, if $-a = \wedge$, then a is of the form (1) for $b = a$. Conversely, for every element $b \in \boldsymbol{A}$,

$$-(b \cup -b) = -b \cap --b = \wedge$$

on account of I, 12.3, (31), (27).

Every pseudo-Boolean algebra has at least one dense element since the unit element is dense.

5.4. *A pseudo-Boolean algebra $\boldsymbol{A}$ is a Boolean algebra if and only if the unit element is the only dense element.*

By 5.3, the unit element is the only dense element if and only if

$$a \cup -a = \vee \quad \text{for every} \quad a \in \boldsymbol{A},$$

i.e. if the pseudo-complement $-a$ of a is the complement of a. By II, 1.1, and 1.2 this holds if and only if $\boldsymbol{A}$ is a Boolean algebra.

5.5. *If elements a, b are dense, then $a \cap b$ is also dense. If a is dense, and $a \leqslant b$, then b is dense.*

In other words:

The set of all dense elements in a pseudo-Boolean algebra $\boldsymbol{A}$ is a filter in $\boldsymbol{A}$.

Suppose that a and b are dense. Let $c \neq \wedge$. By 5.2, $a \cap c \neq \wedge$. By the same argument, $b \cap (a \cap c) \neq \wedge$, i.e. $(a \cap b) \cap c \neq \wedge$ for every $c \neq \wedge$. By 5.2 this proves that the element $a \cap b$ is dense.

If $a \leqslant b$, then $-b \leqslant -a$. Thus if $-a = \wedge$, then also $-b = \wedge$.

5.6. *If a and $a \Rightarrow b$ are dense, then b is also dense.*

This follows immediately from the second formulation of 5.5 and from I, 13.1.

5.7. *The element $--a \Rightarrow a$ is dense for every $a \in \boldsymbol{A}$.*

By I, 12.3, (33), (30) we have

$$--a \Rightarrow a \geqslant ---a \cup a = -a \cup a .$$

Since $-a \cup a$ is dense by 5.3, the element $--a \Rightarrow a$ is dense by 5.5.

5.8. *If a filter ∇ in a pseudo-Boolean algebra $\boldsymbol{A}$ contains all the dense elements, then $\boldsymbol{A}/\nabla$ is a Boolean algebra.*

By I, 13.2, $\boldsymbol{A}/\nabla$ is a pseudo-Boolean algebra. By II, 1.2, in order to prove that $\boldsymbol{A}/\nabla$ is Boolean it suffices to show that, for every $|a| \in \boldsymbol{A}/\nabla$,

the pseudo-complement $-|a|$ of $|a|$ is the complement of $|a|$. Indeed,

$$|a| \cup -|a| = |a| \cup |-a| = |a \cup -a| = V_{A|V}$$

since $a \cup -a \in V$ by 5.3.

§ 6. Regular elements. Let A be a pseudo-Boolean algebra.

An element $a \in A$ is said to be *regular* [1] provided $a = --a$.

To illustrate the notion of regular elements let us consider the case in which A is the algebra $\mathfrak{G}(B)$ of all open elements of a topological Boolean algebra B. Then an open element a is regular if and only if $a = \mathbf{IC}a$. Indeed, denoting the complement in B by $-$, we have

$$--a = \mathbf{I}-\mathbf{I}-a = \mathbf{IC}a \quad \text{for every open } a$$

(see III, § 1, (1)).

For instance, if $A = \mathfrak{G}(X)$ where X is the set of all real numbers, then each of the infinite open intervals $(-\infty, 0)$, $(0, \infty)$ is regular, but their union is not regular.

6.1. *An element a is regular if and only if it is of the form*

$$a = -b \tag{1}$$

where b is an element in A.

In fact, if $a = --a$, then $a = -b$ for $b = -a$. Conversely, if (1) holds, then $--a = ---b = -b = a$ by I, 12.3, (30).

6.2. *If a and b are regular, then $a \cap b$ is also regular. If b is regular, then $a \Rightarrow b$ is also regular for every a.*

In fact, if a and b are regular, i.e. $a = -a'$ and $b = -b'$ for some $a', b' \in A$, then $a \cap b = -(a' \cup b')$ by I, 12.3, (31). Thus $a \cap b$ is regular by 6.1.

If b is regular, i.e. $b = -b'$, then $a \Rightarrow b = -(a \cap b')$ by I, 12.3, (35). Thus $a \Rightarrow b$ is regular.

6.3. *For all elements a, b the elements $a \Rightarrow -b$ and $-b$ are regular.*

This follows immediately from 6.1 and 6.2 (or from I, 12.3, (30) and (36)).

6.4. *For every element a, the element $--a$ is the least regular element b such that $a \leqslant b$.*

In fact, $a \leqslant --a$ by I, 12.3, (29). On the other hand, if $a \leqslant b$ and b is regular, i.e. $b = -b'$, then $-b \leqslant -a$ and consequently

$$--a \leqslant --b = ---b' = -b' = b$$

by I, 12.3, (30). This proves 6.4.

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[1] For an investigation of regular elements, see McKinsey and Tarski [2].

For every pseudo-Boolean algebra A the symbol $\mathfrak{R}(A)$ will denote the set of all regular elements in A. This set is not empty since it contains the unit element and the zero element of A. We shall consider $\mathfrak{R}(A)$ to be an ordered set, the ordering relation in $\mathfrak{R}(A)$ being that in A. In general, $\mathfrak{R}(A)$ is not any sublattice of A because the join of two regular elements is not always regular (see the example quoted before theorem 6.1).

However, the ordered set $\mathfrak{R}(A)$ is a lattice. Indeed, for any given elements $a, b \in \mathfrak{R}(A)$, the element $a \cap b$ is the greatest lower bound of a, b in the ordered set $\mathfrak{R}(A)$ (see the first part of 6.2), and the element $--(a \cup b)$ is the least upper bound of a, b in the ordered set $\mathfrak{R}(A)$ (see 6.4). Of course, the unit element and the zero element of A are also the unit element and the zero element of $\mathfrak{R}(A)$, respectively. Since $a \Rightarrow b \in \mathfrak{R}(A)$ for $a, b \in \mathfrak{R}(A)$ (see the second part of 6.2) and since the meet in $\mathfrak{R}(A)$ is the restriction of the meet in A, the element $a \Rightarrow b$ is the pseudo-complement of $a \in \mathfrak{R}(A)$ relative to $b \in \mathfrak{R}(A)$ in the lattice $\mathfrak{R}(A)$. Taking $b = \bigwedge$ we assert that the pseudo-complement $-a$ of $a \in \mathfrak{R}(A)$ in A is also the pseudo-complement of a in the lattice $\mathfrak{R}(A)$. Thus $\mathfrak{R}(A)$ is a pseudo-Boolean algebra with operations $\cap$, $\Rightarrow$, $-$ the same as in A and with a join different than in A, in general. The join in $\mathfrak{R}(A)$ will be denoted by $\cup^*$ to distinguish it from the join $\cup$ in A. As we have observed,

$$a \cup^* b = --(a \cup b) \quad \text{for} \quad a, b \in \mathfrak{R}(A). \tag{2}$$

If $a, b \in \mathfrak{R}(A)$, then $a = -a'$, $b = -b'$ where $a' = -a$, $b' = -b$. Consequently

$$--(a \cup b) = --(-a' \cup -b') = -(--a' \cap --b') = -(-a \cap -b)$$

because of I, 12.3, (31), (30). Thus the definition (2) of $\cup^*$ can also be written in the form

$$a \cup^* b = -(-a \cap -b) \quad \text{for} \quad a, b \in \mathfrak{R}(A). \tag{3}$$

We shall now prove that

6.5. *$\mathfrak{R}(A)$ is a Boolean algebra.*

By II, 1.2, it suffices to prove that the pseudo-complement $-a$ of any element $a \in \mathfrak{R}(A)$ is the complement of a in $\mathfrak{R}(A)$, i.e., that

$$a \cup^* -a = \bigvee.$$

By (2), we have to prove that $--(a \cup -a) = \bigvee$. This equation holds by 5.1 since the element $a \cup -a$ is dense by 5.3.

6.6. *The unit element is the only dense regular element.*

In fact, both equations $--a = \mathsf{V}$ and $a = ---a$ hold simultaneously if and only if $a = \mathsf{V}$.

6.7. *Let V be the filter of all dense elements in $\boldsymbol{A}$. The mapping*

$$h(a) = |a| \in \boldsymbol{A}/\mathsf{V} \qquad (a \in \mathfrak{R}(\boldsymbol{A}))$$

is a Boolean isomorphism from the Boolean algebra $\mathfrak{R}(\boldsymbol{A})$ onto the Boolean algebra $\boldsymbol{A}/\mathsf{V}$.

We recall that $\boldsymbol{A}/\mathsf{V}$ is a Boolean algebra on account of 5.8.

First we shall prove that h maps $\mathfrak{R}(\boldsymbol{A})$ onto $\boldsymbol{A}/\mathsf{V}$. For this purpose it suffices to show that, for every element $|b| \in \boldsymbol{A}/\mathsf{V}$, there exists a regular element a such that $|b| = |a|$. The element $a = --b$ has this property. In fact, since $b \leqslant --b$ (see I, 12.3, (29)), we have $b \Rightarrow --b = \mathsf{V} \in \mathsf{V}$. By 5.7, $--b \Rightarrow b \in \mathsf{V}$. Thus $|b| = |--b|$.

It is obvious that $a \leqslant a'$ implies $h(a) \leqslant h(a')$. Conversely, if $h(a) \leqslant h(a')$, then $a \leqslant a'$. In fact, we have then $|a \Rightarrow a'| = |a| \Rightarrow |a'| = \mathsf{V}$, i.e. $a \Rightarrow a'$ is dense. Since $a \Rightarrow a'$ is regular by the second part of 6.2, we infer from 6.6 that $a \Rightarrow a' = \mathsf{V}$. This proves $a \leqslant a'$.

Since for all $a, a' \in \mathfrak{R}(\boldsymbol{A})$

$$a \leqslant a' \quad \text{if and only if} \quad h(a) \leqslant h(a'),$$

h is an isomorphism (see I, 6.2).

§ 7. Infinite joins and meets. We have proved in I, 12.1, that the finite distributive laws (l_5) hold in every relatively pseudo-complemented lattice and consequently also in every pseudo-Boolean algebra. Also one of the infinite distributive laws always holds:

7.1. *If an infinite join $\bigcup_{t \in T} b_t$ exists in a relatively pseudo-complemented lattice $\boldsymbol{A}$, then for every $a \in \boldsymbol{A}$ the join $\bigcup_{t \in T} a \cap b_t$ also exists in $\boldsymbol{A}$ and*

$$a \cap \bigcup_{t \in T} b_t = \bigcup_{t \in T} a \cap b_t . \tag{1}$$

This follows immediately from I, 11.2.

Observe that the dual identity

$$a \cup \bigcap_{t \in T} b_t = \bigcap_{t \in T} a \cup b_t \tag{1'}$$

does not hold, in general, in pseudo-Boolean algebras. For instance, let $\boldsymbol{A} = \mathfrak{G}(X)$ where X is the set of all real numbers, let T be the set of all positive integers, let b_t be the open interval $-1/t < \xi < 1/t$ and let a be the union of the open intervals $-1 < \xi < 0$ and $0 < \xi < 1$. Then the infinite meet of all b_t in the lattice $\boldsymbol{A}$ is the empty set, consequently $a \cup \bigcap^{\boldsymbol{A}}_{t \in T} b_t = a$. On the other hand, $a \cup b_t$ is the open interval $-1 < \xi < 1$ for every $t \in T$, and consequently the infinite meet of all $a \cup b_t$ in the lattice $\boldsymbol{A}$ is equal to this interval. Thus $a \cup \bigcap^{\boldsymbol{A}}_{t \in T} b_t \neq \bigcap^{\boldsymbol{A}}_{t \in T} a \cup b_t$.

The following two theorems will be useful in Chapter X, § 6.

7.2. *In every relatively pseudo-complemented lattice* A,

$$a \cup \bigcap_{t\in T} a_t \leqslant \bigcap_{t\in T} (a \cup b_t)\,, \tag{2}$$

$$\bigcup_{t\in T} (a_t \Rightarrow b) \leqslant (\bigcap_{t\in T} a_t) \Rightarrow b\,, \tag{3}$$

$$\bigcup_{t\in T} (a \Rightarrow b_t) \leqslant a \Rightarrow \bigcup_{t\in T} b_t\,, \tag{4}$$

$$\bigcap_{t\in T} (a_t \Rightarrow b_t) \leqslant (\bigcap_{t\in T} a_t) \Rightarrow (\bigcap_{t\in T} b_t)\,, \tag{5}$$

$$\bigcap_{t\in T} (a_t \Rightarrow b_t) \leqslant (\bigcup_{t\in T} a_t) \Rightarrow (\bigcup_{t\in T} b_t)\,, \tag{6}$$

provided all the infinite joins and meets in question exist.

If the join $\bigcup_{t\in T} a_t$ *exists in* A, *then the meet* $\bigcap_{t\in T} (a_t \Rightarrow b)$ *exists for every* $b \in A$ *and*

$$\bigcap_{t\in T} (a_t \Rightarrow b) = (\bigcup_{t\in T} a_t) \Rightarrow b\,. \tag{7}$$

If the meet $\bigcap_{t\in T} b_t$ *exists in* A, *then the meet* $\bigcap_{t\in T} (a \Rightarrow b_t)$ *exists for every* $a \in A$ *and*

$$\bigcap_{t\in T} (a \Rightarrow b_t) = a \Rightarrow \bigcap_{t\in T} b_t\,. \tag{8}$$

(2) follows directly from the inequality

$$a \cup \bigcap_{t\in T} b_t \leqslant a \cup b_t \quad \text{for every} \quad t \in T\,.$$

Proof of (3). By § 1, (l_7), we have $a_t \cap (a_t \Rightarrow b) \leqslant b$. Hence $(\bigcap_{t\in T} a_t) \cap (a_t \Rightarrow b) \leqslant b$, i.e. $a_t \Rightarrow b \leqslant (\bigcap_{t\in T} a_t) \Rightarrow b$ for every $t \in T$. The last inequality implies (3) directly.

Proof of (4). Since $a \cap (a \Rightarrow b_t) \leqslant b_t$ (see § 1, (l_7)), we have $a \cap (a \Rightarrow b_t) \leqslant \bigcup_{t\in T} b_t$, i.e. $a \Rightarrow b_t \leqslant a \Rightarrow \bigcup_{t\in T} b_t$ for every $t \in T$. This implies (4).

Proof of (5). Since $a_t \cap (a_t \Rightarrow b_t) \leqslant b_t$, we have (see I, 7.1, (8), (15))

$$\bigcap_{t\in T} a_t \cap \bigcap_{t\in T} (a_t \Rightarrow b_t) \leqslant \bigcap_{t\in T} b_t\,,$$

which implies (5) by § 1, (1).

Proof of (6). Since $a_t \cap (a_t \Rightarrow b_t) \leqslant b_t$, we have $a_t \cap \bigcap_{t\in T} (a_t \Rightarrow b_t) \leqslant b_t$ for every $t \in T$. Hence by (1) and I, 7.1, (7), $(\bigcup_{t\in T} a_t) \cap \bigcap_{t\in T} (a_t \Rightarrow b_t) \leqslant \bigcup_{t\in T} b_t$, which implies (6).

Proof of (7). Writing $a = \bigcup_{t\in T} a_t$, we have $a_t \leqslant a$ and consequently $a \Rightarrow b \leqslant a_t \Rightarrow b$ for every $t \in T$, by I, 12.2, (11). Conversely, if $c \leqslant a_t \Rightarrow b$ for every $t \in T$, then $a_t \cap c \leqslant b$ and consequently $a \cap c \leqslant b$ by (1) and I, 7.1, (4). This implies $c \leqslant a \Rightarrow b$, which completes the proof of (7).

Proof of (8). Writing $b = \bigcap_{t\in T} b_t$, we have $a \Rightarrow b \leqslant a \Rightarrow b_t$ for every $t \in T$, by I, 12.2, (12). Conversely, if $c \leqslant a \Rightarrow b_t$ for every $t \in T$, i.e. $a \cap c \leqslant b_t$, then $a \cap c \leqslant b$, i.e. $c \leqslant a \Rightarrow b$. This proves (8).

Note that (7) and (8) are generalizations of identities I, 12.2, (17) and (16) respectively.

7.3. *In every pseudo-Boolean algebra* A,

$$\bigcup_t -a_t \leqslant -\bigcap_{t\in T} a_t \tag{9}$$

provided the join and the meet exist. If the join $\bigcup_{t\in T} a_t$ *exists in* A, *then also the meet* $\bigcap_{t\in T} -a_t$ *exists and*

$$\bigcap_{t\in T} -a_t = -\bigcup_{t\in T} a_t\,. \tag{10}$$

Proof of (9). Since $\bigcap_{t\in T} a_t \leqslant a_t$, we have $-a_t \leqslant -\bigcap_{t\in T} a_t$ for every $t \in T$, which implies (9).

Proof of (10). Writing $a = \bigcup_{t\in T} a_t$, we have $a_t \leqslant a$ and consequently $-a \leqslant -a_t$ for every $t \in T$. On the other hand, if $c \leqslant -a_t$ for every $t \in T$, then $c \cap a_t = \Lambda$ by the definition of pseudo-complement. Thus $c \cap a = \Lambda$ by (1), i.e. $c \leqslant -a$. This proves (10).

(9) and (10) are generalizations of identities I, 12.3, (32) and (31) respectively.

7.4. *Let* B *be a topological Boolean algebra,* $A = \mathfrak{G}(B)$, *and* $a_t \in A$ *for every* $t \in T$.

The join $\bigcup^A_{t\in T} a_t$ *exists if and only if the join* $\bigcup^B_{t\in T} a_t$ *exists. Then*

$$\bigcup^A_{t\in T} a_t = \bigcup^B_{t\in T} a_t\,.$$

If the meet $\bigcap^B_{t\in T} a_t$ *exists, then the meet* $\bigcap^A_{t\in T} a_t$ *also exists and*

$$\bigcap^A_{t\in T} a_t = \mathbf{I}\bigcap^B_{t\in T} a_t\,.$$

Suppose that $a = \bigcup^A_{t\in T} a_t$ exists and that $a_t \leqslant b \in B$ for every $t \in T$. Then $a_t = \mathbf{I}a_t \leqslant \mathbf{I}b \in A$. Hence $a \leqslant \mathbf{I}b$ and consequently $a \leqslant b$, which proves that $a = \bigcup^B_{t\in T} a_t$.

Suppose now that $a = \bigcup^B_{t\in T} a_t \in B$ exists. The join of any number of open elements being open, we have $a \in A$, which implies $a = \bigcup^A_{t\in T} a_t$.

Suppose that $b = \bigcap^B_{t\in T} a_t$ exists, $b \in B$. Since $a_t \geqslant b$, we have $\mathbf{I}a_t \geqslant \mathbf{I}b$, i.e. $a_t \geqslant \mathbf{I}b$ for every $t \in T$. On the other hand, if $a_t \geqslant a \in A$ for every $t \in T$, then $b \geqslant a$ and consequently $\mathbf{I}b \geqslant \mathbf{I}a = a$. This proves that $\mathbf{I}b = \bigcap^A_{t\in T} a_t$.

7.5. *If* B *is a complete topological Boolean algebra, then the pseudo-Boolean algebra* $A = \mathfrak{G}(B)$ *is complete lattice and, for any indexed set* $a_t \in A$, $t \in T$,

$$\bigcup^A_{t\in T} a_t = \bigcup^B_{t\in T} a_t\,, \qquad \bigcap^A_{t\in T} a_t = \mathbf{I}\bigcap^B_{t\in T} a_t\,.$$

In particular, $\mathfrak{G}(X)$ *is complete for every topological space* X.

Theorem 7.5 follows directly from 7.4.

§ 8. Relativization. Let g be a fixed element in a pseudo-Boolean algebra A. According to I, § 11, p. 57, the symbol A_g will denote the sublattice of all $a \in A$ such that $a \leqslant g$. We recall that the ordering relation $\leqslant$ and operations $\cup, \cap$ in A_g are the same as in A. The zero element of A is also the zero element of A_g, the element g is the unit element of A_g. The following theorem is an analogue to II, 6.1,

8.1. *The lattice A_g is pseudo-Boolean. For every element $a \in A_g$, the element $g \cap -a$ is the pseudo-complement of a in A_g. For all elements $a, b \in A_g$, the element $g \cap (a \Rightarrow b)$ is the pseudo-complement of a relative to b in A_g.*

Of course, $-$ and $\Rightarrow$ here denote operations in A.

To prove 8.1 let us observe that for arbitrary $x, a, b \in A_g$

$$x \cap a \leqslant b \quad \text{if and only if} \quad x \leqslant g \cap (a \Rightarrow b).$$

This proves that $g \cap (a \Rightarrow b)$ is the pseudo-complement of a relative to b in the lattice A_g. Thus A_g is pseudo-Boolean. Assuming $b = \wedge$ we infer that $g \cap -a$ is the pseudo-complement of $a \in A_g$ in the lattice A_g.

Note that A_g is never a pseudo-Boolean subalgebra of A, except for the trivial case in which $g = \vee$.

If A is the algebra $\mathfrak{G}(X)$ of all open subsets of a topological space X and G is an open subset of X, then A_G is the algebra $\mathfrak{G}(G)$ of all open subsets of G.

8.2. *If A is a pseudo-Boolean algebra and $g \in A$, then the mapping*

$$h_0(a) = g \cap a \tag{1}$$

is a pseudo-Boolean homomorphism from the pseudo-Boolean algebra A onto the pseudo-Boolean algebra A_g. The homomorphism (1) *preserves all infinite joins and meets in A.*

Since A is distributive, we have

$$h_0(a \cup b) = g \cap (a \cup b) = (g \cap a) \cup (g \cap b) = h_0(a) \cup h_0(b).$$

Similarly

$$h_0(a \cap b) = g \cap (a \cap b) = (g \cap a) \cap (g \cap b) = h_0(a) \cap h_0(b).$$

By I, 12.2, (24),

$$h_0(a \Rightarrow b) = g \cap \big(h_0(a) \Rightarrow h_0(b)\big)$$

and the right-hand side of this equation is the pseudo-complement of $h_0(a)$ relative to $h_0(b)$ in the lattice A_g (see 8.1). This proves that h_0 also preserves the operation $\Rightarrow$. Since $h_0(\wedge) = \wedge$, h_0 is a pseudo-Boolean homomorphism. It is obvious that h_0 maps A onto A_g.

To prove the second part of 8.2, let us first observe that for every indexed set $\{a_t\}_{t\in T}$ of elements in A_g

$$\bigcup^{A}_{t\in T} a_t = \bigcup^{A_g}_{t\in T} a_t\,, \qquad \bigcap^{A}_{t\in T} a_t = \bigcap^{A_g}_{t\in T} a_t \tag{2}$$

which justifies the avoidance of superscripts at $\bigcup$ and $\bigcap$. More precisely, (2) should be understood as follows: if the join (meet) on one side exists, then the join (meet) on the second side also exists and the equation holds.

To prove that h_0 preserves all infinite joins and meets, it suffices to show that

$$g \cap \bigcup_{t\in T} a_t = \bigcup_{t\in T} (g \cap a_t)\,, \qquad g \cap \bigcap_{t\in T} a_t = \bigcap_{t\in T} (g \cap a_t)$$

provided the join or meet on the left-hand side exists. This follows from the distributive law 7.1 and from I, 7.1, (10).

It is not difficult to verify that the mapping

$$h(a) = |a| \qquad (a \in A_g) \tag{3}$$

is an isomorphism from A_g onto A/V where V is the principal filter generated by the element g.

§ 9. Imbeddings and extensions of pseudo-Boolean algebras. The following theorem is an analogue of III, 4.2:

9.1. *For every pseudo-Boolean algebra A there exists a complete pseudo-Boolean algebra A^* and an isomorphism of A into A^*, preserving all infinite joins and meets in A* (1).

By theorem 3.1, we can assume that $A = \mathfrak{G}(B)$ where B is a topological Boolean algebra. By theorem III, 4.2, there exists a complete topological Boolean algebra B^* and a topological isomorphism h of B into B^* such that h preserves all infinite joins and meets in B, and the elements $h(a)$, where $a \in A$, constitute an open basis for B^*. By 7.5, the pseudo-Boolean algebra $A^* = \mathfrak{G}(B^*)$ is complete. By 2.1, the isomorphism h restricted to A is a pseudo-Boolean isomorphism of A into A^*. We shall prove that h preserves all infinite joins and meets in A.

Suppose $a = \bigcup^{A}_{t\in T} a_t$ $(a, a_t \in A)$. By 7.4, $a = \bigcup^{B}_{t\in T} a_t$. Hence $h(a) = \bigcup^{B^*}_{t\in T} h(a_t)$. By theorem 7.4, $h(a) = \bigcup^{A^*}_{t\in T} h(a_t)$.

Now suppose that $a = \bigcap^{A}_{t\in T} a_t$ $(a, a_t \in A)$. We have

$$a \leqslant a_t \quad \text{for} \quad \text{every} \quad t \in T\,. \tag{1}$$

Hence $h(a) \leqslant h(a_t)$ for every $t \in T$. Let $b = \bigcap^{A^*}_{t\in T} h(a_t)$. By (1), $h(a) \leqslant b$. On the other hand, $b = \bigcup^{B}_{s\in S} h(a'_s)$ for some elements $a'_s \in A$ since b is

(1) Rasiowa [2].

open and elements $h(c)$, $c \in A$, constitute an open basis for B^*. Since $h(a'_s) \leqslant h(a_t)$, we have $a'_s \leqslant a_t$ for every $s \in S$ and $t \in T$. Hence $a'_s \leqslant a$ and, consequently, $h(a'_s) \leqslant h(a)$ for every $s \in S$. This proves that $b \leqslant h(a)$. Thus $b = h(a)$, i.e. $h(a) = \bigcap^{A^*}_{t \in T} h(a_t)$, q. e. d.

Let A be a pseudo-Boolean algebra and let

$$a_s = \bigcup^{A}_{t \in T_s} a_{s,t}\,, \qquad s \in S\,, \tag{q}$$

be a given set of infinite joins in A. Let X be a topological space. A pseudo-Boolean homomorphism (isomorphism) of A into $\mathfrak{G}(X)$ is said to be an $(\mathrm{q}, \cap)$-*homomorphism* ($(\mathrm{q}, \cap)$-*isomorphism*) provided

$$h(a_s) = \bigcup_{t \in T_s} h(a_{s,t}) \quad \text{for every} \quad s \in S, \tag{2}$$

$$h(b) = \mathbf{I} \bigcap_{t \in T} h(b_t) \tag{3}$$

for every indexed set $b_t \in A$ ($t \in T$) such that the meet $b = \bigcap^{A}_{t \in T} b_t$ exists. The signs $\bigcup$ and $\bigcap$ on the right-hand side of (2) and (3) denote the set-theoretical union and intersection, respectively.

9.2. *Every pseudo-Boolean algebra A is isomorphic to a subalgebra of the algebra of open subsets of a topological space, i.e. there exist a topological space X and a pseudo-Boolean isomorphism h of A into $\mathfrak{G}(X)$* [1]. *We can assume, in addition, that the sets $h(a)$, where $a \in A$, constitute a basis for X, and that the cardinal of X is $\leqslant 2^{\mathfrak{m}}$ provided the cardinal of A is $\leqslant \mathfrak{m}$ ($\mathfrak{m} \geqslant \aleph_0$).*

Moreover, given any enumerable set (q) *of infinite joins in A, we can also assume that h is a $(\mathrm{q}, \cap)$-isomorphism of A into $\mathfrak{G}(X)$* [2].

By 3.1 we may assume that $A = \mathfrak{G}(B)$ where B is a topological Boolean algebra such that the cardinal of B is $\leqslant \mathfrak{m}$. By 7.4 we have

$$a_s = \bigcup^{B}_{t \in T_s} a_{s,t} \quad \text{for} \quad s \in S\,. \tag{4}$$

By III, 4.3, there exists a topological space X of a cardinal $\leqslant 2^{\mathfrak{m}}$, and a topological isomorphism h of B into $\mathfrak{B}(X)$. Moreover, we can assume that

$$h(a_s) = \bigcup^{\mathfrak{B}(X)}_{t \in T_s} h(a_{s,t}) \quad \text{for every} \quad s \in S\,, \tag{5}$$

and that the class of all sets $h(a)$, $a \in A$, constitute a basis for X. It follows from the last property that, for any $b_t \in A$, $t \in T$,

$$b = \bigcap^{A}_{t \in T} b_t \quad \text{implies} \quad h(b) = \mathbf{I} \bigcap^{\mathfrak{B}(X)}_{t \in T} h(b_t)\,.$$

The exact proof of this implication is analogous to the corresponding part in the proof of 9.1. By 7.4, (5) implies

$$h(a) = \bigcup^{\mathfrak{G}(X)}_{t \in T_s} h(a_{s,t}) \quad \text{for every} \quad s \in S\,.$$

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[1] McKinsey and Tarski [2].

[2] Rasiowa and Sikorski [3].

Thus, the isomorphism h, restricted to A, is a pseudo-Boolean $(\mathrm{q}, \cap)$-isomorphism of A into $\mathfrak{G}(X)$ (see 2.1).

Theorem 9.2 just proved is an analogue of theorem III, 4.3. The next theorem 9.3 is an analogue of theorem III, 4.4.

9.3. *Let A be a pseudo-Boolean algebra, let $A_0 \subset A$ be a finite set containing r elements. Then there exists a finite pseudo-Boolean algebra A' containing at most 2^{2^r} elements, such that $V \in A'$, $A_0 \subset A'$ and the operations in A' are extensions of operations in $A'_0 = A_0 \cup (V)$, i.e. for any $a, b, c \in A'_0$:*

(a) *if c is the join of a, b in A, then c is the join of a, b in A';*

(b) *if c is the meet of a, b in A, then c is the meet of a, b in A';*

(c) *if c is the pseudo-complement of a relative to b in A, then c is the pseudo-complement of a relative to b in A';*

(d) *if c is the pseudo-complement of a in A, then c is the pseudo-complement of a in A'* [1].

By theorem 3.1 we may assume that $A = \mathfrak{G}(B)$ where B is a topological Boolean algebra. Let B' be the Boolean subalgebra of B generated by A_0. By II, 2.3, B' has at most 2^{2^r} elements. By III, 4.4, there exists an interior operation $\mathbf{I}'$ in B' such that, for every $c \in B'$,

$$\text{if } \mathbf{I}c \in A_0, \text{ then } \mathbf{I}'c = \mathbf{I}c. \tag{6}$$

We shall consider B' as a topological Boolean algebra with the interior operation $\mathbf{I}'$. Let $A' = \mathfrak{G}(B')$. Denote the relative pseudo-complement and pseudo-complement in A' by $\Rightarrow'$ and $-'$ respectively, i.e.

$$a \Rightarrow' b = \mathbf{I}'(-a \cup b) \quad \text{and} \quad -'a = \mathbf{I}'-a \quad \text{for} \quad a, b \in A'$$

where $-$ denotes the Boolean complementation in B. If $a, b, c \in A'_0$ and $c = a \Rightarrow b = \mathbf{I}(-a \cup b)$, then, by (6), $c = \mathbf{I}'(-a \cup b) = a \Rightarrow' b$, which proves (c). The proof of (d) is similar. (a) and (b) follows from the fact that A and A' are sublattices of B.

§ 10. Enumerable pseudo-Boolean algebras. The following theorem is analogue of theorem III, 10.1:

10.1. *For every enumerable pseudo-Boolean algebra A and for every given enumerable set* (q) *of infinite joins in A there exist a set X_0 of irrational numbers and a pseudo-Boolean* $(\mathrm{q}, \cap)$*-isomorphism of A into $\mathfrak{G}(X_0)$* [2].

The proof is similar to that of III, 10.1.

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[1] McKinsey and Tarski [2].

[2] Sikorski [7].

By 9.2 there exist a separable topological space X with a cardinal $\leqslant 2^{\aleph_0}$ and a $(\mathrm{q}, \cap)$-isomorphism h of $\boldsymbol{A}$ into $\mathfrak{G}(X)$. By III, 9.4, there exist a set X_0 of irrational numbers and an interior mapping φ of X_0 onto X. Since φ^{-1} preserves all infinite set-theoretical operations in $\mathfrak{B}(X)$ and the interior operation $\mathbf{I}$ (see III, 3.2), it follows from 2.2 and 7.5 that the pseudo-Boolean isomorphism h_1 of $\mathfrak{G}(X)$ into $\mathfrak{G}(X_0)$,

$$h_1(A) = \varphi^{-1}(A) \quad \text{for} \quad A \in \mathfrak{G}(X),$$

preserves all infinite joins and meets in $\mathfrak{G}(X)$. Consequently

$$h_0(a) = \varphi^{-1}\big(h(a)\big) \quad \text{for} \quad a \in \boldsymbol{A}$$

is a pseudo-Boolean $(\mathrm{q}, \cap)$-isomorphism of $\boldsymbol{A}$ into $\mathfrak{G}(X_0)$.

§ 11. Products of pseudo-Boolean algebras. Let $\{A_n\}_{n\in N}$ be an indexed set of pseudo-Boolean algebras conceived as abstract algebras

$$\{\boldsymbol{A}_n, \cup, \cap, \Rightarrow, -\} \tag{1}$$

according to § 1, p. 124. By the general definition in I, § 4, p. 27, the product $\boldsymbol{A}$ of all $\boldsymbol{A}_n$ is the algebra

$$\{\boldsymbol{A}, \cup, \cap, \Rightarrow, -\} \tag{2}$$

where $\boldsymbol{A} = \boldsymbol{P}_{n\in N} \boldsymbol{A}_n$ and the operation in $\boldsymbol{A}$ are defined as follows:

$$\begin{aligned} \{a_n\}_{n\in N} \cup \{b_n\}_{n\in N} &= \{a_n \cup b_n\}_{n\in N}, \\ \{a_n\}_{n\in N} \cap \{b_n\}_{n\in N} &= \{a_n \cap b_n\}_{n\in N}, \\ \{a_n\}_{n\in N} \Rightarrow \{b_n\}_{n\in N} &= \{a_n \Rightarrow b_n\}_{n\in N}, \\ -\{a_n\}_{n\in N} &= \{-a_n\}_{n\in N}. \end{aligned} \tag{3}$$

11.1. *The product* (2) *of pseudo-Boolean algebras* (1) *is a pseudo-Boolean algebra.*

The proof is similar to the proof of the analogous theorem I, 6.3, for lattices, II, 7.1, for Boolean algebras and III, 13.1, for topological Boolean algebras.

Note that the lattice $\{\boldsymbol{A}, \cup, \cap\}$ is then the product of the lattices $\{\boldsymbol{A}_n, \cup, \cap\}$ (see I, § 6, p. 38, and I, § 7, p. 43). Consequently the statement in I, § 6, (14), and theorem I, 7.2, are also valid for products of pseudo-Boolean algebras. In particular, if all $\boldsymbol{A}_n$ are complete Boolean algebras, their product $\boldsymbol{A}$ is also.

11.2. *If, for every* $n \in N$, $\boldsymbol{A}_n$ *is the pseudo-Boolean algebra* $\mathfrak{G}(\boldsymbol{B}_n)$ *of all open elements in a topological Boolean algebra* $\boldsymbol{B}_n$, *then the product* $\boldsymbol{A} = \boldsymbol{P}_{n\in N} \boldsymbol{A}_n$ *is the pseudo-Boolean algebra* $\mathfrak{G}(\boldsymbol{B})$ *of all open elements in the product* $\boldsymbol{B} = \boldsymbol{P}_{n\in N} \boldsymbol{B}_n$ *of all topological Boolean algebras* $\boldsymbol{B}_n$.

This follows directly from III, 13.2, and from the identities § 1, (5), (6).

The following theorem results immediately from 11.2 and III, 13.3:

11.3. *If, for every* $n \in N$, $\boldsymbol{A}_n$ *is the pseudo-Boolean algebra* $\mathfrak{G}(X_n)$ *of all open subsets of a topological space* X_n, *and all the spaces* X_n *are disjoint, then the mapping*

$$h(\{A_n\}_{n\in N}) = \bigcup_{n\in N} A_n \tag{4}$$

(where $A_n \in \boldsymbol{A}_n$ *for every* $n \in N$ *and* $\bigcup$ *denotes the set-theoretical union) is a pseudo-Boolean isomorphism from the product* $\boldsymbol{A} = \boldsymbol{P}_{n\in N} \boldsymbol{A}_n$ *onto the pseudo-Boolean algebra* $\mathfrak{G}(X)$ *of all open subsets of the direct union* X *of all the topological spaces* X_n.

Isomorphism (4) yields a natural set-theoretical representation for the pseudo-Boolean algebra $\boldsymbol{A} = \boldsymbol{P}_{n\in N} \boldsymbol{A}_n$. Consequently we can identify the product $\boldsymbol{A}$ with the pseudo-Boolean algebra $\mathfrak{G}(X)$ by identification of $\{A_n\}_{n\in N}$ with the union $\bigcup_{n\in N} A_n$.

PART TWO

CLASSICAL LOGIC

CHAPTER V

FORMALIZED MATHEMATICAL THEORIES

§ 1. The concept of formalized theories. Until recently, till the end of the 19th century, mathematical theories used to be built in an intuitive or axiomatic way. In other words, they were based either on intuitive ideas concerning the basic notions of the theory—ideas taken from reality—or on the properties of these notions expressed in systems of axioms.

The historical development of mathematics has shown that it is not sufficient to base theories on an intuitive understanding of their notions only. This fact was especially obvious in set theory. The basic concept of this theory, *set*, is certainly taken from reality, for there we come across many examples of various sets, all of which are finite. But in mathematics it is also necessary to consider various infinite sets, such as the set of all integers, the set of all rational numbers, the set of all real numbers, the set of all segments, the set of all triangles, the set of all subsets of a given set, etc. It also seems reasonable to consider concepts such as the set of all topological spaces, the set of all sets, the set of all objects. However, a simple reasoning indicates that it is necessary to impose some limitations on the concept of set. For instance, the notion of the set Z of all sets A such that A is not an element of A, leads to a contradiction. In fact, let us consider whether the set Z is or is not an element of itself. Suppose that Z is an element of itself. It follows from this hypothesis that Z is not an element of itself, since the set Z is composed only of sets which are not elements of themselves. Similarly, the opposite hypothesis, i.e. that Z is not an element of itself, implies that Z is an element of itself, since according to the definition the set Z is composed of all the sets which are not elements of themselves. This paradox [1] arises because the concept of set was not precisely enough defined, and was too liberally interpreted. The example above shows that it is necessary to consider the set of all sets which are

[1] Discovered by B. Russell. For other examples of paradoxes in the intuitive set theory, see e.g. Fraenkel [2].

not elements of themselves as a notion devoid of sense; it is impossible to admit it in set theory. On the other hand, the need to consider such pathological sets does not arise in mathematics.

The paradox described above, as well as many similar ones occurring in intuitive set theory cannot be avoided by referring to intuition only. By characterizing the intuitive concept of set by a suitable system of axioms we obtain an axiomatic set theory without such antinomies. The problem arises what set of axioms should be chosen in order to obtain a sufficiently rich theory of sets. To discuss this question is not the purpose of this book. An example of a system of axioms for set theory will be given in § 13. The theory based on that system is, in practice, sufficient to include all actual mathematical theories.

The axiomatic method is the first step leading to greater precision in the construction of mathematical theories. In intuitive mathematical theories the borderline between that which is obvious and that which requires proof is not exact. In axiomatic theories a system of primitive notions is assumed which are characterized by a set of axioms. Other notions can be defined by means of the primitive notions. All statements which are consequences of the axioms are called theorems of the theory. All properties of any notion of the theory which are not expressed in the axioms, require proof.

Until recently, this degree of exactness in the construction of mathematical theories seemed sufficient. However, it turned out that the assumption of a consistent set of axioms does not prevent the occurrence of another kind of paradoxes.

For instance, let us consider the arithmetic based on the well-known system of axioms due to Peano. Let A denote the set of all positive integers which can be defined in the English language by means of a sentence containing at most 1000 letters. The set A is finite since the set of all sentences containing at most 1000 letters is finite. Hence, there exist positive integers which do not belong to A. The sentence

(*) *n is the least positive integer which cannot be defined by means of a sentence of the English language containing at most* 1000 *letters*

contains less than 1000 letters and defines a positive integer n. Therefore n belongs to A. On the other hand, n does not belong to A by the definition of n. This contradicts the first statement.

It is obvious that the reason for this paradox [1] is that in its con-

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[1] Discovered by G. G. Berry. For other examples of paradoxes of this kind (called *semantic paradoxes*) see e.g. Beth [4], Kleene [3].

struction we used some notions (e.g. the English language, letters, sentences) which do not belong to the pure arithmetic. Usually we do not introduce definitions like (*) in mathematics. The paradox resulted entirely from the fact that we did not say precisely what notions and sentences *belong* to the arithmetic and what notions and sentences *concern* the arithmetic, examined as a fixed and closed deductive system. Intuitively we conceive the arithmetic to be a set of sentences expressing certain properties of positive integers and of other notions defined by means of the notion of integer. For instance, the following sentences certainly belong to the arithmetic:

for every prime number p, $(p-1)!+1$ *is divisible by* p;

there exist no positive integers x, y, z, n *such that* $x^{n+2}+y^{n+2}=z^{n+2}$.

On the other hand we can also talk about the arithmetic. That is, assuming that all the sentences in the arithmetic are formulated in the English language, we can formulate sentences concerning the problem how many integers were defined in the arithmetic by means of at most 1000 letters. However, such sentences about the arithmetic do not belong to the arithmetic, and can only be acknowledged as statements belonging to another theory, which examines the arithmetic as a new subject of investigation. This theory is usually called *metaarithmetic*. In particular, the sentence (*) does not belong to arithmetic; it belongs to metaarithmetic.

In every-day mathematical practice, we often formulate and use simultaneously statements belonging to a theory and to its metatheory, i.e. that theory which examines the given theory. For instance, if the theory under consideration is lattice theory, then

$$a \cup (a \cap b) = a, \qquad a \cap (a \cup b) = a$$

are theorems of lattice theory, but the duality principle (see I, § 6) is a theorem of the *metatheory* of lattice theory.

This example shows that sometimes metatheoretical statements are very useful in developing the mathematical theory under consideration. However, they should always be distinguished from statements belonging to the theory.

It follows from the paradox given by (*) that, in order to exclude such contradictions from an axiomatic theory, it is necessary to describe its language precisely, i.e. the set of sentences of the theory and the set of signs used to build these sentences. In this way we avoid contradictions caused by a collision between the theory and its metatheory, that is, by including metatheoretical statements in the theory. This

inclines us to introduce still greater precision in the construction of mathematical theories and leads to the concept of *formalized theories*, in which not only the properties of primitive notions are given in an exact way by means of a set of axioms, but also the language of the theory is precisely defined (1). The formalization of the language also gives the following advantage: it permits us to describe precisely the logical means assumed in the theory, i.e. to define the process of deduction precisely.

In formalized mathematical theories, e.g. in formalized arithmetic and formalized set theory, such paradoxes as those mentioned above cannot be constructed. On the other hand, a mathematician following good intuition in every-day investigations does not reach a contradiction even though the language of the theory and the logical means he employs are not precisely described. This is explained by the fact that his investigations can always in practice be repeated in a corresponding formalized theory. Thus he avoids the practical difficulties of formalized theories, the formalized language of which is complicated and inconvenient in every-day practice. Consequently, in mathematical practice we build theories axiomatically but always in such a way that they can easily be formalized, i.e. that all the reasonings can be repeated in a corresponding formalized theory. However, the formalization of the language and the logical means are necessary if we intend to develop the metatheory of a given theory because only in this way such important notions as the existence of a proof of a given statement or the set of all theorems of the theory can be precisely defined (in non-formalized axiomatic theory they are far from this precision).

To construct a formalized language of a theory we have first to distinguish a set of *signs* which will form the alphabet of the theory and which, roughly speaking, will play the same role as letters in the colloquial language. All notions, expressions and sentences of the theory will be finite sequences of the admitted signs. The method of forming the sequences of signs corresponding to notions and sentences of the intuitive theory is defined in the corresponding formalized theory by means of simple rules.

The signs in the alphabet of the formalized language of a theory should be divided into several categories.

In any intuitive or axiomatic mathematical theory we investigate a fixed set of *objects* or *individuals* (e.g. positive integers in arithmetic, elements of a fixed lattice in lattice theory, elements of a fixed group in group theory, etc.), relations between objects (e.g. the relation of

(1) The first formalized theory was built by Frege [3].

equality) and some mappings (functions) from objects to objects. Consequently, the alphabet of the corresponding formalized theory should contain signs which are symbols of arbitrary objects (individuals) under investigation. These signs are called *individual variables*. Other signs in the formalized language correspond to the relations under examination. They are called *predicates*, or—more precisely—*m-argument predicates* provided they correspond to m-argument relations ($m = 1, 2, \ldots$). Finally, some signs in the formalized language should correspond to the mappings and functions being examined. These signs are called *functors*, or—more precisely—*m-argument functors* provided they correspond to m-argument mappings from objects to objects ($m = 1, 2, \ldots$). Some individuals (e.g. the zero and the unit element in the theory of Boolean algebras) play special role and therefore they require the introduction of special signs in the formalized language. Signs corresponding to such individuals are called *individual constants*. For purely technical reasons it is convenient to treat individual constants as 0-argument mappings (i.e. constant mappings) from individuals to individuals. Consequently, individual constants in the formalized language are treated as 0-argument functors.

As a rule, it is convenient to introduce into the alphabet of a formalized theory some auxiliary signs corresponding to parantheses, commas, etc. in the colloquial language. The purpose of introducing these auxiliary signs is to avoid mistakes in reading the more complicated expressions.

By means of the functors, individual variables, individual constants and auxiliary signs, using the admitted formation rules, we form certain finite sequences of signs corresponding to the values of functions (or, more generally, of superpositions of functions) at fixed or arbitrary values of their arguments. All the sequences mentioned, together with all the individual variables and constants, are called *terms* of the formalized theory. For instance,

$$1, \quad x, \quad (x+y), \quad ((1+x)\cdot y) \quad \text{etc.}$$

are examples of terms in a formalized language of arithmetic.

The language of an intuitive theory contains sentences and propositional functions. In reality, we deal more often with propositional functions than with sentences. Propositional functions are expressions which have the structure of sentences and which contain variables ranging over a fixed set of objects. If all these variables are replaced by names of concrete objects, then the propositional function becomes a sentence (true or false). For instance,

$$x < y, \quad z = x+y, \quad z = x\cdot y, \quad x \leqslant y+1$$

are examples of propositional functions in arithmetic. Sentences are particular cases of propositional functions: they are propositional functions with empty sets of variables.

By means of such words as

(1) "or", "and", "if ..., then ...", "not"

and

(2) "there exists an ξ such that", "for every ξ"

from given propositional functions α, β we can form more complicated ones. For brevity we shall replace the words (1) by the symbols

(1′) $\cup, \cap, \Rightarrow, -$

respectively, and expressions (2) by

(2′) $\bigcup_\xi, \bigcap_\xi$

respectively. Symbols (1′) are called, respectively, the *disjunction sign*, the *conjunction sign*, the *implication sign* and the *negation sign*. They are all called *propositional connectives*, viz. $\cup, \cap, \Rightarrow$ are binary propositional connectives and $-$ is a unary propositional connective. The symbols $\bigcup$ and $\bigcap$ are called respectively the *existential quantifier* and the *universal quantifier*. Thus

$$\alpha \cup \beta, \quad \alpha \cap \beta, \quad \alpha \Rightarrow \beta, \quad -\alpha, \quad \bigcup_\xi \alpha, \quad \bigcap_\xi \alpha$$

should be read, respectively:

$$\alpha \text{ or } \beta; \quad \alpha \text{ and } \beta; \quad \text{if } \alpha, \text{ then } \beta; \quad \text{not } \alpha;$$
$$\text{there exists } \xi \text{ such that } \alpha; \quad \text{for every } \xi, \alpha.$$

Examples of more complicated propositional functions in arithmetic are

$$(3) \quad \begin{array}{l} (x < y) \Rightarrow \bigcup_\xi (x + \xi = y), \quad \bigcap_\xi \bigcap_\eta - (\xi = \eta) \Rightarrow \big((\xi < \eta) \cup (\eta < \xi)\big), \\ (x \cdot x = 2 \cdot y + 1) \cup (x + x = 2 \cdot y), \quad \bigcap_\xi \bigcup_\eta (\xi = 2 \cdot \eta + x), \quad \text{etc.} \end{array}$$

The variables in propositional functions in colloquial mathematical language coincide with individual variables in the corresponding formalized language. Observe that in examples (3) the Greek letters play a different part from the Latin letters. The Greek letters ξ, η are variables bounded by the quantifiers $\bigcap_\xi, \bigcup_\xi, \bigcap_\eta, \bigcup_\eta$ and any substitution of fixed numbers for ξ, η makes no sense. The Latin letters x, y are free variables; the substitution of names of fixed numbers for x and y is allowed and then (3) become true or false sentences. This suggests that in formalized languages also we have to distinguish between free and bound individual variables. It is not necessary

but it is very convenient, in formalized languages, to have separate sets of free individual variables and bound individual variables.

In order to be able to reproduce all propositional functions of any mathematical theory in its formalized language, we must also have in the alphabet of the theory the signs $\vee, \wedge, \Rightarrow, \neg$ of propositional connectives and the signs $\bigcup, \bigcap$ of quantifiers.

By means of terms, predicates and some auxiliary signs (e.g. parentheses), in the formalized language of a theory we form finite sequences of signs which correspond to the simplest sentences and propositional functions of the intuitive theory. They are called *elementary formulas of the formalized language.* For instance, the expressions

$$(x < (y+z)), \quad ((1+2) = 3), \quad ((x+z) = y)$$

are examples of elementary formulas in a formalized language of arithmetic. Roughly speaking, elementary formulas are a formalization of sentences stating that relations represented by predicates hold between individuals represented by terms.

Joining elementary formulas by means of propositional connectives, quantifiers and auxilliary signs, we form other finite sequences of signs representing more complicated sentences and propositional functions of the theory. These sequences of signs, together with elementary formulas, are called *formulas.* For instance, the sequence formed of the signs $x, y, \xi, =, <, +, (,), \Rightarrow, \bigcup$ given in the following arrangement:

$$((x < y) \Rightarrow \bigcup_{\xi}((x+\xi) = y))$$

is an example of a non-elementary formula in a formalization of arithmetic (this sequence is a formalization of the first of the propositional functions (3)).

The exact definition of the alphabet A, the set T of terms, the set F of all formulas and the exact definition of the notion of a formalized language $\mathfrak{L}$ of any mathematical theory will be given in § 3.

As we have seen, terms are formal substitutes for individuals (including the values of functions under investigation, and their superpositions). Formulas are formal substitutes for propositional functions. Formulas without free individual variables, called *closed formulas*, have a special meaning in the process of formalization of a mathematical theory. Namely they are formal substitutes for propositional functions without free variables, i.e. for the sentences of the theory.

The formalization of a mathematical theory still requires a precise description of the logical methods assumed in the theory. The process of deduction consists of the recognition of some sentences or propositional functions as logical consequences of the others. To formalize the

process of deduction we have to define for any given formalized language $\mathfrak{L}$ the *consequence operation* in $\mathfrak{L}$ which, to every set S of formulas in $\mathfrak{L}$, associates a set $C(S)$ of all logical consequences of formulas belonging to S. This operation should be introduced in such a way that it constitutes the formalization of the intuitive concept of consequence. For this purpose we shall assume a set $\mathscr{A}_\mathfrak{l}$ of *logical axioms* which are formulas in $\mathfrak{L}$, true by virtue of their syntax only, and we shall admit certain *rules* of *inference*, e.g. operations which associate to certain finite sequences of formulas new formulas which are, in the intuitive interpretation, their immediate logical consequences. For an arbitrary set S of formulas we define the set $C(S)$ as the least set of formulas containing S, the set $\mathscr{A}_\mathfrak{l}$ and closed with respect to the rules of inference. Any pair $\mathfrak{S} = \{\mathfrak{L}, C\}$ where $\mathfrak{L}$ is a formalized language and C is the consequence operation in $\mathfrak{L}$, will be called a *formalized deductive system*, or briefly a *deductive system*. The precise description of deductive systems will be given in § 10 and § 11.

By a *formalized theory* we shall understand any triple $\mathfrak{T} = \{\mathfrak{L}, C, \mathscr{A}\}$ where $\{\mathfrak{L}, C\}$ is a formalized deductive system and $\mathscr{A}$ is a set of mathematical axioms of this theory. The formulas belonging to the set $C(\mathscr{A})$ are said to be the *theorems* of $\mathfrak{T}$. The exact definition of formalized theories will be given in § 10 and § 11.

In every formalized theory, the sentences and the propositional functions are represented as formulas, i.e. as finite sequences of signs, and the rules of inference reduce to simple operations on the finite sequences of signs. In this way the process of deduction, i.e. of obtaining some sentences or propositional functions as conclusions from the axioms of the theory is reduced to simple mechanical operations on the finite sequences of signs.

§ 2. Operations on expressions. As we explained in § 1, terms and formulas in a formalized language are finite sequences of certain elements called *signs* which form the alphabet of the language. To facilitate the description of formalized languages and of some operations on terms or formulas, i.e. on finite sequences of signs, we assume the following terminology and notation.

The sequence formed of signs s_i $(i = 1, \ldots, n)$ will usually be written as follows:

$$s_1 s_2 \ldots s_n . \tag{1}$$

Finite sequences of signs will usually be denoted by Greek letters $\alpha, \beta, \gamma, \delta, \tau, \vartheta, \Theta$ (with indices, if necessary) and called *expressions*. If ϑ denotes a sequence $s_1 s_2 \ldots s_n$ and Θ denotes a sequence $t_1 t_2 \ldots t_n$, then $\vartheta\Theta$ will denote the sequence $s_1 s_2 \ldots s_n t_1 t_2 \ldots t_m$. The meaning of symbols

like $\vartheta_1 \vartheta_2 \vartheta_3$, $s\vartheta\Theta$, $\vartheta s \Theta t$ (where s and t denote signs and $\vartheta, \vartheta_1, \vartheta_2, \vartheta_3, \Theta$ denote finite sequences of signs) is similar.

An expression ϑ_1 is said to be a *part* of an expression ϑ, if ϑ has the form

$$\vartheta_0 \vartheta_1 \vartheta_2 \tag{2}$$

where ϑ_0 and ϑ_2 may be empty. If ϑ is of the form (2) and Θ is any expression, then the expression

$$\vartheta_0 \Theta \vartheta_2$$

is said to be obtained from ϑ by *replacing the part* ϑ_1 *by* Θ.

In particular, if ϑ denotes an expression of the form $\vartheta_1 s \vartheta_2$ (where ϑ_1, ϑ_2 may be empty) then $\vartheta_1 \Theta \vartheta_2$ is said to be obtained from ϑ by replacing this occurrence of s by Θ. More generally, if ϑ denotes an expression of the form $\vartheta_1 s_1 \vartheta_2 s_2 \dots \vartheta_n s_n \vartheta_{n+1}$ (where ϑ_i, $i = 1, \dots, n+1$, may be empty), then $\vartheta_1 \Theta_1 \vartheta_2 \Theta_2 \dots \vartheta_n \Theta_n \vartheta_{n+1}$ is said to be obtained from ϑ by the *simultaneous replacement* of these occurrences of $s_1, s_2, \dots, s_n$ by $\Theta_1, \Theta_2, \dots, \Theta_n$, respectively. The expression obtained from an expression ϑ by simultaneously replacing all occurrences of the sign s_1 by the expression Θ_1, of all occurrences of the sign s_2 by the expression $\Theta_2, \dots$, and of all occurrences of the sign s_n by the expression Θ_n will be denoted by

$$\vartheta(s_1/\Theta_1, \dots, s_n/\Theta_n) \tag{3}$$

and called *the result of substitution* (or briefly: *the substitution*) *of* $\Theta_1, \dots, \Theta_n$ *for* $s_1, \dots, s_n$ *in* ϑ.

Sometimes, in order to emphasize that an expression ϑ contains certain signs s_i $(i = 1, \dots, n)$ we shall denote it by

$$\vartheta(s_1, \dots, s_n)\,. \tag{4}$$

Then the substitution of $\Theta_1, \dots, \Theta_n$ for these signs $s_1, \dots, s_n$ respectively, which appear in $\vartheta(s_1, \dots, s_n)$ will often be denoted by

$$\vartheta(\Theta_1, \dots, \Theta_n)\,. \tag{5}$$

We explain that $(|,|) \dots$ never denote signs in any formalized language. Therefore the notation (3), (4), (5) does not cause any confusion as to notation (1).

§ 3. Formalized languages of elementary mathematical theories. By the intuitive consideration in § 1, the set of all signs of the formalized language of a formalized mathematical theory is the union of sets of signs of the following disjoint categories:

(a) a set V of free individual variables,

(b) a set Ξ of bound individual variables,

(c) sets Φ_m of m-argument functors, $m = 0, 1, 2, \dots,$

(d) sets P_m of m-argument predicates, $m = 1, 2, \ldots$,

(e) a set L_1 of unary propositional connectives and a set L_2 of binary logical connectives,

(f) a set Q containing two signs: the existential and universal quantifiers,

(g) a set U of auxiliary signs, such as parentheses, etc.

This suggests the following precise definition: By an *alphabet of an elementary mathematical theory*, or: by an *alphabet of the first order*, we mean an ordered system

$$A = \{V, \Xi, \{\Phi_m\}_{m \in M_0}, \{P_m\}_{m \in M}, L_1, L_2, Q, U\}$$

where

1° M denotes the set of all positive integers, and M_0 denotes the set of all non-negative integers;

2° all the sets $V, \Xi, \Phi_0, \Phi_1, \Phi_2, \ldots, P_1, P_2, \ldots, L_1, L_2, Q, U$ are disjoint;

3° the sets V and Ξ are infinite;

4° the union of the sets $P_1, P_2, \ldots$ and the union of L_1, L_2 are not empty, and the set Q contains two elements.

Elements in V will be called *free individual variables* and denoted by x, y, z (with indices if necessary).

Elements in Ξ will be called *bound individual variables* and denoted by ξ, η, ζ (with indices if necessary).

Elements in the union Φ of the sets

$$\Phi_0, \Phi_1, \Phi_2, \ldots$$

will be called *functors* and usually denoted by φ, ψ (with indices if necessary). More precisely, elements in Φ_m will be called *m-argument functors*. Element in Φ_0 will also be called *individual constants*.

Elements in the union P of the sets

$$P_1, P_2, \ldots$$

will be called *predicates* and usually denoted by ϱ, π (with indices, if necessary). More precisely, elements in P_m will be called *m-argument predicates*.

Elements in L_1 are called *unary propositional connectives* and elements in L_2 are called *binary propositional connectives*. In Chapters V-X we shall always suppose that

5° L_1 contains only one element denoted by $-$ and called the *negation sign*; L_2 contains three elements denoted by $\cup, \cap, \Rightarrow$ and called the *disjunction sign*, the *conjunction sign*.and the *implication sign*, respectively.

6° Q contains two elements denoted by $\bigcup, \bigcap$ and called signs of the *existential quantifier* and of the *universal quantifier*, respectively.

As we shall see later (p. 158), in all formulas of every formalized language, the signs $\bigcup$ and $\bigcap$ are always followed by a sign of a bound individual variable ξ, η etc., i.e. $\bigcup$ and $\bigcap$ always appear in configurations like

$$\vartheta_1 \bigcup \xi \vartheta_2 \quad \text{or} \quad \vartheta_1 \bigcap \xi \vartheta_2$$

where ϑ_1 and ϑ_2 are certain sequences of signs. For purely technical reasons the pairs of signs

$$\bigcup \xi \quad \text{and} \quad \bigcap \xi$$

composed of a quantifier and a bound individual variable will always be denoted by

$$\bigcup_\xi \quad \text{and} \quad \bigcap_\xi$$

respectively. Thus

$$\vartheta_1 \bigcup_\xi \vartheta_2, \quad \vartheta_1 \bigcap_\xi \vartheta_2$$

are another notation for $\vartheta_1 \bigcup \xi \vartheta_2$ and $\vartheta_1 \bigcap \xi \vartheta_2$ respectively. The pairs $\bigcup_\xi, \bigcap_\xi$ will be called *quantifiers binding the variable* ξ. Formulas can contain many occurrences of the same expression $\bigcup_\xi$ or $\bigcap_\xi$.

Elements in U will be called *auxiliary signs*. In this book we shall always assume that

7° the set U contains only two elements denoted by $($, $)$ and called the *parentheses of the alphabet* A (or: of the formalized language in question).

The signs $($, $)$ should be distinguished from (,), [,]. The first are parantheses of a formalized theory, the second—of the metatheory of this theory (see also the remark at the end of § 2).

We always suppose that

8° the signs $-, \cup, \cap, \Rightarrow, \bigcup, \bigcap, (,)$ are the same in all alphabets (of the first order) considered in this book.

Note that some of the sets $\Phi_0, \Phi_1, \Phi_2, \ldots, P_1, P_2, \ldots$, may be empty.

The union of the sets

$$V, \Xi, \Phi, P, L_1, L_2, Q, U \tag{1}$$

is the *set of all signs* of the alphabet A. Elements in the union are *signs* of the alphabet A.

By an *elementary term* we shall understand any finite sequence of signs in A of the following form:

$$\varphi(x_1 \ldots x_m), \tag{2}$$

where φ denotes an m-argument functor ($m = 1, 2, \ldots$) and $x_1, \ldots, x_m$ denote free individual variables.

Sometimes it is convenient to assume another method of forming elementary terms, according to the usual notation adopted in mathematics. For instance, in formalized arithmetic we shall adopt as elementary terms the expressions $(x+y)$ and $(x \cdot y)$ instead of $+(xy)$ and $\cdot(xy)$ according to the formation rule (2). However, for every m-argument functor φ in A ($m = 1, 2, \ldots$) a formation rule for elementary terms composed of φ and free individual variables $x_1, \ldots, x_m$ has to be uniquely determined. Except for some concrete examples of formalized theories described in § 13, we shall always use the formation rule (2).

The *set of all terms* formed from signs in A is the least set T (of finite sequences of signs in A) such that

(t_1) all the free individual variables and all the 0-argument functors (i.e. individual constants) are in T;

(t_2) if τ is an elementary term with free individual variables $x_1, \ldots, x_m$ and $\tau_1, \ldots, \tau_m$ are in T, then the result of substitution (see § 2) of $\tau_1, \ldots, \tau_m$ for $x_1, \ldots, x_m$, respectively, in τ is in T.

Terms will usually be denoted by the letter τ with indices if necessary.

If there are no functors in A, then the set T of all terms is equal to the set of all free individual variables in A. If there are no m-argument functors for $m > 0$, then the set T of all terms is the union of the set of all free individual variables and the set of all 0-argument functors (i.e. of individual constants) in A.

By a *primitive formula* we shall understand any finite sequence of signs in A of the form

$$\varrho(x_1 \ldots x_m) \tag{3}$$

where ϱ denotes an m-argument predicate ($m = 1, 2, \ldots$) and $x_1, \ldots, x_m$ denote free individual variables.

Sometimes it is convenient to assume another method of forming primitive formulas, according to the usual notation adopted in mathematics. For instance, the alphabets of formalized theories usually contain a two-argument predicate denoted by $=$ which represents the relation of equality in the intuitive interpretation. Then we are inclined to adopt the expression $(x = y)$ as the primitive formula formed of the predicate $=$ and of any free individual variables x, y, instead of the expression $=(x\,y)$ formed according to (3). However, for every m-argument predicate ϱ in A, a formation rule of primitive formulas from ϱ and free individual variables $x_1, \ldots, x_m$ has to be uniquely determined. Except for some concrete examples of formalized theories

described in § 13, we shall always use the formation rule (3) for primitive formulas.

The *set F of all formulas* formed of signs in A is the least set of finite sequences of signs in A such that

(f_1) if α is a primitive formula with free individual variables $x_1, \ldots, x_m$ and $\tau_1, \ldots, \tau_m$ are terms, then the result of substitution of $\tau_1, \ldots, \tau_m$ for $x_1, \ldots, x_m$, respectively, in α is in F (in particular, every primitive formula is in F);

(f_2) if α is in F and o is a sign of a unary propositional connective in A, then $o\alpha$ is in F;

(f_3) if α and β are in F and o is a sign of a binary propositional connective belonging to A, then $(\alpha\, o\, \beta)$ is in F;

(f_4) if $\alpha(x)$ (where x denotes a free individual variable) is in F, then for every bound individual variable ξ which does not appear in $\alpha(x)$, the expressions $\bigcup_\xi \alpha(\xi)$ and $\bigcap_\xi \alpha(\xi)$ are in F.

The formulas mentioned in (f_1) are called *elementary*.

The letters $\alpha, \beta, \gamma, \delta$ will usually denote formulas.

The system $\mathfrak{L} = \{A, T, F\}$ will be called the *elementary formalized language* (or: *formalized language of the first order*) *based on the alphabet* A.

It follows from the definition of terms and formulas that, in formalized languages, the signs $($, $)$ of parenthenes play the same part as parentheses in the intuitive mathematical language: they assure uniqueness in reading the terms and formulas. Without the signs $($, $)$ expressions like $\alpha \cap \beta \cup \gamma$, where α, β, γ are formulas, are ambiguous: they can be interpreted either as the conjunction of α and the disjunction of β and γ, or as the disjunction of the conjunction of α and β and of the formula γ. In our formalization of the language such expressions do not appear in the set of formulas. Instead we have only expressions of the form $(\alpha \cap (\beta \cup \gamma))$ or $((\alpha \cap \beta) \cup \gamma)$ (where α, β, γ are formulas), which can be interpreted in only one way.

As we have mentioned, sometimes we use other formation rules for elementary terms and primitive formulas than (2) and (3). It should be stated that these rules should always be constructed in such a way that every term and formula can be read in one way only.

Observe the following property of terms and formulas:

3.1. *If τ is a term, $x_1, \ldots, x_n$ are free individual variables and $\tau_1, \ldots, \tau_n$ are terms, then $\tau(x_1/\tau_1, \ldots, x_n/\tau_n)$ is a term.*

If α is a formula, $x_1, \ldots, x_n$ are free individual variables, and $\tau_1, \ldots, \tau_n$ are terms, then $\alpha(x_1/\tau_1, \ldots, x_n/\tau_n)$ is a formula.

Indeed, the set T_0 of all expressions τ such that $\tau(x_1/\tau_1, \ldots, x_n/\tau_n)$ is a term has the properties:

1) it contains all free individual variables;

2) if $\tau_1', \ldots, \tau_m'$ belong to T_0 and φ is an m-argument functor, then the expression $\varphi(\tau_1' \ldots \tau_m')$ also belongs to T_0. In fact, denoting, for brevity, the last expression by τ we assert that the substitution $\tau(x_1/\tau_1, \ldots, x_n/\tau_n)$ is the expression $\varphi(\tau'(x_1/\tau_1, \ldots, x_n/\tau_n) \ldots \tau_m'(x_1/\tau_1, \ldots, x_n/\tau_n))$ and consequently it is a term because of (t_2).

Hence it follows, by (t_1) and (t_2), that T_0 contains all the terms, which proves the first part of 3.1.

The proof of the second part is similar. First we verify that the set F_0 of all expressions α, such that $\alpha(x_1/\tau_1, \ldots, x_n/\tau_n)$ is a formula, has the properties:

1) it contains all elementary formulas;

2) if α and β are in F_0, so are $(\alpha \cup \beta)$, $(\alpha \cap \beta)$, $(\alpha \Rightarrow \beta)$, $-\alpha$, $\bigcap_\xi \alpha(x/\xi)$, $\bigcup_\xi \alpha(x/\xi)$.

Hence it follows, by (f_1)-(f_4), that F_0 contains all formulas, which proves the second part of 3.1.

Proofs like those of the first and second parts of 3.1 are called *proofs by induction on the length of term* or *formula*. In the sequel, proofs of this kind will often be omitted because they can be obtained by a simple verification only.

To complete the definitions in this section the following remarks are necessary.

From the point of view of logical and mathematical precision, we ought to distinguish between mathematical objects and their names. In the colloquial mathematical language we operate exclusively with names of mathematical objects, but very often we formulate sentences in such a way that the difference between objects and their names is not observed. For instance, in theory of real numbers we formulate the following theorem:

(4) $\sqrt{2}$ *is an irrational real number.*

This sentence is not quite correct, because "$\sqrt{2}$" is not a real number at all. In fact, "$\sqrt{2}$" is a symbol on paper and every real number is, according e.g. to the Dedekind theory, a cut in the set of all rational numbers, i.e. a pair of sets of rational numbers, etc. The correct formulation of (4) should be:

the real number called $\sqrt{2}$ *is irrational,*

or

$\sqrt{2}$ *is the name of an irrational number.*

However, the exact distinction between mathematical objects and their names is, in practice, very inconvenient. On the other hand, it is not necessary in practice, because it does not cause any misunderstanding if all sentences and expressions are reasonably understood. That is why the strict distinction between mathematical objects and their names is not assumed in colloquial mathematical language.

The definition of formalized language, as well as other definitions, theorems, proofs, etc. in this book belong to metamathematics and are formulated in colloquial mathematical language using all the customary idioms of this language. Consequently the strict distinction between the objects being examined and their names is not maintained. For instance, phrases like "s is a sign", „a formula α", etc. will often be used even though they are not quite correct.

The correct formulation should be "s denotes a sign" (or: "s is the name of a sign"), "a formula denoted by α", etc. However, it does not cause misunderstanding since we have stated that in our language of metamathematics we operate exclusively with the names of the objects under consideration and phrases like "s is a sign" should always be understood as a colloquial abbreviations of the correct formulation "s denotes a sign", etc.

It should be explained that signs of a formalized language do not have to be signs in the colloquial meaning of this word. In particular, they need not be printed signs. The set of signs of a formalized language is any set of mathematical objects satisfying the hypotheses mentioned above. The nature of these objects is completely arbitrary; signs of the elementary formalized languages are primitive notions of metamathematics, or more exactly, of the metatheory of elementary formalized theories. In all expressions such as "x is an individual variable", "α is a formula", etc., the letters x, α are only names of individual variables or formulas, respectively, etc.

However, if we wish, we can suppose that all signs (or some of them) of the formalized language being considered are concrete printed signs (or, more exactly, equivalence classes of signs having the same form). For instance, we may assume that the printed signs

$$x_1, x_2, x_3, \ldots$$

are all free individual variables,

$$\xi_1, \xi_2, \xi_3, \ldots$$

are all bound individual variables, etc. Then formulas can be interpreted as concrete printed sequences of signs. This concrete interpretation also consists of a kind of abstraction and leads to some difficulties (e.g. it

is easy to define formulas as long as they cannot be printed on any existing piece of paper, etc.). However, at least a fragment of language can really be interpreted in this concrete way.

This interpretation has a sense in the case of *enumerable languages*, i.e. of languages with an enumerable set of signs. The case of enumerable languages is most important, because all mathematical theories can be formalized by means of an enumerable language (observe that the process of formalization of an intuitive axiomatic theory is not unique—see e.g. examples in § 13). In particular, set theory can be formalized by means of an enumerable language (see § 13) and this formalized theory contains the whole of mathematics.

On the other hand, formalized theories with non-enumerable languages (i.e. with a non-enumerable set of signs) recently play a stronger and stronger part in metamathematical investigations. Some theorems on formalized theories with a non-enumerable language are, in fact, stronger than their enumerable analogues (see VII, § 11, and VIII, § 25) and have interesting applications (1).

Therefore, if the opposite is not explicitely stated, no hypotheses about the power of the sets of the signs of formalized languages are assumed in the sequel.

The formalized languages defined in this section are called *elementary* (or: *of the first order*) to distinguish them from other possible formalized languages (called *non-elementary*, or: *of higher order*), whose structure is more complicated. In elementary languages we have only quantifiers binding individual variables. In non-elementary languages we also admit quantifiers of other types (e.g. quantifiers binding predicates or functions). Therefore, in non-elementary languages we have expressions corresponding to the following sentences of the colloquial language:

"for every relation ϱ ..."

"there exists a function φ such that ...", etc.

The restriction of formalized languages to elementary ones is not very essential. The most important mathematical theories, such as set theory can be formalized on the basis of elementary formalized languages (see § 13). The formalization based on formalized languages of the first order is, in a sense, the simplest because it uses the most elementary logical means.

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(1) See e.g. A. Robinson [1], Lyndon [1] and the bibliography in Lyndon [1]. Recently, also formalized deductive systems based on languages in which formulas are infinite sequences of signs were introduced and examined by the Berkeley school; see Scott and Tarski [1], Tarski [14], Karp [1], [2], Henkin [6], Maehara and Takeuti [1].

The described formalization of the languages of elementary theories in not the only possible one. However, this method of formalization is most often assumed in metamathematics.

§ 4. Interpretations. Let $\mathfrak{L} = \{A, T, F\}$ be a formalized language described in § 3.

The terms and formulas of the language $\mathfrak{L}$ are finite sequences of signs in the alphabet A of the language $\mathfrak{L}$. The terms are expressions which have the form of mappings or functions but they are not concrete mappings. Formulas are expressions which have the form of propositional functions or sentences but they are not propositional functions. However, it is the fundamental property of terms and formulas that they can be conceived as concrete mappings and as concrete propositional functions, respectively, if functors and predicates of the language $\mathfrak{L}$ are interpreted as signs of concrete mappings and concrete relations in a given non-empty set J, i.e. if a concrete interpretation of the language $\mathfrak{L}$ is fixed.

To explain this more precisely, let us introduce the following terminology.

Let J be a non-empty set. Assign to every m-argument functor φ a mapping $\varphi_{\mathfrak{I}}: J^m \to J$ (in particular, assign to every individual constant φ a fixed element $\varphi_{\mathfrak{I}} \in J$). Assign to every m-argument predicate ϱ an m-argument relation $\varrho_{\mathfrak{I}}$ in J. Such an assignment $\mathfrak{I}$ will be called an *interpretation* [1] *of the language* $\mathfrak{L}$ *in the set* J because it determines uniquely

1) an interpretation of every term τ in $\mathfrak{L}$ as a concrete mapping $\tau_{\mathfrak{I}}: J \times J \times \ldots \times J \to J$ or (the case of individual constants) as a concrete element in J;

2) an interpretation of every formula α in $\mathfrak{L}$ as a concrete propositional function $\alpha_{\mathfrak{I}}$ defined in the set J.

To obtain the mapping $\tau_{\mathfrak{I}}$ it suffices

a) to interpret every free individual variable in τ as a variable ranging over the set J;

b) to interpret every functor φ appearing in τ as the mapping $\varphi_{\mathfrak{I}}$.

To obtain $\alpha_{\mathfrak{I}}$ it suffices

(a) to interpret all free and bound variables appearing in α as variables ranging over the set J;

(b) to interpret every functor φ appearing in α as the mapping $\varphi_{\mathfrak{I}}$

(c) to interpret every predicate ϱ appearing in α as the relation $\varrho_{\mathfrak{I}}$

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[1] Interpretations in the sense admitted in this book are often called *pseudo models*, or *frames* by other authors.

(d) to interpret propositional connectives and quantifiers as corresponding connectives and quantifiers from the colloquial mathematical language.

The auxiliary signs (and) are, of course, interpreted as parentheses. They will often be omitted in interpretations of terms and formulas, if without them the uniqueness of the interpretations is assured.

We shall explain the notion of interpretation by an example. Suppose that $\mathfrak{L}$ contains only three functors: an individual constant ν and two 2-argument functors σ and μ, and moreover, two 2-argument predicates ϱ and π. Let τ be the term

$$\sigma(\sigma(xy)\mu(\sigma(\nu z)\mu(yz)))\,,$$

where x, y, z are individual variables. Let α be the formula

$$(\bigcup_\xi \varrho(x\sigma(\xi\xi)) \Rightarrow (\bigcup_\eta \varrho(x\mu(\sigma(\nu\nu)\eta)) \frown \pi(\nu x)))\,,$$

where x is a free individual variable, and ξ, η are bound individual variables. Let J be the set of all positive integers. Let $\mathfrak{I}$ be the following interpretation of $\mathfrak{L}$ in J: $\nu_{\mathfrak{I}}$ is the number 1; $\sigma_{\mathfrak{I}}$ is the operation of addition; $\mu_{\mathfrak{I}}$ is the operation of multiplication; $\varrho_{\mathfrak{I}}$ is the relation of equality $=$; $\pi_{\mathfrak{I}}$ is the inequality relation $<$. Then $\tau_{\mathfrak{I}}$: $J \times J \times J \to J$ is the function

$$(x+y)+(1+z)yz$$

and $\alpha_{\mathfrak{I}}$ is the propositional function

$$\bigcup_\xi (x = \xi+\xi) \Rightarrow \left(\bigcup_\eta \left(x = (1+1)\eta\right) \frown (1 < x)\right),$$

where $\bigcup_\xi$, $\bigcup_\eta$, $\Rightarrow$, $\frown$ have the meaning established in § 1, p. 149.

From the mathematical point of view, an interpretation $\mathfrak{I}$ of a formalized language $\mathfrak{L}$ described in § 3 is a mapping defined on the set of all functors and predicates. If φ is an m-argument functor, then the image $\varphi_{\mathfrak{I}}$ of φ, given by the mapping $\mathfrak{I}$, is a transformation $\varphi_{\mathfrak{I}}$: $J^m \to J$ ($m = 0, 1, 2, \ldots$). If ϱ is an m-argument predicate, then the image $\varrho_{\mathfrak{I}}$ of ϱ is an m-argument relation in J ($m = 1, 2, \ldots$). The mapping $\mathfrak{I}$ has the property that it can be extended, in a natural way, over the set of all terms and formulas: it maps terms τ and formulas α onto transformations $\tau_{\mathfrak{I}}$: $J \times J \times \ldots \times J \to J$ and propositional functions $\alpha_{\mathfrak{I}}$ in J, respectively, in the way described above.

This natural extension of $\mathfrak{I}$ onto the set of all terms can be described, by induction, as follows:

A) if x is an individual variable, then $x_{\mathfrak{I}}$ is the identity mapping $f(x) = x$ of J onto itself; if φ is an individual constant, then $\varphi_{\mathfrak{I}}$ is the constant mapping $f(x) = \varphi_{\mathfrak{I}}$;

B) if τ is a term of the form $\varphi(\tau_1 \dots \tau_m)$ where φ is an m-argument functor ($m = 1, 2, \dots$), and τ_i is a term with individual variables $x_{i1}, \dots, x_{in}$ ($i = 1, \dots, m$), then $\tau_{\mathfrak{I}}$ is the mapping

$$\varphi_{\mathfrak{I}}\big(\tau_{1\mathfrak{I}}(x_{11}, \dots, x_{1n_1}), \dots, \tau_{m\mathfrak{I}}(x_{m1}, \dots, x_{mn_m})\big),$$

i.e. $\tau_{\mathfrak{I}}$ is the superposition of the mappings $z = \varphi_{\mathfrak{I}}(y_1, \dots, y_m)$ and the mappings $y_i = \tau_{i\mathfrak{I}}(x_{i1}, \dots, x_{in_i})$, $i = 1, \dots, m$.

It follows from A) and B) that, in particular, if τ is of the form $\varphi(x_1 \dots x_m)$, where φ is an m-argument functor and $x_1, \dots, x_m$ are individual variables, then $\tau_{\mathfrak{I}}$ is the mapping $\varphi_{\mathfrak{I}}(x_1, \dots, x_m)$ of J^m into J.

The natural extension of $\mathfrak{I}$ onto the set of all formulas can be described, by induction, as follows:

C) if α is an elementary formula, i.e. a formula of the form

$$\varrho(\tau_1 \dots \tau_m)$$

where ϱ is an m-argument predicate ($m = 1, 2, \dots$) and τ_i is a term with free individual variables $x_{i1}, \dots, x_{in_i}$ ($i = 1, \dots, m$), then $\alpha_{\mathfrak{I}}$ is the propositional function

$$\varrho_{\mathfrak{I}}\big(\tau_{1\mathfrak{I}}(x_{11}, \dots, x_{1n_1}), \dots, \tau_{m\mathfrak{I}}(x_{m1}, \dots, x_{mn_m})\big);$$

D) for non-elementary formulas the following identities hold:

(1) $$(\beta \cup \gamma)_{\mathfrak{I}} = (\beta_{\mathfrak{I}} \cup \gamma_{\mathfrak{I}}),$$

(2) $$(\beta \cap \gamma)_{\mathfrak{I}} = (\beta_{\mathfrak{I}} \cap \gamma_{\mathfrak{I}}),$$

(3) $$(\beta \Rightarrow \gamma)_{\mathfrak{I}} = (\beta_{\mathfrak{I}} \Rightarrow \gamma_{\mathfrak{I}}),$$

(4) $$(-\beta)_{\mathfrak{I}} = -(\beta_{\mathfrak{I}}),$$

(5) $$\big(\bigcup_\xi \beta(\xi)\big)_{\mathfrak{I}} = \bigcup_\xi \beta_{\mathfrak{I}}(\xi),$$

(6) $$\big(\bigcap_\xi \beta(\xi)\big)_{\mathfrak{I}} = \bigcap_\xi \beta_{\mathfrak{I}}(\xi).$$

Of course, the symbols $\cup$, $\cap$, $\Rightarrow$, $-$, $\bigcup$, $\bigcap$ on the left-hand and on the right-hand sides of these equations have different meanings. On the left-hand side they denote propositional connectives and quantifiers from the formalized language $\mathfrak{L}$, whereas on the right-hand side they denote propositional connectives and quantifiers from the colloquial mathematical language.

Sometimes it is convenient to consider a mapping $\mathfrak{I}$ which assigns, to every m-argument functor φ, a transformation $\varphi_{\mathfrak{I}}$: $J^m \to J$ ($m = 0, 1, 2, \dots$). Then $\mathfrak{I}$ is called an *interpretation* or *realization of terms* of the language $\mathfrak{L}$. It follows from the above consideration that

the mapping $\mathfrak{I}$ can be extended, in a natural way, over the set of all terms: to every term τ it assigns a transformation $\tau_{\mathfrak{I}}$: $J \times J \times \ldots \times J \to J$. The exact inductive definition of $\tau_{\mathfrak{I}}$ is given by A) and B).

§ 5. The intuitive concept of propositional tautologies. In all deductive reasonings we use some sentences which are considered to be true by virtue of their syntax only. Such sentences, called *tautologies*, play a very important role, as they help us to describe precisely the logical means applied in formalized mathematical theories.

In this section we shall be interested in the tautologies which are considered to be true only by their connective syntax, i.e. which are built of sentences $a, b, \ldots$ by means of the connectives $\cup$, $\cap$, $\Rightarrow$, $-$ (see p. 149) in such a way that we have to accept them as true statements because of their special form of construction and independently of whether sentences $a, b, \ldots$ are true or not. Tautologies of this type are called *propositional tautologies*. For instance, our intuition suggests that the following sentences are true for their connective syntax alone, i.e. that they are propositional tautologies:

$$-(a \cap -a), \tag{1}$$

$$((a \cup b) \Rightarrow (-a \Rightarrow b)), \tag{2}$$

$$(((a \Rightarrow b) \cap a) \Rightarrow b). \tag{3}$$

We shall explain more exactly what we mean by a propositional tautology, from the intuitive point of view. Our logical intuition inclines us to associate with sentences their logical values: *truth* or *falsity*. We shall not consider here in what way the criterions for the truth or falsity of given sentences $a, b, \ldots$ can be established. Nor shall we discuss whether the associating of logical values with sentences $a, b, \ldots$ is always possible in concrete cases. We shall be interested only in what can be said about the truth or falsity of sentences $\alpha, \beta, \ldots$ built of sentences $a, b, \ldots$ by means of connectives $\cup$, $\cap$, $\Rightarrow$, $-$ if the logical values of sentences $a, b, \ldots$ are known.

Let α and β be sentences whose logical values are known. According to the intuitive meaning of the propositional connectives the disjunction $(\alpha \cup \beta)$ is true if and only if at least one of the sentences α, β is true. Similarly the conjunction $(\alpha \cap \beta)$ is true if and only if both sentences α and β are true. The negation $-\alpha$ is true, if and only if α is false. The implication $(\alpha \Rightarrow \beta)$ is considered false if and only if α is true and β is false. In all other cases the implication is true. This way of understanding the implication differs somewhat from that in the colloquial language. We are inclined to conceive the implication $(\alpha \Rightarrow \beta)$ in the colloquial language in such a way that β can be deduced from the sentence α. Therefore, at

first glance, the usual colloquial understanding of the implication gives no suggestion as to the logical value of the sentences

(4) if 1 is divisible by 4, then 1 is divisible by 2,

(5) if 2 is divisible by 4, then 2 is divisible by 2.

The proposed manner of treating the implication is, indeed, generally used in mathematics. For instance, it is evident that the following statement from arithmetic:

(6) if x is divisible by 4, then x is divisible by 2

is true for every positive integer x. Consequently, we must also admit that the implications (4) and (5) are true sentences, according to the agreement assumed above.

Notice that the implication in our meaning has the following important property, called the *rule of detachment* (*modus ponens*):

(7) if $(a \Rightarrow \beta)$ is true and a is true, then β is true.

This is the contents of (3). We can easily verify that (3) is a tautology according to the meaning of the connectives established above. In fact, (3) could be false only if $((a \Rightarrow b) \cap a)$ were true and b false. But the first sentence is true if and only if a is true and $(a \Rightarrow b)$ is true. However, if a is true and b is false the sentence $(a \Rightarrow b)$ is false. Hence sentence (3) is always true, independently of the logical values of a and b.

The rules establishing the truth or the falsity of the disjunction, the conjunction, the implication and the negation according to the truth or falsity of their members may be expressed very clearly in the form of the following schemes, where Λ and V denote any false and any true sentence, respectively:

(8) $\Lambda \cup \Lambda = \Lambda, \quad \Lambda \cup V = V \cup \Lambda = V \cup V = V,$

(9) $V \cap V = V, \quad V \cap \Lambda = \Lambda \cap V = \Lambda \cap \Lambda = \Lambda,$

(10) $V \Rightarrow \Lambda = \Lambda, \quad V \Rightarrow V = \Lambda \Rightarrow V = \Lambda \Rightarrow \Lambda = V,$

(11) $-\Lambda = V, \quad -V = \Lambda.$

Note that the schemes given above are identical with equations II, § 4, (1), if we interpret Λ and V as the zero and the unit of the two-element Boolean algebra $\boldsymbol{A}_0$, and $\cap$, $\cup$, $\Rightarrow$, $-$ as the corresponding Boolean operations. This property permits us to decide, in a very simple way, the truth or the falsity of any sentence a formed of sentences $a, b, \ldots$ by means of the propositional connectives according to the truth or the falsity of sentences $a, b, \ldots$ occurring in a. For this purpose it suffices

a) to replace each of the primitive sentences $a, b, \ldots$ appearing in α by V if this sentence is true and by Λ if it is false,

b) to interpret Λ and V as the zero and unit of the two-element Boolean algebra,

c) to conceive the propositional connectives $\cup$, $\cap$, $\Rightarrow$, $-$ as the corresponding Boolean operations in this algebra.

If we obtain V as a result of performing the Boolean operations in the Boolean expression obtained from α in the way described above, then α is true. If we obtain Λ, then α is false.

For instance, let us suppose that in the sentence

$$(12) \qquad (-(a \cap b) \Rightarrow (-a \cup -b))$$

a is a true sentence and b is a false one. Then sentence (12) is true, since

$$(-(V \cap \Lambda) \Rightarrow (-V \cup -\Lambda)) = V \Rightarrow V = V .$$

This method of verifying the truth or falsity of composed sentences is said to be the *truth-table method.*

If a sentence α formed of $a, b, \ldots$ by means of propositional connectives is always true, independently of the truth or falsity of $a, b, \ldots$ then α is said to be a *propositional tautology.* In other words, α is called a propositional tautology if the corresponding Boolean expression in the two-element Boolean algebra $\boldsymbol{A}_0$ assumes the value V exclusively for all substitutions of V and Λ for $a, b, \ldots$

For instance (1), (2), (3), (12) are tautologies in the sense just defined.

To verify whether a sentence α formed of n primitive sentences $a_1, \ldots, a_n$ is a tautology, we must perform all possible substitutions of V, Λ for $a_1, \ldots, a_n$, i.e. 2^n substitutions. In general, the following method of seeking a substitution giving the value Λ is simpler because it permits to avoid performing a great number of substitutions. We shall now explain this method by two examples.

Consider the sentence

$$(13) \qquad ((a_1 \Rightarrow (a_2 \Rightarrow a_3)) \Rightarrow ((a_1 \Rightarrow a_2) \Rightarrow (a_1 \Rightarrow a_3))) .$$

Suppose that for some a_1, a_2, a_3 in the two-element Boolean algebra $\boldsymbol{A}_0$ we obtain the value Λ:

$$\big(a_1 \Rightarrow (a_2 \Rightarrow a_3)\big) \Rightarrow \big((a_1 \Rightarrow a_2) \Rightarrow (a_1 \Rightarrow a_3)\big) = \Lambda .$$

This is possible if and only if

$$(14) \qquad \big(a_1 \Rightarrow (a_2 \Rightarrow a_3)\big) = V \quad \text{and} \quad \big((a_1 \Rightarrow a_2) \Rightarrow (a_1 \Rightarrow a_3)\big) = \Lambda .$$

The last equation is possible if and only if

$$(a_1 \Rightarrow a_2) = V \tag{15}$$

and $(a_1 \Rightarrow a_3) = \Lambda$, i.e.

$$a_1 = V \quad \text{and} \quad a_3 = \Lambda . \tag{16}$$

It follows from (16) and (15) that $a_2 = V$. Hence

$$\big(a_1 \Rightarrow (a_2 \Rightarrow a_3)\big) = V \Rightarrow (V \Rightarrow \Lambda) = V \Rightarrow \Lambda = \Lambda$$

in contradiction to (14). Thus no substitution of V, Λ for a_1, a_2, a_3 gives Λ, and (13) is a tautology.

Now consider the following example:

$$((a_1 \Rightarrow -a_1) \Rightarrow a_2) . \tag{17}$$

This expression, interpreted in the two-element Boolean algebra, assumes the value Λ if and only if

$$a_2 = \Lambda \quad \text{and} \quad (a_1 \Rightarrow -a_1) = V ,$$

i.e. if $a_2 = \Lambda$ and $a_1 = \Lambda$. Thus (17) is not a tautology.

Propositional tautologies constitute the basis of all deductive reasonings. For instance, suppose that we have to prove a statement of the form $(a \Rightarrow b)$. It is easy to verify that the expression

$$((-b \Rightarrow -a) \Rightarrow (a \Rightarrow b)) \tag{18}$$

is a propositional tautology. Hence, by the detachment rule, in order to prove $(a \Rightarrow b)$ it suffices to prove $(-b \Rightarrow -a)$. This method is often applied in mathematical practice.

Similarly, the propositional tautology

$$(((a \cap -b) \Rightarrow (c \cap -c)) \Rightarrow (a \Rightarrow b)) \tag{19}$$

constitutes the basis of the proofs by *reductio ad absurdum*.

§ 6. Formalized languages of propositional calculi. The importance of propositional tautologies in deductive reasonings suggests to construct formal axiomatic systems which will be a precise description of this section of logic. These formal systems will be called *propositional calculi*.

Languages of propositional calculi are different from those of elementary mathematical theories. First we shall describe their alphabets.

By an *alphabet* of a propositional calculus, or: an *alphabet of the zero order* we shall understand any ordered system

$$A_0 = \{V_0, L_1, L_2, U\}$$

where

1° V_0, L_1, L_2, U are disjoint sets;

2° the set V_0 is infinite; the union of the sets L_1, L_2 is not empty.

Elements in V_0 are called *propositional variables* and denoted by a, b, c with indices if necessary.

Elements in L_1 are called *unary propositional connectives.* Elements in L_2 are called *binary propositional connectives.* In Chapters V-X we shall always suppose (similarly to § 3, 5°) that

3° L_1 contains only one element denoted by $-$ and called the *negation sign*; L_2 contains three elements denoted by $\cup, \cap, \Rightarrow$ and called the *disjunction sign*, the *conjunction sign* and the *implication sign*, respectively.

Elements in U are called *auxiliary signs.* In this book we shall always assume (similarly to § 3, 6°) that

4° the set U contains only two elements denoted by $($, $)$ and called *parentheses* in A_0.

Partly generalizing § 3, 7° we shall always suppose that

5° the signs $-, \cup, \cap, \Rightarrow, (,)$ are the same in all alphabets of the first or zero order.

The union of the sets

$$V_0, L_1, L_2, U \tag{1}$$

is the *set of all signs* of the alphabet A_0. Elements in (1) are *signs* of the alphabet A_0.

The set F_0 of all *formulas* formed of signs of A_0 is the smallest set of finite sequences of signs in A_0, such that

(f) all propositional variables (considered as one-element sequences) are in F_0;

(f′) if α is in F_0 and o is a unary propositional connective in A_0, then $o\alpha$ is in F_0;

(f″) if α, β are in F_0 and o is a binary propositional connective in A_0, then $(\alpha\, o\, \beta)$ is also in F_0.

The pair

$$\mathfrak{L}_0 = \{A_0, F_0\}, \tag{2}$$

where A_0 is an alphabet of the zero order and F_0 is the set of all formulas formed of signs in A_0, is called a *formalized language of the zero order* or a *formalized language of a propositional calculus.*

By a *propositional calculus* we shall understand any deductive system $S_0 = \{\mathfrak{L}_0, C\}$ where $\mathfrak{L}_0$ is a formalized language of the zero order and C is a consequence operation in $\mathfrak{L}_0$.

Propositional calculi will be examined in detail in Chapters VII, IX, and XI. At this point we shall state only some of the basic properties of any language $\mathfrak{L}_0$ of the zero order (see (2)).

If α is any formula in $\mathfrak{L}_0$ and all the propositional variables $a, b, c, \ldots$ in α are interpreted as sentences, then the whole formula α can also be interpreted as a sentence, the signs $\cup, \cap, \Rightarrow, -$ being interpreted as logical connectives "or", "and", "if... then...", "not" (see p. 149).

A formula α in $\mathfrak{L}_0$ is said to be a *propositional tautology* (or simply: a *tautology*) if α is a true sentence independently of whether $a, b, c, \ldots$ are interpreted as false or true sentences (see § 5, p. 165).

By an *interpretation* of a language $\mathfrak{L}_0$ of the zero order we shall mean any mappings $\mathfrak{I}$ which assigns to every propositional variable a sentence, true or false. Thus every interpretation $\mathfrak{I}$ of $\mathfrak{L}_0$ turns every formula α (in $\mathfrak{L}_0$) into a sentence $\alpha_{\mathfrak{I}}$, true or false. A formula α is a propositional tautology if and only if $\alpha_{\mathfrak{I}}$ is a true sentence for every interpretation $\mathfrak{I}$ of $\mathfrak{L}_0$.

On the other hand, if all propositional variables $a, b, c, \ldots$ appearing in a formula α are interpreted as variables ranging over the two-element Boolean algebra $\boldsymbol{A}_0$, and the signs $\cup, \cap, \Rightarrow, -$ are interpreted as signs of the Boolean operations in $\boldsymbol{A}_0$, then α can be interpreted as a mapping from $\boldsymbol{A}_0 \times \boldsymbol{A}_0 \times \ldots \times \boldsymbol{A}_0$ into $\boldsymbol{A}_0$. This mapping will be denoted by $\alpha_{\boldsymbol{A}_0}$ and called the *Boolean polynomial determined by α in $\boldsymbol{A}_0$*.

For instance, if α is the formula

$$(((a \Rightarrow b) \cup c) \cap a),$$

then $\alpha_{\boldsymbol{A}_0}$ is the mapping which, to every triple $(a, b, c) \in \boldsymbol{A}_0^3$, assigns the element

$$\alpha_{\boldsymbol{A}_0}(a, b, c) = \big((a \Rightarrow b) \cup c\big) \cap a = (b \cup c) \cap a .$$

Using the above interpretation of formulas as Boolean polynomials, we can formulate the definition of propositional tautologies in a more precise form as follows: a formula α in $\mathfrak{L}_0$ is a *propositional tautology* if the mapping $\alpha_{\boldsymbol{A}_0}$: $\boldsymbol{A}_0 \times \boldsymbol{A}_0 \times \ldots \times \boldsymbol{A}_0 \to \boldsymbol{A}_0$ is identically equal to the unit element of $\boldsymbol{A}_0$.

The equivalence of the two definitions of tautologies in $\mathfrak{L}_0$ follows immediately from considerations in § 5. The examination of the Boolean polynomial $\alpha_{\boldsymbol{A}_0}$ is nothing more than the truth-table method described in § 5.

In particular, the following formulas (where a, b, c are different propositional variables) are tautologies:

(t_1) $$((a \Rightarrow b) \Rightarrow ((b \Rightarrow c) \Rightarrow (a \Rightarrow c))),$$

(t_2) $$(a \Rightarrow (a \cup b)),$$

(t_3) $$(b \Rightarrow (a \cup b)),$$

(t_4) $$((a \Rightarrow c) \Rightarrow ((b \Rightarrow c) \Rightarrow ((a \cup b) \Rightarrow c)))\,,$$

(t_5) $$((a \cap b) \Rightarrow a)\,,$$

(t_6) $$((a \cap b) \Rightarrow b)\,,$$

(t_7) $$((c \Rightarrow a) \Rightarrow ((c \Rightarrow b) \Rightarrow (c \Rightarrow (a \cap b))))\,,$$

(t_8) $$((a \Rightarrow (b \Rightarrow c)) \Rightarrow ((a \cap b) \Rightarrow c))\,,$$

(t_9) $$(((a \cap b) \Rightarrow c) \Rightarrow (a \Rightarrow (b \Rightarrow c)))\,,$$

(t_{10}) $$((a \cap -a) \Rightarrow b)\,,$$

(t_{11}) $$((a \Rightarrow (a \cap -a)) \Rightarrow -a)\,,$$

(t_{12}) $$(a \cup -a)\,.$$

The proof is by an easy verification that the corresponding Boolean polynomials are identically equal to the unit of A_0. As an example we shall calculate the values of the Boolean polynomial corresponding to the formula (t_1):

$$\begin{aligned}
&(a \Rightarrow b) \Rightarrow \big((b \Rightarrow c) \Rightarrow (a \Rightarrow c)\big) = -(-a \cup b) \cup \big(-(-b \cup c)\big) \cup \big(-a \cup c\big)\\
&= (a \cap -b) \cup (b \cap -c) \cup (-a \cup c) \geqslant -a \cup (a \cap \big(-b \cup (b \cap -c) \cup c\big)\\
&\geqslant -a \cup \Big(a \cap \big(-b \cup \big(b \cap (-c \cup c)\big)\big)\Big)\\
&= -a \cup \big(a \cap (-b \cup b)\big) = -a \cup a = V\,.
\end{aligned}$$

Hence it follows that

$$(a \Rightarrow b) \Rightarrow \big((b \Rightarrow c) \Rightarrow (a \Rightarrow c)\big) = V$$

for all $a, b, c \in A_0$, i.e. (t_1) is a tautology.

Formula (t_1) is called the *syllogism law*. Formula (t_8) is the *importation law*, formula (t_9) the *exportation law*. Formula (t_{10}) is the *Duns Scotus law*. Formula (t_{12}) is known as the *law of the excluded middle* or *tertium non datur*.

We shall apply to the language $\mathfrak{L}_0$ the notation and all operations mentioned in § 2.

6.1. *If α and $\beta_1, \ldots, \beta_m$ are formulas in $\mathfrak{L}_0$, and $a_1, \ldots, a_m$ are propositional variables, then $\alpha(a_1/\beta_1, \ldots, a_m/\beta_m)$ is also a formula in $\mathfrak{L}_0$.*

The proof of 6.1 (by induction on the length of α) is similar to the proof of the analogous theorem 3.1.

6.2. *If α is a propositional tautology in $\mathfrak{L}_0$, $\beta_1, \ldots, \beta_m$ are formulas in $\mathfrak{L}_0$, and $a_1, \ldots, a_m$ are propositional variables, then the formula $\alpha(a_1/\beta_1, \ldots, a_m/\beta_m)$ is also a propositional tautology.*

For any formulas $\alpha, \beta_1, \ldots, \beta_m$, the Boolean polynomial determined by the substitution $\gamma = \alpha(a_1/\beta_1, \ldots, a_m/\beta_m)$ is the superposition of the

Boolean polynomial determined by α and of the Boolean polynomials determined by $\beta_1, \ldots, \beta_m$; in symbols:

$$\gamma_{A_0} = \alpha_{A_0}(\beta_{1A_0}, \ldots, \beta_{mA_0})$$

(the exact proof is by induction on the length of α [1]). If α is a tautology, then α_{A_0} assumes only the value V. Hence γ_{A_0} assumes only the value V, i.e. γ is a tautology.

6.3. *If α and $(\alpha \Rightarrow \beta)$ are propositional tautologies, then β is also a tautology.*

In fact,

$$\alpha_{A_0} \Rightarrow \gamma_{A_0} = (\alpha \Rightarrow \beta)_{A_0} = V \quad \text{and} \quad \alpha_{A_0} = V .$$

Hence it follows (see II, § 4, (1)) that $\beta_{A_0} = V$ identically.

6.4. *If α is a formula in a formalized language $\mathfrak{L}_0$ of the zero order, $a_1, \ldots, a_m$ are all propositional variables appearing in α, and $\beta_1, \ldots, \beta_m$ are formulas in a formalized language $\mathfrak{L}$ of the first order, then $\alpha(a_1/\beta_1, \ldots, a_m/\beta_m)$ is a formula in $\mathfrak{L}$.*

The proof (by induction on the length of α) is similar to that of 3.1.

§ 7. The intuitive concept of predicate tautologies. In § 3 we introduced the notion of a tautology as a sentence true by virtue of its syntax only and we have considered a special kind of tautologies—propositional tautologies, i.e. sentences which are true because of the manner in which their parts are joined by the propositional connectives. In this section we shall deal with the more complicated notion of *predicate tautologies*, i.e. of expressions which are true because of the manner in which their parts are joined by the propositional connectives and the universal and existential quantifiers.

We proceed to the more exact description of this notion by some remarks of an intuitive character.

Let $\gamma(x_1, \ldots, x_m)$ be a propositional function (in intuitive mathematics) with free variables $x_1, \ldots, x_m$ ranging over a set $J \neq 0$. The propositional function γ is said to be true in the set J if every sentence obtained from γ by replacing $x_1, \ldots, x_m$ by names of arbitrary elements in J is true.

To give a more mathematical character to the notion of *truth* of propositional functions we introduce the following definition: by the *characteristic* of a propositional function $\gamma(x_1, \ldots, x_m)$ we shall understand the following function denoted by γ^* or by $\gamma(x_1, \ldots, x_m)^*$, defined on

[1] This identity follows also directly from a general theorem VI, 2.5.

the Cartesian product J^m and assuming values in the two-element Boolean algebra $\boldsymbol{A}_0$:

$$(1) \qquad \gamma(j_1, \ldots, j_m)^* = \begin{cases} V & \text{if the sentence } \gamma(j_1, \ldots, j_m) \text{ is true,} \\ \Lambda & \text{if the sentence } \gamma(j_1, \ldots, j_m) \text{ is false,} \end{cases}$$

where $j_1, \ldots, j_m$ are names of any elements in J.

Thus γ is true in J if and only if γ^* assumes only the value V.

Let $\gamma(x_1, \ldots, x_m)$, $\delta(y_1, \ldots, y_n)$ denote respectively an m-argument propositional function and an n-argument propositional function defined in a fixed set $J \neq 0$. Then it follows from (1), § 5, (8)-(11), and II, § 4, (1), (3) that the following equations hold:

$$(2) \qquad \big(\gamma(x_1, \ldots, x_m) \cup \delta(y_1, \ldots, y_n)\big)^* = \gamma(x_1, \ldots, x_m)^* \cup \delta(y_1, \ldots, y_n)^*,$$

$$(3) \qquad \big(\gamma(x_1, \ldots, x_m) \cap \delta(y_1, \ldots, y_n)\big)^* = \gamma(x_1, \ldots, x_m)^* \cap \delta(y_1, \ldots, y_n)^*,$$

$$(4) \qquad \big(\gamma(x_1, \ldots, x_m) \Rightarrow \delta(y_1, \ldots, y_n)\big)^* = \gamma(x_1, \ldots, x_m)^* \Rightarrow \delta(y_1, \ldots, y_n)^*,$$

$$(5) \qquad \big(-\gamma(x_1, \ldots, x_m)\big)^* = -\big(\gamma(x_1, \ldots, x_m)^*\big),$$

$$(6) \qquad \Big(\bigcup_\xi \gamma(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_m)\Big)^* = \bigcup_{j \in J} \gamma(x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_m)^*,$$

$$(7) \qquad \Big(\bigcap_\xi \gamma(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_m)\Big)^* = \bigcap_{j \in J} \gamma(x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_m)^*,$$

where the signs $\cup$, $\cap$, $\Rightarrow$, $-$, $\bigcup$, $\bigcap$ on the left-hand sides of the equations denote propositional connectives and quantifiers in the colloquial mathematical language, and these signs on the right-hand sides of the equations denote the Boolean operations in the two-element Boolean algebra $\boldsymbol{A}_0$.

Identities (6) and (7) should be explained in more detail. The propositional function $\bigcup_\xi \gamma(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_m)$ is true from the intuitive point of view (for fixed values of its arguments $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m$) if and only if there exists an element j in J such that $\gamma(x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_m)$ is true. By (1) this holds if and only if there exists a j in J such that $\gamma(x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_m)^* = V$ in $\boldsymbol{A}_0$. This statement is equivalent by II, § 4, (3), to the following condition:

$$\bigcup_{j \in J} \gamma(x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_m)^* = V.$$

Thus (6) is clear.

To explain (7) let us observe that, from the intuitive point of view, the propositional function $\bigcap_\xi \gamma(x_1, \ldots, x_{i-1}, \xi, x_{i+1}, \ldots, x_m)$ is true (for fixed $x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_m$) if and only if for every j in J the propositional function $\gamma(x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_m)$ is true. By (1) this holds

if and only if $\gamma(x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_m)^* = V$ for every j in J. By II, § 4, (3), the last condition is equivalent to the following:

$$\bigcap_{j \in J} \gamma(x_1, \ldots, x_{i-1}, j, x_{i+1}, \ldots, x_m)^* = V \quad \text{in} \quad \boldsymbol{A}_0 ,$$

which makes (7) clear.

Identities (2)-(7) assert that $\cup, \cap, \Rightarrow, -, \bigcup_\xi, \bigcap_\xi$ commute with *.

Consider now a fixed formalized language of the first order $\mathfrak{L} = \{A, T, F\}$ described in § 3. If $\mathfrak{I}$ is an interpretation of $\mathfrak{L}$ in a non-empty set J, then, for every formula α in $\mathfrak{L}$, $\alpha_{\mathfrak{I}}$ is a propositional function in J (see § 4). The propositional function $\alpha_{\mathfrak{I}}$ is or is not true in J. It is true if and only if the characteristic $(\alpha_{\mathfrak{I}})^*$ of $\alpha_{\mathfrak{I}}$ is identically equal to the unit element of the two-element Boolean algebra $\boldsymbol{A}_0$. In symbols:

$$(\alpha_{\mathfrak{I}})^* \equiv V . \tag{8}$$

A formula α in $\mathfrak{L}$ is said to be a *predicate tautology* (or simply: a *tautology*) if, for every interpretation $\mathfrak{I}$ of $\mathfrak{L}$ in any set $J \neq 0$, the propositional function $\alpha_{\mathfrak{I}}$ is true, i.e. identity (8) holds.

Roughly speaking, α is a predicate tautology, if it is true without any hypothesis about the signs of relations, functions and individual constants appearing in α. A formula is a tautology only because all its signs are connected in a special way that assures the truth of the formula in every interpretation. Thus we see that the intuitive notion of predicate tautology thus introduced agrees with our intuitive concept of propositional functions true by virtue of their syntax only.

We shall explain the above definition of predicate tautologies by some examples.

Let α be a formula of the following form:

$$(\bigcap_\xi (\varrho(\xi) \cup \pi(\xi)) \Rightarrow (\bigcap_\xi \varrho(\xi) \cup \bigcap_\xi \pi(\xi))) ,$$

where ϱ and π denote one-argument predicates. We shall show that it is not a tautology. Let J be the set of all positive integers and let us associate with ϱ the one-argument relation denoted by $\varrho_{\mathfrak{I}}$, such that for every positive integer j, $\varrho_{\mathfrak{I}}(j)$ holds if and only if j is even. Let us associate with π the one-argument relation denoted by $\pi_{\mathfrak{I}}$ which, for every positive integer j, holds if and only if j is odd. Then

$$\bigcap_{j \in J} (\varrho_{\mathfrak{I}}(j)^* \cup \pi_{\mathfrak{I}}(j)^*) = V$$

and

$$\bigcap_{j \in J} \varrho_{\mathfrak{I}}(j)^* \cup \bigcap_{j \in J} \pi_{\mathfrak{I}}(j)^* = \Lambda .$$

Hence, by (2), (4), (7) we obtain

$$(\alpha_{\mathfrak{I}})^* = V \Rightarrow \Lambda = \Lambda .$$

Thus α is not a tautology.

On the other hand, it is easy to see that the following formulas are predicate tautologies:

$$(\bigcap_\xi \varrho(\xi) \Rightarrow \varrho(x)), \quad (\varrho(x) \Rightarrow \bigcup_\xi \varrho(\xi)).$$

In fact, for every interpretation $\mathfrak{I}$ in every set $J \neq 0$ we have

$$((\bigcap_\xi \varrho(\xi) \Rightarrow \varrho(x))_\mathfrak{I})^* = \bigcap_{j \in J} \varrho_\mathfrak{I}(j)^* \Rightarrow \varrho_\mathfrak{I}(x)^* \equiv V$$

and

$$((\varrho(x) \Rightarrow \bigcup_\xi \varrho(\xi))_\mathfrak{I})^* = \varrho_\mathfrak{I}(x)^* \Rightarrow \bigcup_{j \in J} \varrho_\mathfrak{I}(j)^* \equiv V.$$

Now let $\alpha(a_1, \ldots, a_n)$ be a formula in a formalized language $\mathfrak{L}_0$ of the zero order, $a_1, \ldots, a_n$ being all propositional variables in this formula. Let $\alpha_1, \ldots, \alpha_n$ be any formulas in a formalized language $\mathfrak{L}$ of the first order. By 6.4, the substitution

$$\alpha(a_1/\alpha_1, \ldots, a_n/\alpha_n) \tag{9}$$

is a formula in $\mathfrak{L}$. It can easily be proved by induction on the length of α that, for every interpretation $\mathfrak{I}$ of $\mathfrak{L}$ in any set $J \neq 0$,

$$(\alpha(a_1/\alpha_1, \ldots, a_n/\alpha_n)_\mathfrak{I})^* = \alpha_{A_0}((\alpha_1)_\mathfrak{I})^*, \ldots, (\alpha_n)_\mathfrak{I})^*), \tag{10}$$

i.e. the characteristic of the interpretation of substitution (9) is the superposition of the Boolean polynomial α_{A_0} determined by α, and of the characteristics of the interpretations of $\alpha_1, \ldots, \alpha_n$ [1].

By the above hypothesis on α,

7.1. *If $\alpha(a_1, \ldots, a_n)$ is a propositional tautology in a formalized language $\mathfrak{L}_0$ of the zero order, then for all formulas $\alpha_1, \ldots, \alpha_n$ in a formalized language $\mathfrak{L}$ of the first order the substitution $\alpha(a_1/\alpha_1, \ldots, a_n/\alpha_n)$ is a predicate tautology in $\mathfrak{L}$.*

Indeed, $\alpha_{A_0} \equiv V$ since α is a propositional tautology. Hence it follows from (10) that the characteristic of any interpretation of (9) is identically equal to V, i.e. (9) is a predicate tautology.

§ 8. Rules of inference. All reasoning in mathematical proofs is composed of simple steps, each of which consists in acknowledging some sentences as immediate logical consequences of others. These steps also have analogues in formalized mathematical theories. These analogues, which are a formalization of the elementary intuitive links of deductive reasoning, are called rules of inference.

By a *rule of inference* we mean an operation which with a finite sequence of formulas $\alpha_1, \ldots, \alpha_n$ $(n \geqslant 1)$ in a formalized language associates another formula β in this language in such a way that, according

[1] Identity (10) follows also directly from a general theorem VI, 6.9.

to our intuition, β is a logical consequence of formulas $\alpha_1, \ldots, \alpha_n$. We shall note this fact as follows:

(1) $$\frac{\alpha_1, \ldots, \alpha_n}{\beta}.$$

(1) should be read:

$$\beta \quad \text{since} \cdot \alpha_1, \ldots, \alpha_n .$$

Formulas $\alpha_1, \ldots, \alpha_n$ will be called the *premises* and β the *conclusion* of (1).

We must explain more exactly what we mean by saying that a formula β is a logical consequence of a set S of formulas. From the intuitive point of view it means that if every formula α belonging to S is acknowledged to be true, then β must also be accepted as true. A little more precise explanation of this statement is given by definition (2) and (3) below.

Let $\mathfrak{L}$ be an elementary formalized language described in § 3. A formula β in $\mathfrak{L}$ is a logical consequence, from the intuitive point of view, of a set S of formulas of the language $\mathfrak{L}$, provided that

(2) *for every interpretation* $\mathfrak{I}$ *of* $\mathfrak{L}$ *in any set* $J \neq 0$, *if all the propositional functions* $\alpha_{\mathfrak{I}}$, *where* α *is in* S, *are true in* J (*i.e.* $(\alpha_{\mathfrak{I}})^* \equiv V$), *then* $\beta_{\mathfrak{I}}$ *is also true in* J (*i.e.* $(\beta_{\mathfrak{I}})^* \equiv V$).

Now we shall describe some rules of inference which will be applied to formalized elementary languages of the first order. Observe that all the rules of inference given below reduce to simple mechanical operations on finite sequences of signs.

(r_1) *Modus ponens* (*rule of detachment*). Let us consider any formulas α, β in $\mathfrak{L}$ and let us suppose that, in an interpretation $\mathfrak{I}$ in a set $J \neq 0$, α and $(\alpha \Rightarrow \beta)$ are true. Thus (see § 7, (8)) we have

$$(\alpha_{\mathfrak{I}})^* \equiv V \quad \text{and} \quad ((\alpha \Rightarrow \beta)_{\mathfrak{I}})^* \equiv V .$$

By § 4, (3), and § 7, (4),

$$(\alpha_{\mathfrak{I}})^* \Rightarrow (\beta_{\mathfrak{I}})^* \equiv V .$$

Hence, by II, § 4, (1), $(\beta_{\mathfrak{I}})^* \equiv V$, i.e. $\beta_{\mathfrak{I}}$ is true in J. Hence, according to (2), β is a consequence of formulas α and $(\alpha \Rightarrow \beta)$. The method of inference leading from α, $(\alpha \Rightarrow \beta)$ as premises to the conclusion β is universal in deductive reasoning. Therefore, the rule of inference

$$\frac{\alpha, (\alpha \Rightarrow \beta)}{\beta},$$

called *modus ponens* or the *rule of detachment*, will be assumed in all deductive systems considered in this book.

(r_2) *Rule of substitution for free individual variables.* Let $\alpha = \alpha(x_1, \ldots, x_m)$ be a formula in $\mathfrak{L}$ and let $\alpha' = \alpha(\tau_1, \ldots, \tau_m)$ be the formula obtained from α by substitution of some terms $\tau_1, \ldots, \tau_m$ for free individual variables $x_1, \ldots, x_m$, respectively. For every interpretation $\mathfrak{I}$ of $\mathfrak{L}$ in any set $J \neq 0$, the set of values of the characteristic $(\alpha'_{\mathfrak{I}})^*$ is contained in the set of values of the characteristic $(\alpha_{\mathfrak{I}})^*$ since $(\alpha'_{\mathfrak{I}})^*$ is the superposition of $(\alpha_{\mathfrak{I}})^*$ and the mappings $\tau_{1\mathfrak{I}}, \ldots, \tau_{m\mathfrak{I}}$. Consequently, if α is true in an interpretation $\mathfrak{I}$, i.e. if $(\alpha_{\mathfrak{I}})^*$ assumes only the value $\vee$, then $(\alpha'_{\mathfrak{I}})^*$ also assumes only the value $\vee$, i.e. α' is true in the interpretation $\mathfrak{I}$.

This reasoning suggests the following rule of inference:

$$\frac{\alpha(x_1, \ldots, x_m)}{\alpha(\tau_1, \ldots, \tau_m)}$$

called the *rule of substitution for free individual variables.*

(r_3) *Rule of introduction of an existential quantifier.* Let $\alpha(x), \beta$ be formulas in $\mathfrak{L}$. Suppose that β contains no occurrence of x and that no quantifier binding the variable ξ appears in $\alpha(x)$. Then the formula

$$(\bigcup_\xi \alpha(\xi) \Rightarrow \beta)$$

is a logical consequence of the formula

$$(\alpha(x) \Rightarrow \beta) .$$

Indeed, consider an interpretation $\mathfrak{I}$ of $\mathfrak{L}$ in a set $J \neq 0$. Suppose that $(\alpha(x) \Rightarrow \beta)$ is a propositional function true in J. Then its characteristic

$$((\alpha(x) \Rightarrow \beta)_{\mathfrak{I}})^* = (\alpha(x)_{\mathfrak{I}})^* \Rightarrow (\beta_{\mathfrak{I}})^* \equiv \vee .$$

Hence, by II, § 4, (2),

$$(\alpha(x)_{\mathfrak{I}})^* \leqslant (\beta_{\mathfrak{I}})^* \quad \text{for every } x \text{ in } J .$$

Since no occurrence of x appears in β we infer (II, § 4, (3)) that

$$\bigcup_{j \in J} \alpha_{\mathfrak{I}}(j)^* \leqslant (\beta_{\mathfrak{I}})^* .$$

Thus by II, § 4, (2), $\bigcup_{j \in J} \alpha_{\mathfrak{I}}(j)^* \Rightarrow (\beta_{\mathfrak{I}})^* \equiv \vee$, i.e.

$$((\bigcup_\xi \alpha(\xi) \Rightarrow \beta)_{\mathfrak{I}})^* \equiv \vee ,$$

which shows that the propositional function $(\bigcup_\xi \alpha(\xi) \Rightarrow \beta)_{\mathfrak{I}}$ is true in J.

The rule of inference

$$\frac{(\alpha(x) \Rightarrow \beta)}{(\bigcup_\xi \alpha(\xi) \Rightarrow \beta)}$$

where β contains no occurrence of x, and $\alpha(x)$ contains no quantifiers binding ξ, is called the *rule of introduction of an existential quantifier.*

(r_4) *Rule of introduction of a universal quantifier.* Let α and $\beta(x)$ be formulas of $\mathfrak{L}$. Let us suppose that no occurrence of x appears in α and that no sign of a quantifier binding a bound variable ξ appears in α. Then the formula

$$(\alpha \Rightarrow \bigcap_\xi \beta(\xi))$$

is a logical consequence of the formula

$$(\alpha \Rightarrow \beta(x)) .$$

Indeed, consider an interpretation $\mathfrak{J}$ of the language $\mathfrak{L}$ in a set J and suppose that $(\alpha \Rightarrow \beta(x))_\mathfrak{J}$ is a true propositional function in J. Then its characteristic satisfies

$$((\alpha \Rightarrow \beta(x))_\mathfrak{J})^* = (\alpha_\mathfrak{J})^* \Rightarrow (\beta(x)_\mathfrak{J})^* \equiv V .$$

Hence, by II, § 4, (2),

$$(\alpha_\mathfrak{J})^* \leqslant (\beta(x)_\mathfrak{J})^* \quad \text{for every} \quad x \in J .$$

Since no occurrence of x appears in α we infer (II, § 4, (3)) that

$$(\alpha_\mathfrak{J})^* \leqslant \bigcap_{j \in J} \beta_\mathfrak{J}(j)^* .$$

Thus, by II, § 4, (2), $(\alpha_\mathfrak{J})^* \Rightarrow \bigcap_{j \in J} \beta_\mathfrak{J}(j)^* \equiv V$, i.e.

$$((\alpha \Rightarrow \bigcap_\xi \beta(\xi)_\mathfrak{J}))^* \equiv V ,$$

i.e. $(\alpha \Rightarrow \bigcap_\xi \beta(\xi))_\mathfrak{J}$ is also true in J.

The rule of inference

$$\frac{(\alpha \Rightarrow \beta(x))}{(\alpha \Rightarrow \bigcap_\xi \beta(\xi))} ,$$

where α contains no occurrence of x and $\beta(x)$ contains no quantifier binding ξ, is called the *rule of introduction of a universal quantifier.*

(r_5) *Rule of elimination of an existential quantifier.* Let $\alpha(x)$ and β be formulas of the language $\mathfrak{L}$ and let a bound individual variable ξ not appear in $\alpha(x)$. The formula

$$(\alpha(x) \Rightarrow \beta)$$

is—from the intuitive point of view—a consequence of the formula

$$(\bigcup_\xi \alpha(\xi) \Rightarrow \beta) .$$

Indeed, consider its interpretation $(\bigcup_\xi \alpha(\xi) \Rightarrow \beta)_\mathfrak{J}$ in a non-empty set J. If this interpretation is a true propositional function in J, then

$$\left((\bigcup_\xi \alpha(\xi) \Rightarrow \beta)_\mathfrak{J}\right)^* \equiv V,$$

i.e., by § 4, (3), (5) and § 7, (4), (6)

$$\bigcup_{j \in J} \alpha_\mathfrak{J}(j)^* \Rightarrow (\beta_\mathfrak{J})^* \equiv V.$$

Thus by II, § 4, (2)

$$\bigcup_{j \in J} \alpha_\mathfrak{J}(j)^* \leqslant (\beta_\mathfrak{J})^*.$$

Consequently, by II, § 4, (3),

$$\left(\alpha(x)_\mathfrak{J}\right)^* \leqslant (\beta_\mathfrak{J})^*.$$

Hence

$$\left((\alpha(x) \Rightarrow \beta)_\mathfrak{J}\right)^* \equiv V,$$

which shows that $(\alpha(x) \Rightarrow \beta)_\mathfrak{J}$ is a propositional function true in J.

The rule of inference

$$\frac{(\bigcup_\xi \alpha(\xi) \Rightarrow \beta)}{(\alpha(x) \Rightarrow \beta)}$$

is called the *rule of elimination of an existential quantifier.*

(r_6) *Rule of elimination of a universal quantifier.* Let α and $\beta(x)$ be any formulas of the language $\mathfrak{L}$. From the intuitive point of view the formula

$$(\alpha \Rightarrow \beta(x))$$

is a consequence of the formula

$$(\alpha \Rightarrow \bigcap_\xi \beta(\xi)),$$

where ξ does not appear in $\beta(x)$.

Indeed, consider an interpretation $\mathfrak{J}$ of the language $\mathfrak{L}$ in a non-empty set J. If the interpretation $(\alpha \Rightarrow \bigcap_\xi \beta(\xi))_\mathfrak{J}$ of (3) is a true propositional function in the set J, then its characteristic

$$\left((\alpha \Rightarrow \bigcap_\xi \beta(\xi))_\mathfrak{J}\right)^* \equiv V,$$

i.e., by § 4, (6), (3) and § 7, (7), (4),

$$(\alpha_\mathfrak{J})^* \Rightarrow \bigcap_{j \in J} \beta_\mathfrak{J}(j)^* \equiv V.$$

Consequently, by II, § 4, (2),

$$(\alpha_\mathfrak{J})^* \leqslant \bigcap_{j \in J} \beta_\mathfrak{J}(j)^*.$$

Hence, by II, § 4, (3), (2),

$$(\alpha_\mathfrak{J})^* \leqslant \left(\beta(x)_\mathfrak{J}\right)^*, \quad \text{i.e.} \quad (\alpha_\mathfrak{J})^* \Rightarrow \left(\beta(x)_\mathfrak{J}\right)^* \equiv V.$$

Thus

$$(\alpha \Rightarrow \beta(x))_{\mathfrak{J}}^* \equiv V,$$

and in consequence $(\alpha \Rightarrow \beta(x))_{\mathfrak{J}}$ is a propositional function true in J.

The rule of inference

$$\frac{(\alpha \Rightarrow \bigcap_{\xi} \beta(\xi))}{(\alpha \Rightarrow \beta(x))}$$

is called the *rule of elimination of a universal quantifier.*

Besides formalized languages of the first order we shall also examine, as an auxiliary notion, formalized languages $\mathfrak{L}_0$ of the zero order, described in § 6. From the intuitive point of view, a formula β in $\mathfrak{L}_0$ is said to be a logical consequence of a set S of formulas in $\mathfrak{L}_0$ provided that

(3) for any interpretation $\mathfrak{J}$ of $\mathfrak{L}_0$, if all sentences $\alpha_{\mathfrak{J}}$ (see § 6, p. 168), where α is in S, are true, then $\beta_{\mathfrak{J}}$ is also a true sentence.

Interpreting formulas in $\mathfrak{L}_0$ as Boolean polynomials in the two-element Boolean algebra A_0, we can formulate the last definition as follows: a formula β in $\mathfrak{L}_0$ is said to be a logical consequence of a set S of formulas in $\mathfrak{L}_0$ provided that

(3′) if all Boolean polynomials α_{A_0}, where α is any formula in S, assume the value V for a fixed value of variables, then β_{A_0} also assumes the value V for the same value of the variables.

For instance, let α and β be any formulas in $\mathfrak{L}_0$. If for a fixed value of the variables, both the Boolean polynomials α_{A_0} and $(\alpha \Rightarrow \beta)_{A_0}$ (i.e. $\alpha_{A_0} \Rightarrow \beta_{A_0}$) assume the value V, then β_{A_0} also assumes the value V for this value of the variables, because of II, § 4, (1). Thus formula β is a logical consequence of formulas α and $(\alpha \Rightarrow \beta)$. In other words, the rule of detachment (*modus ponens*)

$$(r_1) \qquad \frac{\alpha,\ (\alpha \Rightarrow \beta)}{\beta}$$

is also a rule of inference for arbitrary formalized languages of the zero order.

Of course, none of the rules (r_2)-(r_5) has any sense for languages of the zero order.

It follows from definitions (2) and (3) of consequence in languages of the first and zero order that if all formulas in S are tautologies and β is a logical consequence of S, then β is also a tautology. In fact, for any interpretation $\mathfrak{J}$ of the language, $\alpha_{\mathfrak{J}}$ is true for every α in S. Thus $\beta_{\mathfrak{J}}$ is true for every interpretation $\mathfrak{J}$, i.e. β is a tautology. Hence it follows, by the definition of rules of inference, that

8.1. *If all premises in a rule of inference are tautologies, so is the conclusion.*

We should like to emphasize a property of rules of inference (r_1)-(r_6). They were justified by arguments based on interpretations of formalized languages. However, in order to verify that a given formula β is or is not the conclusion in one of the rules (r_1)-(r_6), we do not have to use interpretations: it suffices to compare only the syntax of the conclusion with the syntax of the premises, i.e. to compare the manner in which the conclusion and the premises are composed of signs. Moreover, this verification is very simple: in all the rules (r_1)-(r_6) the conclusion is obtained from premises by means of simple operations on formulas conceived as finite sequences of signs. This syntactic character and simplicity are required in practice from all rules of inference assumed in formalized languages.

§ 9. Formal proofs. The process of deducing some sentences from others is the most important part of mathematical work. The proof of a sentence (mathematical theorem) consists in deducing this sentence from a given set of sentences recognized as true, by means of a series of elementary steps here called rules of inference. In the process of proving we also make use, more or less consciously, of some logical tautologies.

We shall now define the notion of a formal proof in a formalized language. This notion is a precise analogue of the process of deduction in intuitive mathematics.

Let $\mathfrak{L}$ be a formalized language of the first order or of the zero order (see § 3 and § 6). A formula β in $\mathfrak{L}$ is said to be an *immediate consequence of formulas* $\alpha_1, \ldots, \alpha_m$ *in* $\mathfrak{L}$ *by means of a rule of inference* (r) if β has the form of the conclusion in the scheme defining the rule (r), while the formulas $\alpha_1, \ldots, \alpha_m$ have the form of premises of this scheme. For instance, a formula β is an immediate consequence of formulas α and $(\alpha \Rightarrow \beta)$ by means of the rule (r_1) of detachment.

Let $\mathfrak{A}_0$ be a set of formulas in $\mathfrak{L}$ and let $R_1, \ldots, R_k$ be rules of inference. We say that a formula α is *derivable from* $\mathfrak{A}_0$ *by means of the rules of inference* $R_1, \ldots, R_k$ if there exists a finite sequence of formulas in $\mathfrak{L}$

$$\alpha_1, \ldots, \alpha_n$$

such that:

1) α is identical with α_n;

2) for every $i \leqslant n$, either the formula α_i belongs to $\mathfrak{A}_0$, or it is an immediate consequence of some of the formulas $\alpha_1, \ldots, \alpha_{i-}$ by means of one of the rules of inference $R_1, \ldots, R_k$.

Then the sequence $a_1, \ldots, a_n$ is said to be a *formal proof of* a *from* $\mathcal{A}_0$ *by means of the rules of inference* $R_1, \ldots, R_k$.

Sometimes the set $\mathcal{A}_0$ is represented as the union of a set $\mathcal{A}_1$ of tautologies and of a set $\mathcal{A}$ of other formulas. Then if a is derivable from the union of $\mathcal{A}$ and $\mathcal{A}_1$ by means of the rules $R_1, \ldots, R_k$, we say also that a is *derivable from* $\mathcal{A}$ *by means of the rules of inference* $R_1, \ldots, R_k$ *and tautologies in* $\mathcal{A}_1$. The formal proof $a_1, \ldots, a_n$ of a is then also called a *formal proof of* a *from* $\mathcal{A}$ *by means of the rules of inference* $R_1, \ldots, R_k$ *and tautologies in* $\mathcal{A}_1$.

We shall illustrate the above definitions by examples.

9.1. *Each of the following formulas (where* α, β *are formulas*):

$$(\alpha \Rightarrow (\beta \Rightarrow \alpha)), \tag{1}$$

$$(\alpha \Rightarrow \alpha) \tag{2}$$

is derivable from the set $\mathcal{A}$ *of all formulas of the form*

$$((\gamma_1 \cap \gamma_2) \Rightarrow \gamma_1), \tag{3}$$

$$((\gamma_1 \cap \gamma_2) \Rightarrow \gamma_2), \tag{4}$$

$$(((\gamma_1 \cap \gamma_2) \Rightarrow \gamma_3) \Rightarrow (\gamma_1 \Rightarrow (\gamma_2 \Rightarrow \gamma_3))) \tag{5}$$

(*where* γ_1, γ_2, γ_3 *are any formulas*) *by means of the rule of detachment.*

Indeed, the sequence of formulas

$$((\alpha \cap \beta) \Rightarrow \alpha),$$
$$(((\alpha \cap \beta) \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow (\beta \Rightarrow \alpha))),$$
$$(\alpha \Rightarrow (\beta \Rightarrow \alpha))$$

is a formal proof of (1) since the first formula is of the form (3) and therefore belongs to $\mathcal{A}$, the second is of the form (5) and therefore belongs to $\mathcal{A}$, and the third formula is obtained from the first and the second by *modus ponens*.

The formal proof of (2) is a little longer:

$$((\alpha \cap \alpha) \Rightarrow \alpha),$$
$$((((\alpha \cap \alpha) \Rightarrow \alpha) \cap \alpha) \Rightarrow \alpha),$$
$$(((((\alpha \cap \alpha) \Rightarrow \alpha) \cap \alpha) \Rightarrow \alpha) \Rightarrow (((\alpha \cap \alpha) \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha))),$$
$$(((\alpha \cap \alpha) \Rightarrow \alpha) \Rightarrow (\alpha \Rightarrow \alpha)),$$
$$(\alpha \Rightarrow \alpha).$$

The first and second formulas belong to $\mathcal{A}$ since they are of the form (4). The third formula belongs to $\mathcal{A}$ since it is of the form (5). The fourth

formula is an immediate consequence of the second and third by means of *modus ponens*. The last formula is an immediate consequence of the first and the fourth by means of *modus ponens*.

§ 10. The consequence operations. Formalized deductive systems and theories. Let $\mathfrak{L}$ be a formalized language of the first order or of the zero order (see § 3 and § 6). We know that formulas in $\mathfrak{L}$ are formal substitutes for sentences and propositional functions. An operation C which, to every set S of formulas in $\mathfrak{L}$, assigns the set $C(S)$ of all formulas recognized as logical consequences of S will be called a *consequence operation in* $\mathfrak{L}$. By a *deductive system* based on the language $\mathfrak{L}$ we shall understand the pair $\mathfrak{S} = \{\mathfrak{L}, C\}$, where C is a consequence operation in $\mathfrak{L}$.

The consequence operation in a language $\mathfrak{L}$ should be a formalization *of the* intuitive concept of the consequence, and therefore it should have all the properties which our intuition attribute to this notion. For instance, the following conditions [1] are expected to be satisfied for any sets S, S_1, S_2, S_3 of formulas in $\mathfrak{L}$:

(c_1) $S \subset C(S)$,

(c_2) *if* $S_1 \subset S_2$, *then* $C(S_1) \subset C(S_2)$,

(c_3) *if* $S_1 \subset C(S_2)$ *and* $S_2 \subset C(S_3)$, *then* $S_1 \subset C(S_3)$.

The first two conditions require no explanation. The third expresses the transitivity of consequence.

Condition (c_3) can be replaced by the following:

(c_3') $C\big(C(S)\big) = C(S)$.

Conditions (c_1), (c_2), (c_3') are equivalent to conditions (c_1), (c_2), (c_3). In fact, by substituting

$$S_1 = C\big(C(S)\big), \qquad S_2 = C(S), \qquad S_3 = S$$

in (c_3), we obtain

$$C\big(C(S)\big) \subset C(S).$$

On the other hand, it follows from (c_1) and (c_2) that

$$C(S) \subset C\big(C(S)\big),$$

which with the previous inclusion gives (c_3'). Conversely, suppose that (c_3') is satisfied. If $S_2 \subset C(S_3)$, then by (c_2) we obtain $C(S_2) \subset C\big(C(S_3)\big) = C(S_3)$, which proves (c_3).

Another property to be expected is as follows:

(c_4) *if* β *is in* $C(S)$, *then there exists a finite subset* $S_0 \subset S$ *such that* β *is in* $C(S_0)$.

---

[1] Tarski [2].

This condition expresses the finiteness of the process of deduction if β can be deduced from S, then β can be deduced from a finite number of formulas in S.

A consequence operation C in $\mathfrak{L}$ is a complete substitute for the intuitive concept of logical consequence if it satisfies the following condition:

(C) *For any set S of formulas in $\mathfrak{L}$, the set $C(S)$ contains those and only those formulas β in $\mathfrak{L}$ which are logical consequences of S in the sense explained in* § 8, (2), *or* (3), i.e. *those formulas β which are true in all the interpretations of $\mathfrak{L}$ in which all formulas α in S are true.*

Condition (C) implies, in particular (when the set S is empty) the following property:

(C′) *The set $C(0)$ coincides with the set of all tautologies in $\mathfrak{L}$.*

Statement (C) can, of course, be considered as a definition of a consequence operation C in any formalized language of the first or zero order. That is: for any set S of formulas in a formalized language of the first order (or: of the zero order) we could define the set $C(S)$ as the set of all formulas for which condition § 8, (2) (or: § 8, (3)) holds. This definition of the consequence operation completely agrees with the intuitive meaning of consequence. It has property (C), by definition.

However, we shall not assume this definition of the consequence operation C because it is too complicated: to verify that a formula is a consequence of a set S of formulas we have to examine all possible interpretations of the language! We also have to avoid this definition because it is far removed from the fundamental ideas of the process of formalization: it is based on intuitive interpretations of formulas in formalized languages instead of the direct definition of formulas in formalized languages. Formulas are finite sequences of signs. The fact of whether a formula β is a consequence of a set S of formulas depends only on the syntax of β and of formulas α in S, i.e. it depends only on the manner in which the formulas α, β are constructed signs. Consequently we can expect that it is possible to formulate another equivalent definition of the consequence operation based exclusively on syntactical properties of formulas conceived as finite sequences of signs.

In this book we shall examine various consequence operations. All definitions of consequence operations will always have a purely syntactic character. Namely all consequence operations will be defined by means of the following scheme:

(S) Let $\mathfrak{L}$ be a formalized language of the first order or the zero order. First we fix a set $\mathscr{A}_1$ of tautologies (called the *logical axioms*) and a set $R_1, \ldots, R_k$ of rules of inference. For every set S of formulas in $\mathfrak{L}$

we define $C(S)$ as the set of all formulas (in $\mathfrak{L}$) which are derivable from S by means of the rules of inference $R_1, \ldots, R_k$ and tautologies in $\mathcal{A}_1$ (see § 9, p. 180).

The consequence operation C defined by (S) in $\mathfrak{L}$ is said to be *determined by the rules of inference* $R_1, \ldots, R_k$ *and the set* A_1 *of logical axioms.*

The rules of inference in (S) will always have a syntactic character. The conclusion of each of the rules $R_1, \ldots, R_k$ will always be the result of simple mechanical operations on premises conceived as sequences of signs. Except for a part of Chapter XI, the set of rules of inference adopted will be formed of the rules (r_1)-(r_6) from § 8 in the case of languages of the first order, and of *modus ponens* in the case of languages of the zero order. Consequently the definition of C by means of the scheme (S) will have a purely syntactic character. By definition, we include a formula β in the set $C(S)$ if and only if there exists a formal proof

$$(1) \qquad \alpha_1, \ldots, \alpha_n$$

for β from S by means of $R_1, \ldots, R_k$ and the logical axioms in $\mathcal{A}_1$. The verification that (1) is a formal proof for β is completely mechanical. It reduces to the verification that each of the formulas α_j is in S or in $\mathcal{A}_1$ or is obtained from formulas α_i with $i < j$ by the mechanical operations represented by one of the rules of inference $R_1, \ldots, R_k$. Thus the process of forming the set $C(S)$, which is a formalized substitute for the intuitive process of deduction, consists in performing mechanical operations on formulas conceived as sequences of signs.

It is not obvious whether it is possible to define operation C by means of scheme (S) in such a way that condition (C) is satisfied. The theorem stating that such a formalization of the notion of consequence is feasible, is one of the most important theorems in metamathematics (see VIII, § 9). The definition required will be given in the second part of § 11.

It is easy to see that we can always choose $\mathcal{A}_1$ and $R_1, \ldots, R_k$ so that (C′) holds. For instance, if $\mathcal{A}_1$ is the set of all predicate tautologies and we admit the rules (r_1)-(r_6), § 8, as the rules of inference, then condition (C′) holds. This follows from 8.1. It seems natural to assume all predicate tautologies of a formalized language $\mathfrak{L}$ as the set of all logical axioms. However, for any language $\mathfrak{L}$ described in § 3, the definition of predicate tautology given in § 7 is rather complicated, as it requires examination of all interpretations of $\mathfrak{L}$ in all non-empty sets. We have no general method of deciding by means of a finite number of steps whether an arbitrary formula is a predicate tautology or not. To avoid difficulties

of this kind, in all definitions of a consequence operation by means of scheme (S) we shall assume as $\mathscr{A}_\mathfrak{l}$ only a set of tautologies whose structure is simply described.

Observe that sometimes consequence operations are examined which are defined by scheme (S) and do not have properties (C), (C'). The reason for the examination of such consequence operations is partly methodological and partly philosophical [1]. From the methodological point of view it is sometimes interesting to know what formulas are derivable from a given set S of formulas only by means of a smaller set of rules of inference or by means of a smaller set of logical axioms. Some philosophical reasons for restricting the consequence operation will be explained in Chapter IX, § 1.

The set $C(0)$ plays a special role in any deductive system $\mathfrak{S} = \{\mathfrak{L}, C\}$. Formulas in $C(0)$ are said to be *derivable* in the deductive system $\mathfrak{S}$. If C is defined by means of (S), a formula β is derivable in $\mathfrak{S}$ if and only if it is derivable from $\mathscr{A}_\mathfrak{l}$ by means of $R_1, \ldots, R_k$, i.e. if there exists a formal proof of β from $\mathscr{A}_\mathfrak{l}$ by means of $R_1, \ldots, R_k$. Since all logical axioms in $\mathscr{A}_\mathfrak{l}$ are supposed to be tautologies and since all rules of inference lead from tautologies to tautologies (see 8.1), all formulas in $C(0)$ are tautologies. If C does not have property (C'), $C(0)$ is a proper subset of the set of all tautologies. This is the case when, for some reason, the logical means in the deductive system $\mathfrak{S}$ are restricted. $C(0)$ is then the set of tautologies admitted in $\mathfrak{S}$.

If a consequence operation C in a formalized language $\mathfrak{L}$ is definable by means of the scheme (S), then—in general—it is definable by (S) in various ways, i.e. the set of assumed rules of inference and the assumed set of logical axioms can, in general, be modified without changing operation C. The assumed sets of logical axioms and rules of inference have only an auxiliary character. From the point of view of the investigations made in this book, the choice of the set of logical axioms and rules of inference to define a consequence operation is completely arbitrary. The reason for which a set of rules and axioms is selected is usually of a didactic or methodological character. For example, we may wish to have a rather small set of axioms and rules, or a set from which it is easy to deduce the fundamental properties of the consequence operation. The sets of axioms quoted in this book are not the simplest. They were chosen mainly because lattice-theoretical properties of deductive systems

[1] For instance, the conviction that falsity and truth are not the only logical values of sentences gave rise to the idea of many-valued logics. Then the consequence operations have not the property (C). The first many-valued logic (a 3-valued logic) is due to Łukasiewicz [1]. The m-valued logics (for any positive integer m) were discussed by Post [1]; see also Łukasiewicz and Tarski [1]. Cf. Rosser and Turquette [1].

and formalized theories can easily be deduced from these sets of axioms (see VI, § 10).

Now we shall quote some properties of any consequence operation defined by means of the scheme (S).

10.1. *The consequence operation C in $\mathfrak{L}$ defined by* (S) *has properties* (c_1), (c_2), (c_3), (c_3') and (c_4).

(c_1) follows immediately from the definition of $C(S)$ since, for every formula a in S, the sequence formed only of the formula a is a formal proof of a by means of $R_1, \ldots, R_k$ and $\mathscr{A}_\mathfrak{l}$. (c_2) also follows immediately from the definition since, if $S_1 \subset S_2$, then every formal proof from S_1 is also a formal proof from S_2, by means of the same rules of inference $R_1, \ldots, R_k$ and $\mathscr{A}_\mathfrak{l}$.

In order to prove (c_3') it suffices, because of (c_1) and (c_2), to show that $C\big(C(S)\big) \subset C(S)$. This inclusion follows from the following remark: if $a_1, \ldots, a_n$ is a formal proof of a formula a from $C(S)$ by means of $R_1, \ldots, R_k$ and $\mathscr{A}_\mathfrak{l}$, then some of them, say $a_{p_1}, a_{p_2}, \ldots, a_{p_r}$, belong to $C(S)$ and the remaining formulas, say $a_{q_1}, a_{q_2}, \ldots, a_{q_s}$, either belong to $\mathscr{A}_\mathfrak{l}$ or are immediate consequences of the preceding formulas by means of the rules $R_1, \ldots, R_k$ $(p_1 < \ldots < p_r,\ q_1 < \ldots < q_s,\ r+s = n)$. Let $a_{j1}, \ldots, a_{jn_j}$ be a formal proof of a_{p_j}. Then the sequence

$$a_{11}, \ldots, a_{1n_1}, a_{21}, \ldots, a_{2n_2}, \ldots, a_{r1}, \ldots, a_{rn_r}, a_{q_1}, a_{q_2}, \ldots, a_{q_s}, a$$

is a formal proof of a from S by means of $R_1, \ldots, R_k$ and $\mathscr{A}_\mathfrak{l}$. Consequently, if a is in $C\big(C(S)\big)$, then a belongs to $C(S)$.

As we have stated on p. 181, properties (c_1), (c_2), (c_3') imply (c_3).

To prove (c_4) let us observe that if a is in $C(S)$ then a also belongs to $C(S_0)$ where S_0 is the finite set of all the formulas from a fixed formal proof of a, which belong to S.

In order to give another characterization of the consequence operation C defined by means of (S) we shall introduce the following definition.

A set S_0 of formulas in a language $\mathfrak{L}$ is said to be *closed under a rule of inference* R if all immediate consequences of formulas in S_0 by means of rule R belong to S_0.

For instance, the set of all tautologies in a formalized language of the first order is closed under each of the rules of inference (r_1)-(r_6) described in § 8 by Theorem 8.1. The set of all tautologies of a formalized language of the zero order is closed under *modus ponens* by Theorem 6.3.

Observe that, for any set S of formulas (in a formalized language $\mathfrak{L}$) and for any given rules of inference $R_1, \ldots, R_k$, there exists a smallest set S_0 of formulas in $\mathfrak{L}$ such that

a) S is a subset of S_0,

b) S_0 is closed under all rules $R_1, \ldots, R_k$.

Indeed, there exist sets S_0 satisfying conditions a) and b). For example, the set F of all formulas in $\mathfrak{L}$ has these properties. On the other hand, the intersection of any class of sets S_0 with properties a) and b) also has these properties. Thus the intersection of all sets S_0 satisfying a) and b) is the smallest set satisfying these conditions.

10.2. *If the consequence operation C in a formalized language $\mathfrak{L}$ is defined by means of the scheme* (S), *then, for every set S of formulas in $\mathfrak{L}$, $C(S)$ is the smallest set (of formulas in $\mathfrak{L}$) containing $\mathscr{A}_\mathfrak{l}$ and S and closed under each of the rules of inference $R_1, \ldots, R_k$.*

First observe that if β is an immediate consequence of formulas $\alpha_1, \ldots, \alpha_m$ by means of R_i $(1 \leqslant i \leqslant k)$ and all the formulas $\alpha_1, \ldots, \alpha_m$ belong to $C(S)$, then β also belongs to $C(S)$. In fact, β belong then to $C\big(C(S)\big)$, i.e. to $C(S)$ by (c_3').

Thus $C(S)$ is closed under each of the rules $R_1, \ldots, R_k$, and contains S and $\mathscr{A}_\mathfrak{l}$. On the other hand, all formulas in any formal proof from S by means of $R_1, \ldots, R_k$ and $\mathscr{A}_\mathfrak{l}$ belong to any set B which is closed under each of the rules $R_1, \ldots, R_k$ and contains S and $\mathscr{A}_\mathfrak{l}$. Hence it follows that $C(S)$ is contained in any set B which is closed under each of the rules $R_1, \ldots, R_k$ and contains S and $\mathscr{A}_\mathfrak{l}$. This proves 10.2.

As was stated in § 1 (p. 151), by a formalized theory we shall understand any system $\mathfrak{T} = \{\mathfrak{L}, C, \mathscr{A}\}$ in which $\mathfrak{L}$ is a formalized language, C a consequence operation in $\mathfrak{L}$, and $\mathscr{A}$ a set of formulas called the *mathematical* (or: *specific*) *axioms* of the theory $\mathfrak{T}$. In this book we shall deal mainly with the case in which $\mathfrak{L}$ is a formalized language of the first order, described in § 3. Formalized theories of this kind will be called *elementary* for the reasons explained in § 3, p. 159. As an auxiliary notion we shall also consider the case in which $\mathfrak{L}$ is a formalized language of the zero order, described in § 6.

Roughly speaking, the set $C(\mathscr{A})$ plays the same role in formalized theories as the set $C(0)$ in the case of deductive systems. Formulas in $C(\mathscr{A})$ will be called *theorems* in the theory $\mathfrak{T}$ concerned. In the same deductive system $\{\mathfrak{L}, C\}$, different sets $\mathscr{A}$, $\mathscr{A}'$ of mathematical axioms can give the same sets of theorems: $C(\mathscr{A}) = C(\mathscr{A}')$. Then, from the intuitive point of view, the sets $\mathscr{A}$, $\mathscr{A}'$ of mathematical axioms are equivalent, i.e. they are different sets of axioms for the same mathematical theory. Thus, from the mathematical point of view, it is irrelevant what set of mathematical axioms of the theory was selected. Similarly, as in the case of logical axioms, the reason for which a set $\mathscr{A}$ of mathematical axioms of a theory was selected from the class of all equivalent sets of axioms is usually of didactic or methodological character. For instance, the class of all theorems of the theory in question can be assumed to be

a set of mathematical axioms, but such a procedure is not applied in practice because it is very difficult to describe this set, except for some trivial cases. Therefore the accepted set $\mathcal{A}$ of axioms of a formalized mathematical theory is, in practice, a small subset of the set of all theorems. Such a procedure is applied in intuitive axiomatic mathematics. Our definition of formalized theory is intended to underline this procedure.

If the set of mathematical axioms is empty, there is no reason to distinguish between $\{\mathfrak{L}, C, 0\}$ and $\{\mathfrak{L}, C\}$. Thus we shall always identify the deductive system $\mathfrak{S} = \{\mathfrak{L}, C\}$ with the theory $\{\mathfrak{L}, C, 0\}$.

§ 11. The general notion of logic. The classical logic. It follows from considerations in § 10 and at the end of § 1 that our tendency is to concentrate in the consequence operation C in $\mathfrak{L}$ all logical means applied in any formalized theory $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$. Thus the consequence operation C in a formalized theory $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ should also be called the *logic of* $\mathfrak{T}$.

In applications we do not define the consequence operation for every formalized language separately. Indeed we define a general method $\mathfrak{C}$ which, to every formalized language $\mathfrak{L}$ of the zero or first order, uniquely assigns a consequence operation $C = \mathfrak{C}_{\mathfrak{L}}$ in $\mathfrak{L}$. Such a general method will be called a *consequence operation*. Observe the terminological difference: $\mathfrak{C}$ is a consequence operation; $\mathfrak{C}_{\mathfrak{L}}$ is a consequence operation *in* $\mathfrak{L}$.

In agreement with our intuition connected with the notion of consequence operation, every such general method $\mathfrak{C}$ will also be called a *logic*.

In Chapters VI-X we shall consider only logics $\mathfrak{C}$ defined by means of the following scheme:

(L) We distinguish first a set $\mathfrak{l}$ of propositional tautologies in an auxilliary language of zero order. For every formalized language $\mathfrak{L}$ (of the zero or first order) the set $\mathcal{A}_{\mathfrak{l}}$ of logical axioms in $\mathfrak{L}$ consists of all formulas in $\mathfrak{L}$ which are substitutions of tautologies in $\mathfrak{l}$ [1]. If $\mathfrak{L}$ is a formalized language of zero order, then the consequence operation $\mathfrak{C}_{\mathfrak{L}}$ in $\mathfrak{L}$ is defined by scheme (S) in § 10 where the set of rules of inference is formed of *modus ponens* only. If $\mathfrak{L}$ is a formalized language of the first order, then the consequence operation $\mathfrak{C}_{\mathfrak{L}}$ in $\mathfrak{L}$ is defined by scheme (S) from § 10 where the set of assumed rules of inference is composed of all the rules (r_1), (r_2), (r_3), (r_4), (r_5), (r_6) described in § 8.

Thus if $\mathfrak{L}$ is a formalized language of zero order, then for every set S of formulas in $\mathfrak{L}$ the set $\mathfrak{C}_{\mathfrak{L}}(S)$ of all consequences is the least set (of all formulas in $\mathfrak{L}$) containing all formulas in $\mathcal{A}_{\mathfrak{l}}$ and S, and closed under the rule of detachment. If $\mathfrak{L}$ is a formalized language

[1] That is, formulas in $\mathfrak{L}$ which are results of substitutions of some formulas in $\mathfrak{L}$ for propositional variables in tautologies in $\mathfrak{l}$.

of the first order then, for every set S of formulas in $\mathfrak{L}$, the set $C_{\mathfrak{L}}(S)$ is the least set (of formulas in $\mathfrak{L}$) containing all formulas in $\mathscr{A}_{\mathrm{I}}$ and S, and closed under the rule of detachment, the rule of substitution for free individual variables, the rules of introduction of existential and universal quantifiers, and the rules of elimination of existential and universal quantifiers.

In this book the word *logic* will always denote a consequence operation C given by the scheme (L).

According to the general definition from § 10, if C is a consequence operation and $\mathfrak{L}$ a formalized language, then $\mathfrak{S} = \{\mathfrak{L}, C_{\mathfrak{L}}\}$ is a deductive system and, for any given set $\mathscr{A}$ of formulas in $\mathfrak{L}$, $\mathfrak{T} = \{\mathfrak{L}, C_{\mathfrak{L}}, \mathscr{A}\}$ is a formalized theory. For simplicity, we shall usually write

$$\{\mathfrak{L}, C\} \quad \text{instead of} \quad \{\mathfrak{L}, C_{\mathfrak{L}}\}$$

and

$$\{\mathfrak{L}, C, \mathscr{A}\} \quad \text{instead of} \quad \{\mathfrak{L}, C_{\mathfrak{L}}, \mathscr{A}\}.$$

Moreover, if the language $\mathfrak{L}$ in question is fixed, we shall write, for brevity,

$$C(S) \quad \text{instead of} \quad C_{\mathfrak{L}}(S)$$

for any set S of formulas, if it does not lead to misunderstanding (however the subscript $\mathfrak{L}$ for C will often be necessary if two or more languages are considered simultaneously).

The following example of logic is most important: by the *two-valued* (or: *classical*) *logic* we shall understand the consequence operation C defined by the scheme (L) where the set I is composed of all the tautologies (t_1)-(t_{12}) of § 6 (p. 168).

To the end of this section C will always denote the two-valued logic. By definition, for every formalized language $\mathfrak{L}$ (of the zero or first order), we assume as the set $\mathscr{A}_{\mathrm{I}}$ of all logical axioms the set formed of all formulas of the form [1]:

(T_1) $$((\alpha \Rightarrow \beta) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma))),$$

(T_2) $$(\alpha \Rightarrow (\alpha \cup \beta)),$$

(T_3) $$(\beta \Rightarrow (\alpha \cup \beta)),$$

(T_4) $$((\alpha \Rightarrow \gamma) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow ((\alpha \cup \beta) \Rightarrow \gamma))),$$

(T_5) $$((\alpha \cap \beta) \Rightarrow \alpha),$$

(T_6) $$((\alpha \cap \beta) \Rightarrow \beta),$$

---

[1] The first set of axioms for the classical logic was given by Frege [1]. Many other sets of axioms are known in the literature; see e.g. Łukasiewicz [2], Hilbert and Ackermann [1], Rasiowa [1].

(T_7) $$((\gamma \Rightarrow \alpha) \Rightarrow ((\gamma \Rightarrow \beta) \Rightarrow (\gamma \Rightarrow (\alpha \cap \beta))))\,,$$

(T_8) $$((\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \cap \beta) \Rightarrow \gamma))\,,$$

(T_9) $$(((\alpha \cap \beta) \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow (\beta \Rightarrow \gamma)))\,,$$

(T_{10}) $$((\alpha \cap -\alpha) \Rightarrow \beta)\,,$$

(T_{11}) $$((\alpha \Rightarrow (\alpha \cap -\alpha)) \Rightarrow -\alpha)\,,$$

(T_{12}) $$(\alpha \cup -\alpha)\,,$$

where α, β, γ are arbitrary formulas in $\mathfrak{L}$. If $\mathfrak{L}$ is a formalized language of the zero order, then, for every set S of formulas in $\mathfrak{L}$, the set $C(S)$ (i.e. the set $C_{\mathfrak{L}}(S)$) is the set of all formulas derivable from S by means of *modus ponens* and tautologies (T_1)-(T_{12}). If $\mathfrak{L}$ is a formalized language of the first order then, for every set S of formulas in $\mathfrak{L}$, the set $C(S)$ (i.e. the set $C_{\mathfrak{L}}(S)$) is the set of all formulas derivable from S by means of the rules of inference (r_1)-(r_6) and tautologies (T_1)-(T_{12}).

If $\mathfrak{L}$ is a formalized language of the zero order, then the deductive system

$$\mathfrak{S} = \{\mathfrak{L}, C\} = \{\mathfrak{L}, C_{\mathfrak{L}}\}$$

is called a *two-valued* (or: *classical*) *propositional calculus*. Since all formulas of the form (T_1)-(T_{12}) are propositional tautologies by 6.2, we infer from 6.3 or 8.1 that

11.1. *All formulas derivable in a two-valued propositional calculus are propositional tautologies.*

The converse theorem is also true: the consequence operation $C = C_{\mathfrak{L}}$ in $\mathfrak{L}$ satisfies the condition § 10, (C′). But it will be proved later, in Chapter VII, where two-valued propositional calculi will be systematically examined.

If $\mathfrak{L}$ is a formalized language of the first order, then the deductive system

$$\mathfrak{S} = \{\mathfrak{L}, C\} = \{\mathfrak{L}, C_{\mathfrak{L}}\}$$

is called a *two-valued* (or: *classical*) *predicate calculus*. Since all formulas of the form (T_1)-(T_{12}) are predicate tautologies by 7.1, we infer from 8.1 that

11.2. *All formulas derivable in a two-valued predicate calculus are predicate tautologies.*

The converse theorem is also true but it will be proved later, in Chapter VIII, where two-valued predicate calculi will be systematically examined. More generally, we shall then prove that the operation $C = C_{\mathfrak{L}}$ in $\mathfrak{L}$ satisfies the condition (C) from § 10.

By a *formalized theory based on the two-valued* (or: *classical*) *logic* we shall mean any triple

$$\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\} = \{\mathfrak{L}, C_{\mathfrak{L}}, \mathcal{A}\}$$

where $\mathfrak{C}$ is the two-valued logic, $\mathfrak{L}$ is a formalized language (of the zero or first order) and $\mathcal{A}$ is a set of formulas in $\mathfrak{L}$.

The adjective "two-valued" is added in all the above definitions to emphasize the fact that the logical means employed are based on the hypothesis that sentences can have only two logical values: truth or falsity (see e.g. considerations in § 5). This is the classical point of view on logical problems, therefore the adjective "classical" is sometimes used instead of "two-valued". Both adjectives are added to distinguish the deductive systems and theories under consideration from certain „non-classical" ones which will be examined in Chapters IX, X, XI and which are not based on the hypothesis that sentences assume only two logical values.

§ 12. Axioms for equality. Usually, formalized theories of the first order contain in their languages a binary predicate which corresponds to the relation of equality and is called the *sign of equality*. This predicate will be denoted by $\mathfrak{e}$ or by $=$. The first notation will be used by us in the examination of general formalized theories of the first order. Then the primitive formula stating that certain elements x_1, x_2 (individual variables) are equal will be written in the form

$$\mathfrak{e}(x_1 x_2) \tag{1}$$

according to the general convention assumed in § 3, p. 155. The second notation will be used in the examination of some concrete examples of formalized theories. Then the primitive formula stating that elements x_1, x_2 are equal will be written in the form

$$(x_1 = x_2) \tag{1'}$$

according to the additional convention assumed in § 3, p. 155. Notation (1') is more suggestive since it coincides with the usual notation adopted in mathematics. However, in some theoretical investigations of general formalized theories, notation (1) is sometimes more convenient.

Let $\mathfrak{L}$ be a formalized language (of the first order) containing the sign of equality.

Suppose notation (1) is assumed. Then the letter $\mathfrak{E}$ will denote the set consisting of the following formulas $(\mathfrak{e}_1)$, $(\mathfrak{e}_2)$, $(\mathfrak{e}_3)$, $(\mathfrak{e}_4)$, $(\mathfrak{e}_5)$:

$$\mathfrak{e}(x x) \tag{$\mathfrak{e}_1$}$$

where x is a fixed free individual variable;

$$(\mathfrak{e}(x y) \Rightarrow \mathfrak{e}(y x)) \tag{$\mathfrak{e}_2$}$$

where x, y are fixed distinct free individual variables;

$$(\mathfrak{e}(x y) \Rightarrow (\mathfrak{e}(y z) \Rightarrow \mathfrak{e}(x z))) \tag{$\mathfrak{e}_3$}$$

where x, y, z are fixed distinct free individual variables;

(e_4) $(\mathfrak{e}(x_1 y_1) \Rightarrow (\mathfrak{e}(x_2 y_2) \Rightarrow (\ldots(\mathfrak{e}(x_m y_m) \Rightarrow \mathfrak{e}(\varphi(x_1 \ldots x_m)\varphi(y_1 \ldots y_m)))\ldots)))$

where $x_1, \ldots, x_m, y_1, \ldots, y_m$ are fixed distinct free individual variables, φ ranges over all m-argument functors, $m = 1, 2, \ldots$;

(e_5) $(\mathfrak{e}(x_1 y_1) \Rightarrow (\mathfrak{e}(x_2 y_2) \Rightarrow (\ldots(\mathfrak{e}(x_m y_m) \Rightarrow (\varrho(x_1 \ldots x_m) \Rightarrow \varrho(y_1 \ldots y_m)))\ldots)))$

where $x_1, \ldots, x_m, y_1, \ldots, y_m$ are fixed distinct free individual variables and ϱ ranges over all m-argument predicates, $m = 1, 2, \ldots$

If (1′) is adopted, then $\mathfrak{E}$ is the set formed of the following formulas:

(e_1') $$(x = x),$$

(e_2') $$((x = y) \Rightarrow (y = x)),$$

(e_3') $$((x = y) \Rightarrow ((y = z) \Rightarrow (x = z))),$$

(e_4') $$((x_1 = y_1) \Rightarrow ((x_2 = y_2) \Rightarrow (\ldots(x_m = y_m) \Rightarrow (\varphi(x_1 \ldots x_m) = \varphi(y_1 \ldots y_m)))\ldots)),$$

(e_5') $$((x_1 = y_1) \Rightarrow ((x_2 = y_2) \Rightarrow (\ldots(x_m = y_m) \Rightarrow (\varrho(x_1 \ldots x_m) \Rightarrow \varrho(y_1 \ldots y_m)))\ldots))$$

under the same hypotheses concerning $x, y, z, x_1, \ldots, x_m, y_1, \ldots, y_m, \varphi, \varrho$.

Obviously, if for a functor φ or for a predicate ϱ a different way of forming primitive formulas is assumed (see additional conventions in § 3, p. 155), then the formulation of the corresponding formulas (e_4), (e_5) (or (e_4'), (e_5')) evidently needs a modification.

It should be emphasized that every functor φ determines one formula (e_4) (or (e_4')) and every predicate ϱ determines one formula (e_5) (or (e_5')). Consequently, if the formalized language under consideration contains a finite number of functors and predicates, then the set $\mathfrak{E}$ is finite.

The set $\mathfrak{E}$ is called the *set of axioms for equality* for the language $\mathfrak{L}$.

The mathematical content of the axioms for equality is very simple. Axiom (e_1) (or (e_1')) expresses the reflexivity of the relation of equality. Axioms (e_2) (or (e_2')) and (e_3) (or (e_3')) express respectively the symmetry and the transitivity of the relation of equality. Axiom (e_4) (or (e_4')) asserts that any function represented by a functor assumes equal values for equal arguments. It follows easily from (e_5) (or (e_5')) that, for any m-argument relation ϱ_3 $(m = 1, 2, \ldots)$ represented by a predicate ϱ, $\varrho_3(j_1, \ldots, j_m)$ is equivalent to $\varrho_3(j_1', \ldots, j_m')$ if j_i is equal to j_i' for $i = 1, \ldots, m$.

Usually, if the language $\mathfrak{L}$ of a formalized theory $\mathfrak{T}$ of the first order contains the sign of equality, then the set $\mathfrak{E}$ of axioms for equality for

the language $\mathfrak{L}$ is included to the set of mathematical axioms of the theory $\mathfrak{T}$. Then we say briefly that $\mathfrak{T}$ is a *theory with the sign of equality* or that $\mathfrak{T}$ *contains the sign of equality.*

§ 13. Examples of elementary formalized theories based on classical logic. All theories described in this section are based on the classical logic (cf. § 11, p. 189). According to § 3, the set of all signs of a language $\mathfrak{L}$ of every such theory always contains an infinite set V of free individual variables denoted by $x, y, z, \ldots$ (with indices if necessary), an infinite set Ξ of bound individual variables denoted by $\xi, \eta, \zeta, \ldots$ (with indices if necessary), the propositional connectives $\cup, \cap, \Rightarrow, -$, quantifiers $\bigcup, \bigcap$ and parentheses $(\ ,\ )$. Therefore, when describing formalized languages of various elementary theories, we shall restrict ourselves to mention only the *specific signs* of a theory under consideration, i.e. functors and predicates.

We recall that the set $\mathfrak{A}_1$ of all logical axioms consists of all formulas of the form (T_1)-(T_{12}) (§ 11) in $\mathfrak{L}$ and that the consequence operation in $\mathfrak{L}$ is defined by scheme (S), § 10, where the set of assumed rules of inference is composed of all the rules (r_1)-(r_6) described in § 8.

A) *Theory of ordering.* The specific signs of this theory are: the binary predicate $\leqslant$ and the binary predicate $=$. Since no functor appears among the signs of the alphabet of this theory, the set of all terms in the considered formalized language $\mathfrak{L}$ coincides with the set V of all free individual variables. The following expressions are primitive formulas in $\mathfrak{L}$:

$$(1) \qquad (x \leqslant y), \quad (x = y)$$

for any free individual variables x, y. The set F of all formulas in $\mathfrak{L}$ is the least set (of finite sequences of signs in $\mathfrak{L}$) containing all elementary formulas (1) and satisfying conditions (f_1)-(f_4), § 3, p. 156.

As mathematical axioms for this theory we adopt the axioms for equality (see § 12), i.e. (e_1'), (e_2'), (e_3') and the following one belonging to the group (e_5'):

$$(2) \qquad ((x_1 = y_1) \Rightarrow ((x_2 = y_2) \Rightarrow ((x_1 \leqslant x_2) \Rightarrow (y_1 \leqslant y_2)))),$$

and the following axioms characterizing ordering relations (see I, § 5, (o_1), (o_2), (o_3)):

$$(A_1) \qquad (x \leqslant x),$$

$$(A_2) \qquad ((x \leqslant y) \Rightarrow ((y \leqslant x) \Rightarrow (x = y))),$$

$$(A_3) \qquad ((x \leqslant y) \Rightarrow ((y \leqslant z) \Rightarrow (x \leqslant z))).$$

Clearly, this theory is a formalization of the part of mathematics in which we deal only with the examination of an ordering relation in an arbitrary non-empty fixed set (see I, § 5, (o_1), (o_2), (o_3)).

Omitting axiom (A_2), we obtain the *theory of quasi-ordering* with equality.

Adding to the axioms of the theory of ordering the axiom

$$(A_4) \qquad ((x \leqslant y) \cup (y \leqslant x))$$

we obtain the *theory of linear ordering.*

Omitting from the list of specific signs of the theory of ordering the predicate $=$, and admitting as mathematical axioms the axiom (A_3) and the axiom

$$(A_5) \qquad \bigcap_\xi \bigcap_\eta \bigcup_\zeta ((\xi \leqslant \zeta) \cap (\eta \leqslant \zeta)),$$

we obtain the *theory of directed sets.* The formalized language of this theory is, of course, poorer than the language of the theory of ordering. The theory of directed sets is an example of one of the few formalized theories without the relation of equality which are investigated in mathematical practice. These theories are clearly very primitive and fragmentary.

B) *Lattice Theory.* As specific signs of this theory we admit the binary predicate $=$, and the two two-argument functors $\vee$, $\wedge$. They represent, respectively, the join and the meet in a lattice (see I, § 6). Here we adopt other symbols ($\vee$ instead of $\cup$ and $\wedge$ instead of $\cap$), because in our formalized language $\cup$ and $\cap$ denote propositional connectives. The elementary terms in the formalized language $\mathfrak{L}$ of lattice theory are all expressions in $\mathfrak{L}$ of the form

$$(3) \qquad (x \vee y), \quad (x \wedge y)$$

for any free individual variables x, y. The set T of all terms in $\mathfrak{L}$ is the least set of expressions in $\mathfrak{L}$ containing all free individual variables, all elementary terms, and fulfilling condition (t_2), § 3. The set F of all formulas in $\mathfrak{L}$ is defined by (f_1)-(f_4), § 3, where as primitive formulas we adopt all expressions in $\mathfrak{L}$ of the form $(x = y)$, x, y denoting any free individual variables. The set of mathematical axioms for this theory consists of the axioms for equality (e_1'), (e_2'), (e_3') and the following two axioms belonging to the group (e_4') of axioms for equality

$$((x_1 = y_1) \Rightarrow ((x_2 = y_2) \Rightarrow ((x_1 \vee x_2) = (y_1 \vee y_2)))),$$

$$((x_1 = y_1) \Rightarrow ((x_2 = y_2) \Rightarrow ((x_1 \wedge x_2) = (y_1 \wedge y_2)))),$$

and, moreover, of the following axioms characterizing $\vee$ and $\wedge$ (see I, § 6, (l_1), (l_2), (l_3)):

(B_1) $\quad ((x \vee y) = (y \vee x)), \quad ((x \wedge y) = (y \wedge x))$,

(B_2) $\quad (((x \vee y) \vee z) = ((x \vee y) \vee z)), \quad (((x \wedge y) \wedge z) = ((x \wedge y) \wedge z))$

(B_3) $\quad (((x \wedge y) \vee y) = y), \quad ((x \wedge (x \vee y)) = x)$.

Adding to the mathematical axioms of lattice theory the new axiom

(B_4) $\quad ((x \wedge (y \vee z)) = ((x \wedge y) \vee (x \wedge z)))$

we get the formalized *theory of distributive lattices* (cf. I, § 9).

By adding to the specific signs of lattice theory a sign — of a one-argument functor and joining to the axioms of the theory of distributive lattices the axioms

$$((x = y) \Rightarrow (-x = -y))$$

(i.e. (e_4) for the functor —) and the following ones which characterize the functor —:

(B_4) $\quad (((x \wedge -x) \vee y) = y)$, $\quad (((x \vee -x) \wedge y) = y)$,.

we obtain the formalized *theory of Boolean algebras* (see II, § 1). The functor — corresponds to the complement in Boolean algebras. We use another symbol (— instead of —) for the complement in the formalized language of this theory, because the symbol — denotes the negation sign in the formalized language.

Notice that the formalized theories of ordering, of linear ordering, of lattices, of distributive lattices and of Boolean algebras given above contain only a fragment of the axiomatic theories discussed in the first part of this book. For example, the representation theorems for distributive lattices (I, 9.3, p. 50) and for Boolean algebras (II, 8.1, p. 83) cannot be expressed in the formalized languages of the formalized theories here considered.

C) *Elementary theory of groups.* As specific signs of this theory we shall admit the binary predicate $=$ and the two-argument functor $\cdot$ (corresponding to group multiplication). The elementary terms in the formalized language $\mathfrak{L}$ of this theory are all expressions in $\mathfrak{L}$ of the form

(4) $\quad (x \cdot y)$

where x, y are arbitrary free individual variables. The set T of all terms in $\mathfrak{L}$ is the least set of expressions in $\mathfrak{L}$ containing all free individual variables, all elementary terms, and fulfilling the condition (t_2), § 3. The primitive formulas in $\mathfrak{L}$ are all expressions of the form $(x = y)$

for any free individual variables x, y. The set F of all formulas in $\mathfrak{L}$ is defined by (f_1)-(f_4), § 3.

We assume the following mathematical axioms for the theory considered: the axioms for equality (e_1'), (e_2'), (e_3') and the following one belonging to (e_4'):

$$(5) \qquad ((x_1 = y_1) \Rightarrow ((x_2 = y_2) \Rightarrow ((x_1 \cdot x_2) = (y_1 \cdot y_2))))$$

and, moreover,

$$(C_1) \qquad ((x \cdot (y \cdot z)) = ((x \cdot y) \cdot z)),$$

$$(C_2) \qquad \bigcap_\xi \bigcap_\eta \bigcup_\zeta ((\xi \cdot \zeta) = \eta),$$

$$(C_3) \qquad \bigcap_\xi \bigcap_\eta \bigcup_\zeta ((\zeta \cdot \xi) = \eta).$$

Note that this fragment of the theory of groups could also be formalized in a different way. For example, we can add to the list of specific signs of $\mathfrak{L}$ an individual constant e (corresponding to the unit element in a group) and a one-argument functor $^{-1}$ (corresponding to the creating of the converse element) and to characterize the new specific signs by the suitable axioms. Using the usually accepted method of forming elementary terms by means of the functor $^{-1}$:

$$(x^{-1})$$

we adopt as mathematical axioms of this theory the axioms for equality: (e_1'), (e_2'), (e_3'), (5)

$$((x = y) \Rightarrow ((x^{-1}) = (y^{-1}))),$$

the axiom (C_1) and

$$(C_2') \qquad ((x \cdot e) = x),$$

$$(C_3') \qquad ((x \cdot (x^{-1})) = e).$$

The language of this theory is richer, but the theory is only apparently stronger. The above example shows that axiomatic theories can be formalized in various ways.

In a similar way it is possible to define the formalized theory of Abelian groups, of rings, of fields etc.

D) *Arithmetic.* The specific signs of this theory are: the binary predicate $=$, the individual constant 1 and two two-argument functors $+$ and $\cdot$. As mathematical axioms we assume the axioms for equality: (e_1'), (e_2'), (e_3'),

$$((x_1 = y_1) \Rightarrow ((x_2 = y_2) \Rightarrow ((x_1 + x_2) = (y_1 + y_2)))),$$

$$((x_1 = y_1) \Rightarrow ((x_2 = y_2) \Rightarrow ((x_1 \cdot x_2) = (y_1 \cdot y_2)))),$$

and the following axioms which are a modification of the Peano axioms:

$$(D_1) \qquad -(1 = (x+1)),$$

$$(D_2) \qquad (((x+1) = (y+1)) \Rightarrow (x = y)),$$

$$(D_3) \qquad ((x+(y+1)) = ((x+y)+1)),$$

$$(D_4) \qquad ((x \cdot 1) = x),$$

$$(D_5) \qquad ((x \cdot (y \cdot 1)) = ((x \cdot y) \cdot 1)),$$

with the scheme of axioms of induction in the form

$$(D_6) \qquad ((\alpha(1) \cap \bigcap_\xi (\alpha(\xi) \Rightarrow \alpha((\xi+1))) \Rightarrow \bigcap_\xi \alpha(\xi))$$

where $\alpha(x)$ is any formula with a free individual variable x, $\alpha(1)$ is the result of substitution of 1 for x in $\alpha(x)$ and $\alpha(\xi)$, $\alpha((\xi+1))$ are the results of substitution of ξ and $(\xi+1)$, respectively, for x in $\alpha(x)$.

We emphasize that the set of mathematical axioms of elementary formalized arithmetic is infinite [1].

E) *Set Theory.* There are various methods of axiomatization and formalization of set theory. Some of them deal exclusively with sets. Others also admit the existence of a set of objects which are not sets but serve to form sets of objects, sets of sets of objects, etc. There are also formalizations [2] containing a notion which is more general than the notion of set and which is a precise analogue of intuitive notions like the class of all sets, the class of all cardinals, the class of all ordinals, the class of all topological spaces etc. (these classes are not sets in formalized set theory and have only some properties of sets so that the antynomies mentioned in § 1, p. 144, 145, are excluded).

We quote here only one kind of formalization of set theory [3]. This formalization deals exclusively with sets, i.e. the individual variables should be interpreted intuitively as names of sets. The set of specific signs of set theory to be presented contains only two binary predicates: the sign of equality $=$ and the sign ε. Primitive and elementary formulas of the formalized set theory are

$$(x = y) \quad \text{and} \quad (x \varepsilon y)$$

where x, y are free individual variables. The intuitive meaning of $(x \varepsilon y)$ is:

x is an element of y.

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[1] This set of axioms cannot be replaced by a finite one. See Ryll-Nardzewski [1].

[2] A formalization of this kind is due to Bernays [1].

[3] The quoted set of axioms is a slight modification of that of Zermelo [1] and Fraenkel [1].

The set of axioms of equality is formed of the axioms § 12, (e_1'), (e_2'), (e_3') and the following axiom:

$$((x_1 = y_1) \Rightarrow ((x_2 = y_2) \Rightarrow ((x_1 \varepsilon x_2) \Rightarrow (y_1 \varepsilon y_2))))$$

(see § 12, (e_5')).

The remaining axioms of set theory are rather long and complicated. To write them in a readible form, it is necessary to use some abbreviations.

For any formulas α, β in the formalized language of set theory,

$$(\alpha \Leftrightarrow \beta)$$

will be an abbreviation for the formula

$$((\alpha \Rightarrow \beta) \cap (\beta \Rightarrow \alpha)) .$$

Using this notation we can formulate the three first axioms as follows:

(E_1) $$(\bigcap_\xi ((\xi \varepsilon x) \Leftrightarrow (\xi \varepsilon y)) \Rightarrow (x = y)) ,$$

(E_2) $$\bigcup_\xi \bigcap_\eta ((\eta \varepsilon \xi) \Leftrightarrow \bigcup_\zeta ((\eta \varepsilon \zeta) \cap (\zeta \varepsilon x))) ,$$

(E_3) $$\bigcup_\xi \bigcap_\eta ((\eta \varepsilon \xi) \Leftrightarrow \bigcap_\zeta ((\zeta \varepsilon \eta) \Rightarrow (\zeta \varepsilon x))) .$$

Axiom (E_1) is called the *axiom of equality*. It asserts that two sets are equal if and only if they have the same elements. Axiom (E_2) is called the *axiom of union*. It states that, for any set x, there exists the union of all sets which are elements of x. Axiom (E_3) is called the axiom of the *power-set*. It asserts that for every set there exists the set of all subsets of x.

To formulate the fourth axiom we need the following abbreviation. The symbol $(x \subset y)$ will denote the formula

$$\bigcap_\zeta ((\zeta \varepsilon x) \Rightarrow (\zeta \varepsilon y)) ,$$

and $\chi(x)$ will denote the formula

$$\bigcap_{\xi_1} ((\xi_1 \in x) \Rightarrow \bigcup_{\xi_2} (((\xi_2 \in x) \cap (\xi_1 \subset \xi_2)) \cap -(\xi_1 = \xi_2))) .$$

Of course, $(x \subset y)$ means that x is a subset of y. The formula $\chi(x)$ states that every set which is an element of x is contained in a greater set which is also an element of x.

The fourth axiom:

(E_4) $$\bigcup_\xi (\bigcup_\eta (\eta \varepsilon \xi) \cap \chi(\xi))$$

is called the *axiom of infinity* because the set ξ whose existence is postulated is infinite (there exists an increasing sequence of its elements!).

To formulate the fifth axiom we need the following abbreviations.

$\Theta(x)$ will denote the formula

$$\bigcap_{\xi_1}((\xi_1 \varepsilon x) \Rightarrow \bigcup_{\xi_2}(\xi_2 \varepsilon \xi_1))$$

stating that x is a set of non-empty sets.

$\iota_0(x, y)$ will denote the formula

$$\bigcup_{\zeta}((\zeta \in x) \frown (\zeta \in y))$$

stating that the intersection of sets x and y is not empty.

$\iota_1(x, y)$ will denote the formula

$$\bigcap_{\xi_1}\bigcap_{\xi_2}(((\xi_1 \varepsilon x) \frown (\xi_1 \varepsilon y)) \Rightarrow (((\xi_2 \varepsilon x) \frown (\xi_2 \varepsilon y)) \Rightarrow (\xi_1 = \xi_2)))$$

stating that the intersection of sets x and y has at most one element.

$\delta(x)$ will denote the formula

$$\bigcap_{\xi_1}\bigcap_{\xi_2}(((\xi_1 \varepsilon x) \frown (\xi_2 \varepsilon x)) \Rightarrow -\iota_0(\xi_1, \xi_2))$$

stating that all elements of x are disjoint sets.

The fifth axiom

$$(E_5) \qquad ((\Theta(x) \frown \delta(x)) \Rightarrow \bigcup_{\xi}\bigcap_{\eta}((\eta \varepsilon x) \Rightarrow (\iota_0(\xi, \eta) \frown \iota_1(\xi, \eta))))$$

is the *axiom of choice*. It states that for any set x of non-empty disjoint sets there exists a set which has exactly one common point with every set belonging to x.

For every formula $\alpha(x, y)$, let α^0 denote the formula

$$\bigcap_{\xi}\bigcap_{\eta_1}\bigcap_{\eta_2}((\alpha(\xi, \eta_1) \frown \alpha(\xi, \eta_2)) \Rightarrow (\eta_1 = \eta_2))$$

stating that for every x there exists at most one y such that $\alpha(x, y)$ holds. Thus, assigning to every x in a fixed set z this unique y (if it exists), we define a function in a subset of z. Therefore, if α^0 holds, we say that α is a *function relation*.

Every formula $\alpha(x, y)$ in the formalized language of set theory determines the axiom

$$(E_6) \qquad (\alpha^0 \Rightarrow \bigcup_{\zeta}\bigcap_{\eta}((\eta \varepsilon \zeta) \Longleftrightarrow \bigcup_{\xi}((\xi \varepsilon z) \frown \alpha(\xi, \eta)))),$$

stating that if $\alpha(x, y)$ is a function relation, then for every set z there exists the set of all values of the function determined by $\alpha(x, y)$ on the corresponding subset of z. The scheme (E_6) of axioms is called the *axiom of substitution* or *replacement* [1].

In spite of the fact that the language of the formalized set theory is so poor (only one specific sign!) it is possible to formulate all mathe-

[1] This form of the axiom is due to Skolem [2].

matical notions in this language. This is not surprising because we know that all mathematical notions can be defined by means of the relation

$$x \text{ is an element of } y$$

which is included to our formalization in the form of the primitive formula $(x \varepsilon y)$.

As an example we quote here the definition of the equal power of the sets x, y. The definition is rather long. To facilitate its formulation, we need some abbreviations.

$\pi_0(x, y, z)$ will denote the formula

$$\bigcap_{\xi_1}((\xi_1 \varepsilon z) \Leftrightarrow ((\xi_1 = x) \cup (\xi_1 = y)))$$

stating that z is the set (x, y) consisting of elements x and y only.

$\pi(x, y, z)$ will denote the formula

$$\bigcap_{\xi_2}((\xi_2 \varepsilon z) \Leftrightarrow ((\xi_2 = x) \cup \pi_0(x, y, \xi_2)))$$

stating that z is the set $(x, (x, y))$, i.e. that z is the ordered pair $\langle x, y\rangle$.

$\varrho(z)$ will denote the formula

$$\bigcap_{\zeta}((\zeta \varepsilon z) \Rightarrow \bigcup_{\xi}\bigcup_{\eta}\pi(\xi, \eta, \zeta))$$

stating that z is a set of ordered pairs, i.e. a relation.

$\Phi(z)$ will denote the formula

$$(\varrho(z) \cap \bigcap_{\xi}\bigcap_{\eta_1}\bigcap_{\eta_2}\bigcap_{\zeta}((\zeta \varepsilon z) \Rightarrow (((\pi(\xi, \eta_1, \zeta) \cap \pi(\xi, \eta_2, \zeta)) \Rightarrow (\eta_1 = \eta_2)))))$$

stating that z is a set of ordered pairs such that if $\langle x, y_1\rangle$ and $\langle x, y_2\rangle$ are in z, then $y_1 = y_2$. In other words, $\Phi(z)$ states that z is a function which, for every x in the domain of the relation z, assigns unique y such that $\langle x, y\rangle$ is in z.

$\Phi_1(x, y, z)$ will denote the formula

$$(\Phi(z) \cap \bigcap_{\xi}\bigcap_{\eta}\bigcap_{\zeta}(((\zeta \varepsilon z) \cap \pi(\xi, \eta, \zeta)) \Rightarrow ((\xi \varepsilon x) \cap (\eta \varepsilon y))))$$

stating that z is a function whose domain is contained in x and whose domain is contained in y.

$\Phi_2(x, y, z)$ will denote the formula

$$(\Phi_1(x, y, z) \cap \bigcap_{\xi}((\xi \varepsilon x) \Rightarrow \bigcup_{\eta}\bigcup_{\zeta}((\zeta \varepsilon z) \cap \pi(\xi, \eta, \zeta))))$$

stating that z is a function defined in x with values in y.

$\Phi_3(x, y, z)$ will denote the formula

$$(\Phi_2(x, y, z) \cap \bigcap_{\eta}((\eta \varepsilon y) \Rightarrow \bigcup_{\xi}\bigcup_{\zeta}((\zeta \varepsilon z) \cap \pi(\xi, \eta, \zeta))))$$

stating that z is a function whose domain is x and whose range is y.

$\Phi_4(x, y, z)$ will denote the formula

$$(\Phi_3(x, y, z) \cap \bigcap_{\xi_1} \bigcap_{\xi_2} \bigcap_{\eta} \bigcap_{\zeta} (((\zeta \varepsilon z) \cap (\pi(\xi_1, \eta, \zeta) \cap \pi(\xi_2, \eta, \zeta))) \Rightarrow (\xi_1 = \xi_2)))$$

stating that z is a one-to-one mapping from x onto y.

Consequently the formula

$$\bigcup_{\zeta_1} (\Phi_4(x, y, \zeta_1)),$$

denoted for brevity by $\gamma(x, y)$, states that the sets x, y have the same power.

We can formulate all the theorems of set theory in the formalized language. For example, the well-known Cantor-Bernstein theorem can be formulated as follows:

$$(6) \quad ((\bigcup_{\eta_3} ((\eta_3 \subset y) \cap \gamma(x, \eta_3)) \cap \bigcup_{\xi_3} ((\xi_3 \subset x) \cap \gamma(\xi_3, y))) \Rightarrow \gamma(x, y)).$$

The theorem $2^{\mathfrak{n}} \neq \mathfrak{n}$ on cardinals can be formulated as follows:

$$(7) \quad (\bigcap_{\eta} ((\eta \varepsilon y) \Leftrightarrow (\eta \subset x)) \Rightarrow -\gamma(x, y)).$$

Theorems (6), (7) and all other known theorems from intuitive set theory can be proved in our formalization, i.e. they belong to the set of consequences of the set of axioms quoted.

However, the formulation of theorems in formalized set theory is often different than in intuitive set theory. In fact, (7) expresses the same mathematical content as the inequality $2^{\mathfrak{n}} \neq \mathfrak{n}$. Since the adopted language of formalized set theory is very poor, it contains no symbols to denote cardinals. However, it is possible to formalize set theory by means of a richer language containing signs of some cardinals and ordinals, signs of some special sets and functions, etc.

As we have stated, all mathematical notions can be defined by means of the relation: x is an element of y. Consequently all mathematical notions can be formulated in the formalized set theory under consideration, but often they are introduced in a different way from that used in intuitive mathematics. For instance, we can define a sequence of formulas $\nu_0(x)$, $\nu_1(x)$, $\nu_2(x)$, ... such that $\nu_0(x)$ states that x is the empty set 0, $\nu_1(x)$ states that x is the set (0) (composed of the empty set 0 only), $\nu_2(x)$ states that x is the set $(0, (0))$, $\nu_3(x)$ states that x is the set $(0, (0), (0, (0)))$, etc. The sets

$$(8) \quad 0, \ (0), \ (0, (0)), \ (0, (0), (0, (0))), \ ..$$

are substitutes for the integers

$$0, 1, 2, 3, ...$$

respectively. More exactly, formula $\nu_n(x)$ is the substitute for the statement

$$x \text{ is the integer } n.$$

It is also possible to define a formula $\nu(x)$ stating that x is one of the sets (8). Thus the formula $\nu(x)$ is the substitute for the statement:

$$x \text{ is a non-negative integer.}$$

We can prove all known arithmetical theorems on non-negative integers. Using the well-known method of forming ordered pairs we can define a formula stating that x is an integer, or stating that x is a rational number. We can also define formulas stating that z is a set of integers, of rational numbers, etc. By means of Dedekind cuts we can introduce a formula stating that x is a real number and we can reproduce, in the formalized set theory under consideration, the whole of intuitive analysis, geometry and other branches of mathematics.

As we have mentioned, all known mathematical theorems can be derived from the set of axioms of formalized set theory. We shall not prove this in detail. This statement is contained in any text-book on set theory assuming the above type of axiomatization. Since, in formalized theories, the process of derivation of formulas from axioms consists of mechanical operations on formulas, i.e. on finite sequences of signs, the chief result of this section is rather shocking: all known mathematical theorems can be derived from the set of axioms of formalized set theory and from the set of all logical axioms described in § 11 by means of purely mechanical operations called the rules of inference, mentioned in § 8.

On the other hand, it should be emphasized that this mechanical method of deducing some mathematical theorems has no practical value because it is too complicated in practice. We have seen in this section that even formulas expressing a very simple mathematical content are very long and not readable. Moreover, formalized proofs are usually very long and complicated. It is theoretically possible to construct a machine which will systematically print theorems deduced from the axioms of set sheory. It will print true statements, but only exceptionally will these statements be interesting. The aim of a mathematician is not merely to deduce true statements but to discover interesting true statements. Therefore the theoretically possible mechanization of the process of deduction in mathematics has no practical value.

§ 14. Some fundamental metamathematical notions. We complete this introductory Chapter by giving a few definitions of some very general metamathematical notions. These definitions will be applied to classical deductive systems and the formalized theories examined in Chapters VII,

VIII as well as to some of the non-classical deductive systems and formalized theories examined in Chapters IX, X, XI. In this section we do not assume that the formalized theories under examination are based on the two-valued logic.

The main notion is the consistency of a theory $\mathcal{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ based on a formalized language $\mathfrak{L}$ (of the zero or first order). In agreement with intuition, the theory $\mathcal{T}$ is called *consistent* if, for no formula α, both formulas α and $-\alpha$ are theorems in $\mathcal{T}$.

14.1. *Suppose that the set of all theorems in* $\mathcal{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ *is closed under modus ponens and that all formulas of the form*

$$(T_9) \qquad (((\alpha \cap \beta) \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow (\beta \Rightarrow \gamma))),$$

$$(T_{10}) \qquad ((\alpha \cap -\alpha) \Rightarrow \beta)$$

are theorems in $\mathcal{T}$. *Then* $\mathcal{T}$ *is consistent if and only if there is a formula in* $\mathfrak{L}$ *which is not a theorem of* $\mathcal{T}$.

If the consequence operation C *has the properties* § 10, (c_2), (c_4), *then the theory* $\mathcal{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ *is consistent if, for every finite set* $\mathcal{A}_0 \subset \mathcal{A}$, *the theory* $\{\mathfrak{L}, C, \mathcal{A}_0\}$ *is consistent.*

To prove the first part of 14.1, let us first observe that if $\mathcal{T}$ is consistent, then not all formulas are theorems in $\mathcal{T}$. On the other hand, replacing β by $-\alpha$ in (T_9) by *modus ponens* and (T_{10}) we find that every formula of the form

$$(1) \qquad (\alpha \Rightarrow (-\alpha \Rightarrow \gamma))$$

is a theorem in $\mathcal{T}$. Thus if $\mathcal{T}$ is inconsistent, i.e. if there exists a formula α such that both α and $-\alpha$ are theorems in $\mathcal{T}$, then by twice applying *modus ponens* to (1) we infer that every formula γ is a theorem in $\mathcal{T}$.

To prove the second part of 14.1, let us suppose that $\mathcal{T}$ is inconsistent, i.e. there is a formula α such that both α and $-\alpha$ are theorems in $\mathcal{T} = \{\mathfrak{L}, C, \mathcal{A}\}$. By § 10, (c_4), there are finite subsets $\mathcal{A}_1$, $\mathcal{A}_2$ of $\mathcal{A}$ such that α is in $C(\mathcal{A}_1)$ and $-\alpha$ is in $C(\mathcal{A}_2)$. The union $\mathcal{A}_0$ of $\mathcal{A}_1$ and $\mathcal{A}_2$ is a finite subset of $\mathcal{A}$ and both α and $-\alpha$ are in $C(\mathcal{A}_0)$ by § 10, (c_2), i.e. the theory $\{\mathfrak{L}, C, \mathcal{A}_0\}$ is inconsistent.

Observe that under the single hypothesis that the consequence operation C has property § 10, (c_2), if the theory $\{\mathfrak{L}, C, \mathcal{A}\}$ is consistent, then also $\{\mathfrak{L}, C, \mathcal{A}_0\}$ is consistent for every subset $\mathcal{A}_0$ of $\mathcal{A}$.

Let $\mathfrak{L}$ and $\mathfrak{L}'$ be two formalized languages of the same order. The language $\mathfrak{L}'$ is said to be an *extension* of the language $\mathfrak{L}$ provided the set F of formulas in $\mathfrak{L}$ is a subset of the set F' of all formulas in $\mathfrak{L}'$. Thus, if $\mathfrak{L}$ and $\mathfrak{L}'$ are languages of zero order, then $\mathfrak{L}'$ is an extension of $\mathfrak{L}$ if and only if the set of all propositional variables in $\mathfrak{L}$ is a subset of the

set of all propositional variables in $\mathfrak{L}'$. If $\mathfrak{L}$ and $\mathfrak{L}'$ are of the first order, then $\mathfrak{L}'$ is an extension of $\mathfrak{L}$ if and only if the set of free individual variables in $\mathfrak{L}$, the set of all bound variables in $\mathfrak{L}$, the sets of all m-argument functors in $\mathfrak{L}$ ($m = 0, 1, 2, \ldots$), the sets of all m-argument predicates in $\mathfrak{L}$ are all subsets of the corresponding sets of signs in $\mathfrak{L}'$.

Suppose that $\mathfrak{L}'$ is an extension of $\mathfrak{L}$. A theory $\mathfrak{T}' = \{\mathfrak{L}', C', \mathcal{A}'\}$ is said to be an *extension* of a theory $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ provided that, for every formula α in $\mathfrak{T}$, if α is a theorem in $\mathfrak{T}$, then α is a theorem in $\mathfrak{T}'$; in symbols:

$$C(\mathcal{A}) \subset C'(\mathcal{A}') \,. \tag{2}$$

A theory $\mathfrak{T}' = \{\mathfrak{L}', C', \mathcal{A}'\}$ is said to be an *inessential extension* of a theory $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ provided that for every formula α in $\mathfrak{L}$, α is a theorem in $\mathfrak{T}$ if and only if α is a theorem in $\mathfrak{T}'$; in symbols:

$$C(\mathcal{A}) = C'(\mathcal{A}') \cap F \,, \tag{3}$$

where F is the set of all formulas in $\mathfrak{T}$. An extension $\mathfrak{T}'$ of $\mathfrak{T}$ is said to be *essential* if it is not an inessential extension.

14.2. *Every inessential extension of a consistent theory is consistent.*

For if there is a formula α in $\mathfrak{T}$ which is not a theorem in $\mathfrak{T}$, then α is not a theorem in any inessential extension $\mathfrak{T}'$ of $\mathfrak{T}$. Thus $\mathfrak{T}'$ is consistent.

An extension $\mathfrak{T}' = \{\mathfrak{L}', C', \mathcal{A}'\}$ of $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ is said to be *linguistically invariant* if $\mathfrak{L}'$ coincides with $\mathfrak{L}$. In other words, $\mathfrak{T}'$ is a linguistically invariant extension of $\mathfrak{T}$ provided

$$C(\mathcal{A}) \subset C'(\mathcal{A}') \quad \text{and} \quad \mathfrak{L} = \mathfrak{L}'. \tag{4}$$

Suppose $\mathfrak{C}$ is a consequence operation (logic) defined by means of the scheme (L) in § 11. If $\mathfrak{L}'$ is an extension of a language $\mathfrak{L}$, then the theory $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A}\}$ is an extension of $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ (this follows easily from the definition (L) in § 11). We shall then say that $\mathfrak{T}'$ is a *linguistic extension* of $\mathfrak{T}$. It will be proved in VII-X that, in the logics examined in this books, linguistic extensions are always inessential.

According to § 10, p. 186 two theories $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ and $\mathfrak{T}' = \{\mathfrak{L}, C', \mathcal{A}'\}$ based on the same formalized language $\mathfrak{L}$ are said to be *equivalent* provided they have the same set of theorems:

$$C(\mathcal{A}) = C'(\mathcal{A}') \,. \tag{5}$$

In other words, $\mathfrak{T}$ and $\mathfrak{T}'$ are equivalent if $\mathfrak{T}'$ is an extension of $\mathfrak{T}$ and $\mathfrak{T}$ is an extension of $\mathfrak{T}'$.

A theory $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ is said to be *maximal* [1] provided it is consistent and every consistent linguistically invariant extension $\mathfrak{T}' = \{\mathfrak{L}, C, \mathcal{A}'\}$ of $\mathfrak{T}$ is equivalent to $\mathfrak{T}$ (i.e., if $\mathfrak{T}$ has no essential linguistically invariant consistent extension). The use of the adjective "maximal" can be justified as follows. Consider the set of all theories based on the same language $\mathfrak{L}$. The relation

(6) $\qquad \mathfrak{T}'$ is an extension of $\mathfrak{T}$

is a quasi-ordering in this set. Identify two theories if and only if they are equivalent. By I, § 5, after this identification the relation (6) becomes an ordering. A theory $\mathfrak{T}$ is maximal if it is a maximal element in the ordered set of all consistent theories based on $\mathfrak{L}$.

In general, if a disjunction $(\alpha \cup \beta)$ is a theorem in a theory $\mathfrak{T}$, then neither α nor β need be a theorem in $\mathfrak{T}$. For instance, in any theory $\mathfrak{T}$ based on the two-valued logic, every disjunction

(T_{12}) $\qquad (\alpha \cup -\alpha)$

is a theorem, but in general it is not true that either α or $-\alpha$ is a theorem.

A consistent theory $\mathfrak{T}$ of the zero order is said to be *prime* provided it has the following property:

(P) if $(\alpha \cup \beta)$ is a theorem in $\mathfrak{T}$, then either α is a theorem in $\mathfrak{T}$, or β is a theorem in $\mathfrak{T}$.

A similar terminology is assumed for consistent theories of the first order: a theory $\mathfrak{T}$ is said to be *prime* if it has property (P) for all closed formulas α, β, and it is said to be *strongly prime* if it has property (P) for all formulas α, β.

The adjective "prime" is here justified by analogy to the notion of prime filter (see VIII, § 12, IX, § 12 and X, § 12).

We now formulate some definitions concerning formulas.

A formula α is said to be *refutable* in a theory $\mathfrak{T}$ provided $-\alpha$ is a theorem in $\mathfrak{T}$. Otherwise α is said to be *irrefutable* in $\mathfrak{T}$.

For any finite sequence of formulas

(7) $\qquad \alpha_1, \ldots, \alpha_m \qquad (m \geqslant 1)$

in a formalized language $\mathfrak{L}$, we define formulas $\bigcup_{i=1}^{m} \alpha_i$ and $\bigcap_{i=1}^{m} \alpha_i$ by induction as follows.

For $m = 1$, $\bigcup_{i=1}^{1} \alpha_i$ is the formula α_1. For $m > 1$, $\bigcup_{i=1}^{m} \alpha_i$ is the formula $(\bigcup_{i=1}^{m-1} \alpha_i \cup \alpha_m)$.

For $m = 1$, $\bigcap_{i=1}^{1} \alpha_i$ is the formula α_1. For $m > 1$, $\bigcap_{i=1}^{m} \alpha_i$ is the formula $(\bigcap_{i=1}^{m-1} \alpha_i \cap \alpha_m)$.

[1] Maximal theories are called *complete* by other authors.

$\bigcup_{i=1}^{m} a_i$ is called the *disjunction* of formulas (7), and $\bigcap_{i=1}^{m} a_i$ is called the *conjunction* of formulas (7). If $m = 2$, then

$$\bigcup_{i=1}^{2} a_i = (a_1 \cup a_2), \quad \bigcap_{i=1}^{2} a_i = (a_1 \cap a_2).$$

Let $\mathfrak{L}$ be a language of the zero order. A formula β is said to be a *direct subformula* of a formula α provided α is of one of the forms:

$$(\beta \cup \gamma), \quad (\gamma \cup \beta), \quad (\beta \cap \gamma), \quad (\gamma \cap \beta), \quad (\beta \Rightarrow \gamma), \quad (\gamma \Rightarrow \beta), \quad -\beta, \tag{8}$$

where γ is a formula. A formula β is said to be a *subformula* of α provided there exists a sequence $\beta_1, \ldots, \beta_m$ $(m \geqslant 1)$ of formulas such that β_1 is β, β_m is α and, for $i = 2, \ldots, m$, β_{i-1} is a direct subformula of β_i. It can be proved that β is a subformula of α if and only if α is of the form $\omega_1 \beta \omega_2$ where ω_1, ω_2 are sequences of signs, empty sequences being admitted.

In a similar way we define the notion of subformula when $\mathfrak{L}$ is a language of the first order. The only difference is that we add to sequence (8) the forms $\bigcup_\xi \beta(x/\xi)$ and $\bigcap_\xi \beta(x/\xi)$.

We recall that formula in a formalized language $\mathfrak{L}$ of the first order is said to be *closed* if it does not contain any free individual variable. If α is a formula in $\mathfrak{L}$, and $x_1, \ldots, x_n$ are all free individual variables appearing in α, then every formula of the form

$$\bigcap_{\xi_1} \ldots \bigcap_{\xi_n} \alpha(x_1/\xi_1, \ldots, x_n/\xi_n)$$

is said to be a *closure* of α. Usually we shall denote by $\bar{\alpha}$ a closure of α.

A formula in a formalized language $\mathfrak{L} = \{A, T, F\}$ of the first order is said to be *open* if it does not contain any quantifier, i.e. if it does not contain any bound individual variable. The set F^0 of all open formulas in $\mathfrak{L}$ is the smallest set such that

(f_1) every elementary formula belongs to F^0;

(f_2) if α is in F^0 and o is a unary connective, then $o\alpha$ is in F^0;

(f_2) if α and β are in F^0 and o is a binary connective, then $(\alpha o \beta)$ is in F^0.

The triple $\mathfrak{L}^0 = \{A^0, T, F^0\}$, where A^0 is the alphabet obtained from A by omitting quantifiers and bound individual variables, will be called the *language of open formulas in* $\mathfrak{L}$. $\mathfrak{L}^0$ is also a kind of formalized language but of a form different from those discussed in § 3 and § 6. The language $\mathfrak{L}^0$ has an auxilliary meaning in the examination of $\mathfrak{L}$.

A theory $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ of the first order is said to be *open* if all axioms in $\mathcal{A}$ are open formulas.

A formula α is said to be *existential* if it is of the form $\bigcup_\xi \beta$, i.e. if the existential quantifier is the first sign in α.

§ 15. Definitions in formalized theories. The formation of new mathematical notions by assuming some definitions is very useful in intuitive mathematics. Its purpose very often is to point out the importance or essentiality of certain mathematical objects or their properties. The introduction of definitions also has great practical meaning. By a suitable definition we can replace some long statements defining notions by very short phrases usually composed of only one or two words. Consequently the colloquial language of intuitive mathematics becomes much more clear and easy to understand.

On the other hand, in a sense definitions are unnecessary in mathematics since we can everywhere replace the definiendum (i.e. the notion defined) by the definiens (i.e. the defining statements). Thus, without introducing new definitions, we can express the same mathematical content but in a longer and usually much more complicated way. We see that the introduction of a definition has a great practical meaning and also a psychological meaning but does not introduce any new mathematical contents to the theory.

The introduction of definitions is also possible in the case of a formalized theory but, in general, it creates the necessity of augmenting the formalized language of the theory, and also the necessity of passing to an extension of the theory. We shall quote here the three typical cases of definitions applicable to any theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ of the first order.

1) Suppose that

$$\alpha(x_1, \ldots, x_m) \qquad (m > 0) \tag{1}$$

is a formula in $\mathfrak{L}$, the sequence $x_1, \ldots, x_m$ containing all free individual variables appearing in this formula. Let $\mathfrak{L}'$ be an extension of the language $\mathfrak{L}$, such that $\mathfrak{L}'$ contains an m-argument predicate ϱ_α which appears neither in (1) nor in any formula in $\mathfrak{A}$. Let $\mathfrak{A}'$ be the set formed of all formulas in $\mathfrak{A}$ and of the formulas

$$(\varrho_\alpha(x_1, \ldots, x_m) \Rightarrow \alpha(x_1, \ldots, x_m)), \qquad (\alpha(x_1, \ldots, x_m) \Rightarrow \varrho_\alpha(x_1, \ldots, x_m)). \tag{2}$$

The theory $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}, \mathfrak{A}'\}$ is said to be obtained from $\mathfrak{T}$ by assuming the *definition* (1) *of the predicate* ϱ_α.

If both formulas $(\beta \Rightarrow \gamma)$ and $(\gamma \Rightarrow \beta)$ are acknowledged as true, then, from the intuitive point of view, β is equivalent to γ, i.e. both formulas express the same content. The act of including some formulas in the axioms of a theory signifies that they are accepted as true from the point of view of the theory. Thus by assuming definition (2) we assert that $\varrho_\alpha(x_1, \ldots, x_m)$ and $\alpha(x_1, \ldots, x_m)$ have exactly the same meaning; they are two different formalized analogues of the same statement in

any intuitive interpretation of the theory. But, of course, $\varrho_\alpha(x_1, \ldots, x_m)$ is a shorter, and therefore more convenient formula than $\alpha(x_1, \ldots, x_m)$.

2) Suppose that the formula

$$\bigcap_{\xi_1} \ldots \bigcap_{\xi_m} \bigcup_{\eta} \beta(\xi_1, \ldots, \xi_m, \eta) \quad (m \geqslant 0) \tag{3}$$

is closed and is a theorem in $\mathfrak{T}$. Let $\mathfrak{L}'$ be an extension of the language $\mathfrak{L}$ such that $\mathfrak{L}'$ contains an m-argument functor φ_β which appears neither in (3) nor in any formula in $\mathcal{A}$. Let $\mathcal{A}'$ be the set consisting of all formulas in $\mathcal{A}$ and of the formula

$$\bigcap_{\xi_1} \ldots \bigcap_{\xi_m} \beta\left(\xi_1, \ldots, \xi_m, \varphi_\beta(\xi_1 \ldots \xi_m)\right). \tag{4}$$

The theory $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A}'\}$ is said to be obtained from $\mathfrak{T}$ by assuming the *implicit definition* (4) *of the functor* φ_β.

The intuitive meaning of assuming definition (4) is that φ_β is acknowledged as the symbol of an m-argument function satisfying (4).

In practice definition (4) is applied when theory $\mathfrak{T}$ contains the equality sign $\mathfrak{e}$ and, besides (3), the following formula is also a theorem in $\mathfrak{T}$:

$$\bigcap_{\xi_1} \ldots \bigcap_{\xi_m} \bigcap_{\eta_1} \bigcap_{\eta_2} ((\beta(\xi_1, \ldots, \xi_m, \eta_1) \frown \beta(\xi_1, \ldots, \xi_m, \eta_2)) \Rightarrow \mathfrak{e}(\eta_1 \eta_2)). \tag{5}$$

Then, from the intuitive point of view, φ_β is the symbol for the unique function satisfying (4).

3) Suppose that $\mathfrak{T}$ contains the sign of equality $\mathfrak{e}$ and that

$$\tau(x_1, \ldots, x_m) \quad (m \geqslant 0) \tag{6}$$

is a term in $\mathfrak{T}$, the sequence $x_1, \ldots, x_m$ containing all free individual variables appearing in that term. Let $\mathfrak{L}'$ be an extension of the language $\mathfrak{L}$ such that $\mathfrak{L}'$ contains an m-argument functor ψ_τ which appears neither in (6) nor in any formula in $\mathcal{A}$. Let $\mathcal{A}'$ be the set composed of all the formulas in $\mathcal{A}$ and of the formula

$$\mathfrak{e}(\psi_\tau(x_1 \ldots x_m)\tau(x_1, \ldots, x_m)). \tag{7}$$

The theory $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A}'\}$ is said to be obtained from the theory $\mathfrak{T}$ by assuming the *explicit definition* (7) *of the functor* ψ_τ.

From the intuitive point of view, every functor is a symbol of a function, and the terms are superpositions of functions represented by functors. Thus the intuitive meaning of (7) is that ψ_τ is accepted as a shorter notation for the function represented by the term (6).

Clearly we can introduce simultaneously an arbitrary number of definitions of the type 1), 2) and 3). For this purpose let us suppose that we have a set S_1 of formulas (1), a set S_2 of formulas (3) and a set S_3 of terms, satisfying the hypotheses mentioned in 1), 2), 3), respectively.

Some of the sets S_1, S_2, S_3 may be empty. If $\mathfrak{T}$ does not contain the equality sign, the set S_3 is supposed to be empty.

Let $\mathfrak{L}'$ be any extension of the language of $\mathfrak{L}$ such that $\mathfrak{L}'$ contains a set of predicates ϱ_α and terms φ_β, ψ_τ corresponding to formulas (1), (3) and terms (6) respectively, and satisfying the hypotheses mentioned in 1), 2), 3) respectively. We suppose additionally that all predicates ϱ_α differ from one another, all the functors φ_β, ψ_τ differ from one another and none of the signs ϱ_α, φ_β, ψ_τ appears in any formula in $\mathcal{A}$ or in any of the formulas (1), (3) or in any of the terms (6). Let $\mathcal{A}'$ be the set composed of all the formulas in $\mathcal{A}$ and of all the formulas (2), (4), (7). The theory $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A}'\}$ is said to be obtained from $\mathfrak{T}$ by *simultaneously assuming all the definitions* (2), (4), (7).

Note that sometimes we can assume $\mathfrak{L}' = \mathfrak{L}$. This is possible in the case in which $\mathfrak{L}$ has a sufficient number of signs which can be used as predicates ϱ_α and functors φ_β and ψ_τ. In general, in order to assume the definition just defined we need to extend the language $\mathfrak{L}$ by adding some new predicate and functors.

Of course, the act of introducing definitions can be repeated several times, step by step.

CHAPTER VI

THE ALGEBRA OF FORMALIZED LANGUAGES

§ 1. The algebra of formulas. In this Chapter formalized languages will be examined from the point of view of the theory of abstract algebras. The following remark is the basis for this investigation.

1.1. *The set F of all formulas of a formalized language $\mathfrak{L}$ of the zero or first order is an abstract algebra*

$$\{F, \cup, \cap, \Rightarrow, -\}$$

with three binary operations $\cup$, $\cap$, $\Rightarrow$ and one unary operation $-$ defined as follows: the formulas

$$(\alpha \cup \beta), \quad (\alpha \cap \beta), \quad (\alpha \Rightarrow \beta), \quad -\alpha$$

are the results of the operations $\cup$, $\cap$, $\Rightarrow$, $-$ performed on formulas α, β, respectively.

This follows immediately from the definition of sets of formulas in formalized languages (see p. 167 and p. 156).

The algebra $\{F, \cup, \cap, \Rightarrow, -\}$ defined in 1.1 is called the *algebra of formulas of the formalized language* $\mathfrak{L}$ [1].

Despite the fact that some of the operations in F are denoted by the symbols $\cup$, $\cap$ used, in general, for lattice operations, the algebra F is not a lattice with respect to $\cup$ and $\cap$. For instant, if α and β are two distinct formulas, then $(\alpha \cup \beta)$ and $(\beta \cup \alpha)$ are two distinct formulas in F. This implies that the operation $\cup$ does not satisfy the commutative law (see I, § 6, (l_1)). Similarly, the operation $\cap$ does not satisfy the commutative law in F. Also the associative laws (see I, § 6, (l_2)) do not hold in F.

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[1] The idea of treating the set of all formulas of a formalized language as an abstract algebra with operations corresponding to logical connectives was first used by A. Lindenbaum and A. Tarski. In recent years it was applied frequently by many authors (see footnote on p. 245).

The exposition in §§ 1-7 is a slight modification of that in the expository paper of Sikorski [11].

Let $\mathfrak{L}$ and $\mathfrak{L}'$ be formalized languages of the same order. The following remark is obvious:

1.2. *If $\mathfrak{L}'$ is an extension of $\mathfrak{L}$, then F is a subalgebra of F'.*

§ 2. The algebra of formulas of a formalized language of zero order. The interpretation of formulas as mappings. Let us consider first the case of a formalized language $\mathfrak{L}_0 = \{A_0, F_0\}$ of zero order, described in V, § 6.

2.1. *The algebra $\{F_0, \cup, \cap, \Rightarrow, -\}$ of formulas of a language $\mathfrak{L}_0$ of zero order is free in the class $\mathfrak{K}$ of all abstract algebras $\{A, o_1, o_2, o_3, o_4\}$ with three binary operations o_1, o_2, o_3 and one unary operation o_4, the set V_0 of all propositional variables in $\mathfrak{L}_0$ being the set of free generators in F_0.*

Any mapping $f\colon V_0 \to A$, where A is an algebra in $\mathfrak{K}$, can be extended to a mapping $h\colon F_0 \to A$ defined by induction on the length of formulas as follows, the value of h at $a \in F_0$ being denoted by ha:

1) For any propositional variable a, let $ha = f(a)$.

2) Suppose that, for some formulas α, β in F_0, the elements $h\alpha$ and $h\beta$ are defined. Then we define $h(\alpha \cup \beta)$, $h(\alpha \cap \beta)$, $h(\alpha \Rightarrow \beta)$, $h-\alpha$ by the equations:

$$\begin{aligned} h(\alpha \cup \beta) &= h\alpha o_1 h\beta\,, \qquad h(\alpha \cap \beta) = h\alpha o_2 h\beta\,, \\ h(\alpha \Rightarrow \beta) &= h\alpha o_3 h\beta\,, \qquad h-\alpha = o_4 h\alpha\,. \end{aligned} \tag{1}$$

The uniqueness of h follows from the basic property of formulas that every formula in F_0 is of exactly one of the following forms: it is a propositional variable; it is the disjunction of two formulas; it is the conjunction of two formulas; it is the negation of a formula. Moreover, this representation is unique.

It follows from 1) that h is an extension of f.

It follows from (1) that h is a homomorphism. This completes the proof of 2.1.

Now, let $\mathfrak{L}$ be another formalized language of the zero or first order, and let F be the set of all formulas in $\mathfrak{L}$. By a *substitution* from $\mathfrak{L}_0$ into $\mathfrak{L}$ we understand any mapping

$$s\colon V_0 \to F\,, \tag{2}$$

where V_0 denotes, as previously, the set of all propositional variables of the language $\mathfrak{L}_0$. If α is any formula in $\mathfrak{L}_0$ and $a_1, \ldots, a_n$ are all propositional variables in α, let $s\alpha$ be the formula

$$\alpha(a_1/sa_1, \ldots, a_n/sa_n)\,,$$

where sa denotes the value of the mapping s at $a \in V_0$. Denoting α by

$\alpha(a_1, \ldots, a_n)$, more suggestively, we can write the definition of $s\alpha$ in the form of the identity:

$$s\alpha(a_1, \ldots, a_n) = \alpha(sa_1, \ldots, sa_n)\,. \tag{3}$$

For every formula α in $\mathfrak{L}_0$, $s\alpha$ is a formula in $\mathfrak{L}$ (see V, 6.1 and 6.4). Thus (3) defines a mapping

$$s\colon F_0 \to F \tag{4}$$

which is a natural extension of the mapping (2) and, therefore, is denoted by the same letter s.

2.2. *The mapping* (4) *is a homomorphism, i.e.*

$$\begin{aligned} s(\alpha \cup \beta) &= (s\alpha \cup s\beta), \quad s(\alpha \cap \beta) = (s\alpha \cap s\beta)\,, \\ s(\alpha \Rightarrow \beta) &= (s\alpha \Rightarrow s\beta), \quad\quad s-\alpha = -s\alpha\,, \end{aligned} \tag{5}$$

for all formulas α, β *in* $\mathfrak{L}_0$. *The mapping* (4) *is the only homomorphic extension of the mapping* (2).

The proof of (5) is by induction on the length of formulas in $\mathfrak{L}_0$. The uniqueness of the homomorphic extension (4) of (2) follows from the fact that the set V_0 generates the algebra F_0 (see I, 4.4).

Observe that the existence of a homomorphic extension of (2) over the whole algebra F_0 follows immediately from 2.1.

Theorem 2.2 explains that, from the algebraic point of view, the operation of substitution in formulas coincides with a homomorphism.

Let $\{A, \cup, \cap, \Rightarrow, -\}$ be a fixed abstract algebra with three unary operations $\cup, \cap, \Rightarrow$ and one unary operation $-$.

By a *valuation* of $\mathfrak{L}_0$ in A we shall understand any point $v = \{v_a\}_{a \in V_0}$ of the Cartesian product A^{V_0} where, as previously, V_0 is the set of all propositional variables in the language $\mathfrak{L}_0$ of zero order. In other words, a valuation in A is a mapping

$$v\colon V_0 \to A\,. \tag{6}$$

Every formula α in $\mathfrak{L}_0$ uniquely determines an operation (or: can be interpreted as an operation) α_A in A, viz. a mapping (algebraic polynomial) [1]

$$\alpha_A\colon A^{V_\alpha} \to A \tag{7}$$

where V_α is the set of all propositional variables appearing in α. To obtain α_A it suffices to interpret

[1] Mappings α_A are an algebraic form of the truth-table method applied by many authors for examination of the classical and non-classical propositional calculi. This mappings were first defined by Łukasiewicz and Tarski [1] who use the word "matrice" instead of "abstract algebra".

a) the signs $\cup, \cap, \Rightarrow, -$ in α as signs of the corresponding operations in $\boldsymbol{A}$;

b) the propositional variables $a_1, \ldots, a_m$ in α as variables ranging over the algebra $\boldsymbol{A}$.

The signs $($, $)$ should be interpreted, of course, as parentheses.

For instance, if α is the formula

$$((a \Rightarrow b) \cup -(a \cap c)), \tag{8}$$

where a, b, c are propositional variables, then $\alpha_{\boldsymbol{A}}$ is the mapping which, to every triple $(a, b, c) \in \boldsymbol{A}^3$, assigns the element $\alpha_{\boldsymbol{A}}(a, b, c) = (a \Rightarrow b) \cup -(a \cap c)$.

Observe that when $\{\boldsymbol{A}, \cup, \cap, \Rightarrow, -\}$ is the two-element Boolean algebra, the mapping $\alpha_{\boldsymbol{A}}$ just defined coincides with the mapping defined in V, § 6, p. 168.

Every mapping $f\colon \boldsymbol{A}^{V'} \to \boldsymbol{A}$, where $V' = (a_1, \ldots, a_n)$ is a finite subset of V_0, can be interpreted as a mapping $f\colon \boldsymbol{A}^{V_0} \to \boldsymbol{A}$ if we assume

$$f(v) = f(v_{a_1}, \ldots, v_{a_n})$$

for every valuation $v = \{v_a\}_{a \in V_0} \in \boldsymbol{A}^{V_0}$. The element $f(v)$ depends, of course, only on the coordinates $v_{a_1}, \ldots, v_{a_n}$ of the point v.

In particular, for every formula α in $\mathfrak{L}_0$, the mapping $\alpha_{\boldsymbol{A}}$ can be interpreted as a mapping

$$\alpha_{\boldsymbol{A}}\colon \boldsymbol{A}^{V_0} \to \boldsymbol{A}\,. \tag{9}$$

This interpretation of $\alpha_{\boldsymbol{A}}$ is, for purely technical reasons, more convenient than (7) because then all the mappings $\alpha_{\boldsymbol{A}}$ (α in F_0) are defined on the same set $\boldsymbol{A}^{V_0}$ and some identities can be written in a simpler way. Therefore, from now on, we shall always conceive of $\alpha_{\boldsymbol{A}}$ as the mapping (9).

For instance, if α is formula (8), then for every valuation $v \in \boldsymbol{A}^{V_0}$

$$\alpha_{\boldsymbol{A}}(v) = \big((v_a \Rightarrow v_b) \cup -(v_a \cap v_c)\big)\,,$$

where v_a, v_b, v_c are respectively the a-th, b-th and c-th coordinates of v.

A little more precise definition of the mapping (9) is by induction on the length of α:

$$\begin{aligned} \alpha_{\boldsymbol{A}}(v) &= v_a\,, \\ (\beta \cup \gamma)_{\boldsymbol{A}}(v) &= \beta_{\boldsymbol{A}}(v) \cup \gamma_{\boldsymbol{A}}(v)\,, \\ (\beta \cap \gamma)_{\boldsymbol{A}}(v) &= \beta_{\boldsymbol{A}}(v) \cap \gamma_{\boldsymbol{A}}(v)\,, \\ (\beta \Rightarrow \gamma)_{\boldsymbol{A}}(v) &= \beta_{\boldsymbol{A}}(v) \Rightarrow \gamma_{\boldsymbol{A}}(v)\,, \\ (-\beta)_{\boldsymbol{A}}(v) &= -\big(\beta_{\boldsymbol{A}}(v)\big) \end{aligned} \tag{10}$$

for every propositional variable a, formulas β, γ in $\mathfrak{L}_0$ and for every valuation $v = \{v_a\}_{a \in V_0} \in A^{V_0}$.

Another precise definition of a_A can be given as follows, by means of theorem 2.1. Every valuation

$$v\colon V_0 \to A$$

can be extended, by 2.1, to a homomorphism from F_0 into A. Denote this homomorphism by v_A. Thus, for every a in F_0, $v_A(a)$ is a well defined element in A. If a is fixed and v is variable, then the element $v_A(a)$ is a function of the valuation v. Denote this function by a_A. By definition,

$$a_A(v) = v_A(a)\,. \tag{11}$$

The mapping $a_A\colon A^{V_0} \to A$ just defined satisfies identities (10). Thus it coincides with the mapping (9) defined on p. 212.

We note the main results of the above consideration in the form of the following theorem:

2.3. *For every fixed valuation v, the expression $a_A(v)$ considered as a function of a is a homomorphism from the algebra F of formulas into A, such that $a_A(v) = v_a$ for every propositional variable a and for every valuation $v = \{v_a\}_{a \in V_0}$.*

The value of $a_A(v)$ depends only on $v_{a_1}, \ldots, v_{a_n}$ where $a_1, \ldots, a_n$ are all propositional variables in a.

The first part of theorem 2.3 completely characterizes $a_A(v)$ and can be assumed to be another equivalent definition of $a_A(v)$.

Let $\{B, \cup, \cap, \Rightarrow, -\}$ be an algebra similar to $\{A, \cup, \cap, \Rightarrow, -\}$ and let $h\colon A \to B$ be a homomorphism. For every valuation $v = \{v_a\}_{a \in V_0}$ in A let hv be the valuation $\{h(v_a)\}_{a \in V_0}$ in B. Thus hv is the superposition of the mappings $v\colon V_0 \to A$ and $h\colon A \to B$.

2.4. *For every homomorphism $h\colon A \to B$, for every valuation v in A and for every formula a in $\mathfrak{L}_0$,*

$$a_B(hv) = h a_A(v)\,. \tag{12}$$

For every fixed valuation v, both sides of (12), considered as functions of a, are homomorphisms from F into B because of 2.3. They coincide on the set of all propositional variables, i.e. on a set of generators of F. Thus they are equal for every a in F (see I, 4.4).

Theorem 2.4 can also be proved directly, by induction with respect to the length of a, without reference to theorem 2.3. Indeed, both proofs are the same. The above proof of 2.4 is shorter (the direct inductive proof requires a verification of several cases corresponding to all possible forms of a). A similar replacement of inductive proofs of the type just

mentioned by some algebraic arguments will be applied several times in this chapter.

Let now s be a substitution from $\mathfrak{L}_0$ into $\mathfrak{L}_0$. The substitution s induces a mapping

$$s_A\colon A^{V_0} \to A^{V_0}$$

defined as follows: for every valuation $v = \{v_a\}_{a \in V_0}$ in A, $s_A v$ is the valuation $\{sa_A(v)\}_{a \in V_0}$. We recall that sa is the value of s at $a \in V_0$ (i.e. sa is the formula which is substituted for the propositional variable a), and sa_A is the mapping determined by the formula sa. Thus $s_A v$ is the point in A^{V_0} whose a-th coordinate is equal to the value of the mapping sa_A at v.

2.5. *For every formula a, every substitution s from $\mathfrak{L}_0$ into $\mathfrak{L}_0$ and every valuation v in A,*

$$sa_A(v) = a_A(s_A v)\,. \tag{13}$$

The left-hand side of (13) is, of course, the value of the mapping sa_A determined by the formula sa (the result of the substitution s in a) at the point v.

The proof is similar to that of 2.4.

As the algebra A in the algebraic interpretation a_A of formulas a we can take, in particular, the algebra F_0 of all formulas in $\mathfrak{L}_0$ or, more generally, any quotient a'gebra $F_0/\approx$, where $\approx$ is a congruence in F_0.

Consider the last case, i.e. suppose that A is a quotient algebra

$$A = F_0/\approx\,.$$

According to the convention in I, § 3, (p. 21), elements of A will be denoted by $\|a\|$, where a is any formula in F_0.

Every substitution $s\colon V_0 \to F_0$ induces a corresponding valuation v in A:

$$v = \{\|sa\|\}_{a \in V_0}\,. \tag{14}$$

If s is the identity mapping, then the corresponding valuation

$$v^0 = \{\|a\|\}_{a \in V_0} \tag{15}$$

is said to be the *canonical valuation in $A = F_0/\approx$*.

Using the above notation we have the following theorem:

2.6. *If $A = F_0/\approx$, then for every substitution s*

$$a_A(v) = \|sa\|\,, \tag{16}$$

where v is the corresponding valuation (14). *In particular,*

$$a_A(v^0) = \|a\| \tag{17}$$

for the canonical valuation v^0.

For every fixed substitution s, both sides of (16), considered as functions of α, are homomorphisms from F_0 into $\mathcal{A}$ by 2.3 and 2.2. By (14), both homomorphisms coincide on the set V_0 of all propositional variables. Since V_0 generates F_0, they are equal by I, 4.4, i.e. (16) holds.

§ 3. The algebra of terms. Realizations of terms. Now let $\mathfrak{L} = \{A, T, F\}$ be a formalized language of the first order. Let Φ be the set of all functors in $\mathfrak{L}$, and V—the set of all free individual variables.

3.1. *The set T of all terms in $\mathfrak{L}$ forms an abstract algebra*

$$\{T, \{\varphi\}_{\varphi\in\Phi}\}$$

if, for every m-argument functor φ in Φ ($m = 0, 1, \ldots$), the term $\varphi(\tau_1 \ldots \tau_m)$ is conceived as the result of performing the m-argument operation φ on terms $\tau_1, \ldots, \tau_m$; in symbols:

$$\varphi(\tau_1, \ldots, \tau_m) = \varphi(\tau_1 \ldots \tau_m)\,. \tag{1}$$

This follows immediately from the definition of the set T of all terms (see V, § 3, p. 155).

The algebra $\{T, \{\varphi\}_{\varphi\in\Phi}\}$ is called the *algebra of terms* of the language $\mathfrak{L}$. It is an abstract algebra of the type described in I, § 4, p. 22. If Φ is infinite, the algebra T has an infinite number of finite operations. The case in which Φ is empty is also included in the general definition. Then the algebra of terms is simply the set of all individual variables without any operations. If Φ is composed only of individual constants, then T is the set of all individual constants and free individual variables without any m-argument operation ($m > 0$) but with a set of distinguished constants: the individual constants.

Let $\mathfrak{K}(\Phi)$ be the class of all abstract algebras $\{J, \{o_\varphi\}_{\varphi\in\Phi}\}$ such that, for every m-argument functor φ in Φ, o_φ is an m-argument operation in J ($m = 0, 1, 2, \ldots$). Of course, $\{T, \{\varphi\}_{\varphi\in\Phi}\}$ belongs to $\mathfrak{K}(\Phi)$.

3.2. *The algebra $\{T, \{\varphi\}_{\varphi\in\Phi}\}$ of terms in $\mathfrak{L}$ is free in the class $\mathfrak{K}(\Phi)$, the set V of all free individual variables being the set of free generators.*

We have to show that every mapping

$$v\colon V \to J$$

(i.e. a point $v = \{v_x\}_{x\in V} \in J^V$) can be extended to a homomorphism

$$h\colon T \to J\,.$$

The definition of h is by induction on the length of terms, the value of h at $\tau \in T$ being denoted by $h\tau$.

1) For any individual variable x, $hx = v_x$.

2) Suppose that τ is of the form $\varphi(\tau_1 \dots \tau_m)$ where φ is an m-argument functor ($m = 0, 1, 2, \dots$) and that $h\tau_1, \dots, h\tau_m$ are just defined. Then we define h by the equation

$$h\varphi(\tau_1 \dots \tau_m) = o_\varphi(h\tau_1, \dots, h\tau_m) . \tag{2}$$

In particular, if φ is an individual constant (the case $m = 0$), then $h\varphi$ is the constant o_φ.

The uniqueness of h follows from the fact that every term either is an individual variable or can be represented, in one way only, in the form $\varphi(\tau_1 \dots \tau_m)$.

It follows from 1) that h is an extension of v. It follows from (2) that h is a homomorphism. Thus T is free in $\mathfrak{R}(\Phi)$.

By a *substitution in the language* $\mathfrak{L}$ we shall understand any mapping

$$s\colon V \to T , \tag{3}$$

where V denotes, as previously, the set of all free individual variables in $\mathfrak{L}$. If τ is any term in T and $x_1, \dots, x_n$ are all free individual variables in τ, then $s\tau$ will denote the term

$$\tau(x_1/sx_1, \dots, x_n/sx_n) ,$$

where sx denotes the value of the mapping s at $x \in V$. Denoting, more suggestively, τ by $\tau(x_1, \dots, x_n)$, we can write the definition of $s\tau$ in the form of the identity

$$s\tau(x_1, \dots, x_n) = \tau(sx_1, \dots, sx_n) . \tag{4}$$

In other words, $s\tau$ is the term obtained from τ by the substitution of the terms $sx_1, \dots, sx_n$ for the variables $x_1, \dots, x_n$, respectively (see V, § 2 and V, 3.1). Thus (4) defines a mapping

$$s\colon T \to T \tag{5}$$

which is a natural extension of the mapping (3) and, therefore, is denoted by the same letter s.

3.3. *The mapping* (5) *is a homomorphism, i.e. for every* m*-argument functor* φ *and terms* $\tau_1, \dots, \tau_m$ ($m = 0, 1, 2, \dots$),

$$s\varphi(\tau_1 \dots \tau_m) = \varphi(s\tau_1 \dots s\tau_m) . \tag{6}$$

The mapping (5) *is the only homomorphic extension of the mapping* (3).

The proof of (6) is by an easy verification. The uniqueness of the homomorphic extension (5) of (3) follows from the fact that the set V generates the algebra T (see I, 4.4).

Observe that the existence of a homomorphic extension of (3) over the whole algebra T follows immediately from 3.2.

By a *valuation* of $\mathfrak{L}$ *in a non-empty set* J we shall understand any element $v = \{v_x\}_{x \in V}$ of the Cartesian product J^V, i.e. any mapping

$$v\colon V \to J\,,$$

where V denotes, as previously, the set of all individual variables in $\mathfrak{L}$.

We recall (see V, § 4) that by an *interpretation* or *realization of terms in a set* $J \neq 0$ we understand any mapping R which, to every m-argument functor φ in $\mathfrak{L}$, assigns an m-argument operation φ_R in J, i.e. a mapping

$$\varphi_R\colon J^m \to J$$

($m = 0, 1, 2, \ldots$).

We recall (see V, § 4) that, given any realization R of terms, we can consider every term τ as a mapping $\tau_R\colon J \times \ldots \times J \to J$. More precisely, if V_τ is the set of all free individual variables appearing in τ, then τ_R is a mapping

$$\tau_R\colon J^{V_\tau} \to J\,. \tag{7}$$

Every mapping $f\colon J^{V'} \to J$, where $V' = (x_1, \ldots, x_n)$ is a finite subset of V, can be interpreted as a mapping $f\colon J^V \to J$ if we assume

$$f(v) = f(v_{x_1}, \ldots, v_{x_n})$$

for every valuation $v = \{v_x\}_{x \in V}$. The element $f(v)$ depends, of course, only on the coordinates $v_{x_1}, \ldots, v_{x_n}$ of the point v.

In particular, for every term τ in $\mathfrak{L}$, the mapping τ_R can be interpreted as a mapping

$$\tau_R\colon J^V \to J\,. \tag{8}$$

For purely technical reasons, it is more convenient to interprete τ_R as a mapping (8). In this way we omit some difficulties connected with the fact that, for different terms τ, τ_1, the mappings τ_R, τ_{1R} in interpretation (7) are, in general, defined on different sets. In interpretation (8) it is also more convenient to write some identities, especially some formulas for superpositions (see (9) below). Therefore, in the rest of this section, we shall always interpretet τ_R as a mapping (8).

In this interpretation, the rather complicated inductive definition, of τ_R from V, § 4 (see p. 161, A), B) and p. 162) can be formulated in the following simple way:

$$\begin{aligned} x_R(v) &= v_x\,, \\ \varphi(\tau_1 \ldots \tau_m)_R(v) &= \varphi_R\big(\tau_{1R}(v), \ldots, \tau_{mR}(v)\big) \end{aligned} \tag{9}$$

for every individual variable x, for every m-argument functor φ ($m = 0, 1, 2, \ldots$) and arbitrary terms $\tau_1, \ldots, \tau_m$, and for every valuation $v = \{v_x\}_{x \in V}$.

Another precise definition of τ_R can be given as follows, by means of 3.2. Every realization R of terms uniquely determines an abstract algebra in $\mathfrak{R}(\Phi)$:

$$\{J, \{\varphi_R\}_{\varphi \in \Phi}\}$$

denoted in brief by J_R. Every valuation v: $V \to J$ can be extended to a homomorphism from the algebra T into the algebra J_R because of 3.2. Denote this homomorphism by v_R. Thus, for every term τ, $v_R(\tau)$ is well defined element in J. If τ is fixed and v is variable, then the element $v_R(\tau)$ is a function of the valuation v. Denote this function by τ_R. By definition,

$$\tau_R(v) = v_R(\tau) .$$

The mapping τ_R: $J^V \to J$ just defined satisfies identities (9). Thus it coincides with the mapping (8) defined on p. 217.

The chief result of the above consideration can be noted in the form of the following theorem:

3.4. *For every fixed valuation $v = \{v_x\}_{x \in V}$, the expression $\tau_R(v)$ considered as a function of τ is a homomorphism from $\{T, \{\varphi\}_{\varphi \in \Phi}\}$ into $\{J, \{\varphi_R\}_{\varphi \in \Phi}\}$, such that $x_R(v) = v_x$ for every individual variable x.*

From the definition of $\tau_R(v)$ immediately follows

3.5. *The element $\tau_R(v)$ depends only on $\varphi_{1R}, \ldots, \varphi_{nR}$ and on $v_{x_1}, \ldots, v_{x_m}$ where $\varphi_1, \ldots, \varphi_n$ and $x_1, \ldots, x_m$ are, respectively, all functors and all free individual variables appearing in the term τ.*

Now let R be a realization of terms in a set $J \neq 0$ and let s be a substitution in $\mathfrak{L}$. The substitution s induces a mapping

$$s_R\colon J^V \to J^V \tag{10}$$

defined as follows: for every valuation $v = \{v_x\}_{x \in V}$, $s_R v$ is the valuation $\{sx_R(v)\}_{x \in V}$. We recall that sx is the value of s at x (i.e. sx is the term which is substituted for the free individual variable x), and sx_R is the mapping determined by the term sx in the realization R. Thus $s_R v$ is the point in J^V whose x-th coordinate is equal to the value of the mapping sx at v.

3.6. *For every term τ, every substitution s in $\mathfrak{L}$, and every valuation v,*

$$s\tau_R(v) = \tau_R(s_R v) . \tag{11}$$

The left-hand side of (11) is, of course, the value of the mapping $s\tau_R$ determined by the term $s\tau$ (the result of the substitution s in τ) at the point v..

The proof of (11) is similar to that of 2.4. For every fixed v, the left-hand side of (11), considered as a function of τ, is a homomorphism

from T into J_R since it is the superposition of homomorphisms mentioned in 3.3 and 3.4. The left-hand side of (11) is also a homomorphism from T into J_R, by 3.4. Since both homomorphisms coincide on the set V which generates T, they are equal (see I, 4.4).

Now let R and R_1 be two realizations of terms of $\mathfrak{L}$ in non-void sets J and J_1 respectively. Suppose that a mapping

$$f\colon J \to J_1$$

is a homomorphism of J_R into J_{1R_1}, i.e. f has the property

$$f\big(\varphi_R(j_1, \ldots, j_m)\big) = \varphi_{R_1}\big(f(j_1), \ldots, f(j_m)\big) \tag{12}$$

for every m-argument functor φ ($m = 0, 1, 2, \ldots$) and for all $j_1, \ldots, j_m \in J$. For every valuation $v = \{v_x\}_{x \in V} \in J^V$, the symbol fv denotes the valuation $fv = \{f(v_x)\}_{x \in V} \in J_1^V$; i.e. $fv\colon V \to J_1$ is the superposition of $v\colon V \to J$ and $f\colon J \to J_1$.

Under the above hypotheses,

3.7. *For every term τ and any valuation $v \in J^V$,*

$$f\big(\tau_R(v)\big) = \tau_{R_1}(fv)\,. \tag{13}$$

The proof is similar to that of 2.4 and 3.6. For every fixed $v \in J^V$, both sides of (13), considered as functions of τ, are homomorphisms. Since the homomorphisms coincide on the set V which generates the algebra T, they are equal.

We have proved theorems 3.6 and 3.7 in an algebraic way, by interpreting (11) and (13) as equations between homomorphisms of τ. Of course, it is also possible to prove 3.6 and 3.7 directly by induction with respect to the length of τ. Both methods of proof are the same, but the algebraic formulation of the proof is shorter.

§ 4. The algebra and the Q-algebra of a formalized language of the first order. Let $\mathfrak{L} = \{A, T, F\}$ be a fixed formalized language of the first order, described in V, § 3. Let V denote the set of all individual variables in $\mathfrak{L}$.

By § 1, we can form the algebra $\{F, \cup, \cap, \Rightarrow, -\}$ of formulas in $\mathfrak{L}$. This algebra has only an auxiliary meaning since it is an algebraization of the set F of formulas only from the point of view of the logical connectives $\cup, \cap, \Rightarrow, -$. This algebraization is not complete since it does not take into consideration the quantifiers $\bigcup$ and $\bigcap$ which can also be considered as operations leading from certain formulas to other formulas. To complete the algebraization, let us assume the following definitions.

We shall say that formulas α, β in $\mathfrak{L}$ are *congruent* and we shall write

$$\alpha \sim \beta \tag{1}$$

if there exists a formula γ in $\mathfrak{L}$ and bound individual variables $\xi_1, \ldots, \xi_n$, $\eta_1, \ldots, \eta_n$, $\zeta_1, \ldots, \zeta_n$ such that α is the result of the substitution of $\xi_1, \ldots, \xi_n$ for $\zeta_1, \ldots, \zeta_n$ in γ, respectively, and β is the result of the substitution of $\eta_1, \ldots, \eta_n$ for $\zeta_1, \ldots, \zeta_n$ in γ, respectively. Roughly speaking, formulas α and β are congruent if and only if one of them can be obtained from the other by changing some bound variables. For instance, if $\alpha(x, y)$ is a formula, then

$$\bigcup_\xi \alpha(\xi, y) \quad \text{and} \quad \bigcup_\eta \alpha(\eta, y)$$

are congruent, but

$$\bigcup_\xi \alpha(\xi, y) \quad \text{and} \quad \bigcup_\xi \alpha(x, \xi)$$

are not. Similarly, the formulas

$$\bigcup_\xi \bigcup_\eta \alpha(\xi, \eta), \quad \bigcup_\eta \bigcup_\xi \alpha(\eta, \xi)$$

are congruent, but the formulas

$$\bigcup_\xi \bigcup_\eta \alpha(\xi, \eta), \quad \bigcup_\eta \bigcup_\xi \alpha(\xi, \eta)$$

are not.

It is easy to verify that $\sim$ is an equivalence relation in F. Moreover, if

$$\alpha_1 \sim \alpha_2 \quad \text{and} \quad \beta_1 \sim \beta_2,$$

then also

$$(\alpha_1 \cup \beta_1) \sim (\alpha_2 \cup \beta_2), \quad (\alpha_1 \cap \beta_1) \sim (\alpha_2 \cap \beta_2),$$

$$(\alpha_1 \Rightarrow \beta_1) \sim (\alpha_2 \Rightarrow \beta_2), \quad -\alpha_1 \sim -\alpha_2.$$

Thus $\sim$ is a congruence in the algebra $\{F, \cup, \cap, \Rightarrow, -\}$, and consequently we can form the quotient algebra

$$\overline{F} = F/\sim$$

whose elements are equivalence classes $|\alpha|$ of the relation $\sim$ (see I, § 4). The corresponding operations $\cup, \cap, \Rightarrow, -$ in $\overline{F}$ are defined by the equations

$$\begin{aligned} |\alpha| \cup |\beta| &= |(\alpha \cup \beta)|, \quad |\alpha| \cap |\beta| = |(\alpha \cap \beta)|, \\ |\alpha| \Rightarrow |\beta| &= |(\alpha \Rightarrow \beta)|, \quad -|\alpha| = |-\alpha|. \end{aligned} \tag{2}$$

We shall always consider $\overline{F}$ (unless the contrary is explicitely stated) as a generalized algebra

$$\{\overline{F}, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$$

with four finite operations (2) and with two infinite operations $\bigcup$ and $\bigcap$ defined as follows.

Both infinite operations $\bigcup$ and $\bigcap$ have the same domain $\mathfrak{D}$ (see I, § 4, p. 27). A set S is in the domain $\mathfrak{D}$ if and only if there exists a formula $\alpha(x)$ such that S is formed of all elements

$$|\alpha(\tau)| \in \overline{F}, \quad \text{where } \tau \text{ is any term.} \tag{3}$$

In other words, the only sets (3) admissible for $\bigcup$ and $\bigcap$ are sets composed of all substitutions for a fixed individual variable x in a fixed formula $\alpha(x)$. The elements (3) of an admissible set $S \in \mathfrak{D}$ are indexed, in a natural way, by τ. Therefore the results of the operations $\bigcup$ and $\bigcap$ performed on the set S of all elements (3) will be denoted by

$$\bigcup_{\tau \in T} |\alpha(\tau)| \quad \text{and} \quad \bigcap_{\tau \in T} |\alpha(\tau)|$$

respectively, according to I, § 4, (20). The results are defined by the equations

$$\bigcup_{\tau \in T} |\alpha(\tau)| = \left|\bigcup_{\xi} \alpha(\xi)\right|, \quad \bigcap_{\tau \in T} |\alpha(\tau)| = \left|\bigcap_{\xi} \alpha(\xi)\right| \tag{4}$$

where ξ is a bound individual variable which does not appear in $\alpha(x)$.

The generalized algebra

$$\{\overline{F}, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\} \tag{5}$$

will be called the *quantifier algebra* of the language $\mathfrak{L}$, or briefly the Q-*algebra* of $\mathfrak{L}$ [1].

The following theorem is an analogue of 2.1 and 3.2:

4.1. *The* Q-*algebra* (5) *is a generalized free algebra for the class of all complete algebras* $\{A, o_1, o_2, o_3, o_4, O_1, O_2\}$ *with three binary operations* o_1, o_2, o_3, *one unary operation* o_4 *and two infinite operations* O_1 *and* O_2. *The class of all elements* $|\alpha|$, *where* α *is an elementary formula, is a set of free generators for* $\overline{F}$.

The proof of 4.1 (by induction on the length of a formula) is similar to that of 2.1. It is based on the fact that every formula α is either elementary, or it can be uniquely represented as a disjunction or conjunction or implication of two formulas, or as a negation of a formula, or it can be uniquely represented as a quantification of a formula.

For completeness only we shall also quote the following analogue of theorem 4.1 for the algebra F of formulas in $\mathfrak{L}$:

4.2. *The algebra* $\{F, \cup, \cap, \Rightarrow, -\}$ *is a* $\mathfrak{K}$-*free algebra for the class* $\mathfrak{K}$ *of all algebras* $\{A, o_1, o_2, o_3, o_4\}$ *with three binary operations* o_1, o_2, o_3, *and one unary operation* o_4. *The class of all elementary formulas and of all formulas whose first sign is a quantifier is a set of* $\mathfrak{K}$-*free generators.*

[1] Sikorski [11].

The proof is analogous.

The following analogue 4.3 of 2.2 and 3.3 holds for the algebra F but does not hold for the Q-algebra $\overline{F}$.

To formulate theorem 4.3 let us assume the following notation. If s is a substitution in $\mathfrak{L}$, i.e. a mapping

$$s\colon V \to T\,, \tag{6}$$

if α is any formula, and $x_1, \ldots, x_n$ are all free individual variables in α, then $s'\alpha$ will denote the formula

$$\alpha(x_1/sx_1, \ldots, x_n/sx_n)$$

where sx denotes the value of the mapping s at $x \in V$. Denoting, more suggestively, α by $\alpha(x_1, \ldots, x_n)$, we can write the definition of $s'\alpha$ in the form of the identity:

$$s'\alpha(x_1, \ldots, x_n) = \alpha(sx_1, \ldots, sx_n)\,. \tag{7}$$

For every formula α in $\mathfrak{L}$, $s'\alpha$ is a formula in $\mathfrak{L}$ (see V, 3.1). Thus (7) defines a mapping

$$s'\colon F \to F\,. \tag{8}$$

The formula $s'\alpha$ is said to be the *result of the substitution s in α.*

4.3. *The mapping* (8) *is a homomorphism from the algebra F of formulas into itself.*

The proof of 4.3 is by induction on the length of α.

The equation

$$s''(|\alpha|) = |s'\alpha| \tag{9}$$

defines a mapping s'' from $\overline{F}$ into itself. s'' is a homomorphism of the algebra $\{\overline{F}, \cup, \cap, \Rightarrow, -\}$ into itself but it is not a homomorphism of the Q-algebra (5) into itself because it does not preserve the infinite operations $\bigcup$ and $\bigcap$.

§ 5. The I-algebra of a formalized language of the first order. As an auxiliary notion to § 6 and § 7 we shall also introduce another type of algebras of formulas with infinite operations corresponding to logical quantifiers.

Let $\mathfrak{L} = \{A, T, F\}$ be a formalized language of the first order. Adding a set I of individual constants to the language $\mathfrak{L}$ we obtain a new language of the first order, denoted by

$$\mathfrak{L}_I = \{A_I, T_I, F_I\}\,.$$

By definition, A_I is the alphabet of the extension $\mathfrak{L}_I$ of $\mathfrak{L}$, T_I is the algebra of all terms in $\mathfrak{L}_I$, and F_I is the algebra of all formulas in $\mathfrak{L}_I$. Of course, the set I is supposed to be disjoint from the set of all signs in $\mathfrak{L}$.

According to § 4, by $\overline{F}_I$ we shall denote the quotient algebra $\overline{F}_I = F_I/\sim$ where $\sim$ is the equivalence relation § 4, (1), defined in $\mathfrak{L}_I$. Thus $\overline{F}_I$ is obtained from F_I by identification of congruent formulas in $\mathfrak{L}_I$. We shall consider $\overline{F}_I$ (unless the contrary is explicitely stated) as a generalized algebra

$$\{\overline{F}_I, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$$

with four finite operations $\cup$, $\cap$, $\Rightarrow$, $-$ defined by § 4, (2), and with two generalized operations $\bigcup$, $\bigcap$ defined as follows.

Both the generalized operations $\bigcup$ and $\bigcap$ have the same domain $\mathfrak{D}_I$. A set S is in the domain $\mathfrak{D}_I$ if and only if there exists a formula $\alpha(x)$ in F_I such that S is composed of all the elements

$$|\alpha(\iota')| \in \overline{F}_I, \quad \text{where } \iota' \text{ is an element in } I. \tag{1}$$

Roughly speaking, the only sets admissible for $\bigcup$ and $\bigcap$ are sets composed of all substitutions of individual constants $\iota \in I$ for a fixed variable x in a fixed formula $\alpha(x)$ in F_I. The elements (1) of an admissible set $S \in \mathfrak{D}_I$ are indexed, in a natural way, by the element $\iota \in I$. Therefore the results of the operations $\bigcup$ and $\bigcap$ performed on the set S of all elements (1) will be denoted by

$$\bigcup_{\iota \in I} |\alpha(\iota)| \quad \text{and} \quad \bigcap_{\iota \in I} |\alpha(\iota)|$$

respectively, according to I, § 4, (20). The results are defined by the equations

$$\bigcup_{\iota \in I} |\alpha(\iota)| = \Big|\bigcup_{\xi} \alpha(\xi)\Big|, \quad \bigcap_{\iota \in I} |\alpha(\iota)| = \Big|\bigcap_{\xi} \alpha(\xi)\Big|, \tag{2}$$

where ξ is a bound variable which does not appear in $\alpha(x)$.

Definition (2) is correct when the set I has at least two elements since then the set (1) uniquely determines the elements on the right-hand side of equations (2). If I has only one element, say ι_0, then some further explanations are necessary. For instance, if ϱ is a two-argument predicate, then $\big|\bigcup_{\xi} \varrho(\xi\, \iota_0)\big|$ is the result of the operation $\bigcup$ performed on the one-element set S composed of $|\varrho(\iota_0\, \iota_0)|$ only but considered as the indexed set $\{|\varrho(\iota\, \iota_0)|\}_{\iota \in I}$, and the element $\big|\bigcup_{\xi} \varrho(\iota_0\, \xi)\big|$ is the result of the operation $\bigcup$ performed on the same one-element set S now considered as the indexed set $\{|\varrho(\iota_0\, \iota)|\}_{\iota \in I}$. Since we do not intend to complicate the main idea of definition (2) by the particular case of one-element sets, we shall always assume that the set I has at least two elements.

The generalized algebra

$$\{\overline{F}_I, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\} \tag{3}$$

with infinite operations (2) will be called the *I-algebra of the language* $\mathfrak{L}$ (1). Observe that the I-algebra of $\mathfrak{L}$ does not coincide with the Q-algebra of $\mathfrak{L}_I$. Both algebras have the same set of elements and the same finite operations, but they have different operations $\bigcup$ and different operations $\bigcap$.

The following theorem is an analogue of 4.1:

5.1. *The I-algebra* (3) *is a generalized free algebra for the class $\mathfrak{K}$ of all complete algebras $\{A, o_1, o_2, o_3, o_4, O_1, O_2\}$ with three binary operations o_1, o_2, o_3, one unary operation o_4 and two infinite operations O_1 and O_2. The class of all elements $|\alpha|$, where α is an elementary formula in $\mathfrak{L}_I$, is a set of $\mathfrak{K}$-free generators for the algebra* (3).

The proof is the same as that of 4.1.

In contrast to the Q-algebra of $\mathfrak{L}$, the operation of substitution is a homomorphism in the I-algebra of $\mathfrak{L}$. Namely the following theorem holds under the notation § 4, (6)-(9), applied to the language $\mathfrak{L}_I$:

5.2. *For every substitution $s\colon V \to T_I$ in $\mathfrak{L}_I$, the mapping $s''\colon \overline{F}_I \to \overline{F}_I$ is a homomorphism of the I-algebra of $\mathfrak{L}$ into itself.*

The proof is by an easy verification.

Suppose now that ν is a mapping from I onto the set T of all terms in $\mathfrak{L}$.

If τ is a term in $\mathfrak{L}_I$ and the sequence $\iota_1, \ldots, \iota_n$ contains all individual constants from I which appear in τ, then the equation

$$\nu^*\tau = \tau\big(\iota_1/\nu(\iota_1), \ldots, \iota_n/\nu(\iota_n)\big) \tag{4}$$

defines a term $\nu^*\tau$ in $\mathfrak{L}$, viz. $\nu^*\tau$ is the term obtained from τ by replacing all occurrences of ι_i by $\nu(\iota_i)$, $i = 1, \ldots, n$ (see V, § 2).

5.3. *The mapping*

$$\nu^*\colon T_I \to T \tag{5}$$

defined by (4) *is a homomorphism of the algebra $\{T_I, \{\varphi\}_{\varphi\in\Phi}\}$ of all terms in $\mathfrak{L}_I$ onto the algebra $\{T, \{\varphi\}_{\varphi\in\Phi}\}$ of all terms in $\mathfrak{L}$. Moreover,*

$$\nu^*\tau = \tau \quad \textit{for every term } \tau \textit{ in } \mathfrak{L}. \tag{6}$$

The proof is by an easy verification.

The mapping (5) will be called the *natural homomorphism* (from T_I onto T) *determined by* ν.

If α is any formula in $\mathfrak{L}_I$ and the sequence $\iota_1, \ldots, \iota_n$ contains all individual constants from I which appear in α, then the equation

$$\nu'\alpha = \alpha\big(\iota_1/\nu(\iota_1), \ldots, \iota_n/\nu(\iota_n)\big) \tag{7}$$

(1) The I-algebras were introduced by Sikorski [11].

defines a formula $\nu' a$ in $\mathfrak{L}$, viz. $\nu' a$ is the formula obtained from a by replacing all occurrences of ι_i by $\nu(\iota_i)$, $i = 1, 2, \ldots, n$ (see V, § 2). It can be verified that the mapping

$$\nu' \colon F_I \to F \,, \tag{8}$$

defined by (7), is a homomorphism from F_I onto F. It is evident that it has the following property:

$$\text{if} \quad a \sim \beta, \quad \text{then} \quad \nu' a \sim \nu' \beta \,.$$

Consequently, the formula

$$\nu''(|a|) = |\nu' a| \tag{9}$$

defines a mapping

$$\nu'' \colon \overline{F}_I \to \overline{F} \,. \tag{10}$$

5.4. *The mapping* (10) *is a homomorphism of the* I*-algebra* $\{\overline{F}_I, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ *of* $\mathfrak{L}$ *onto the* Q*-algebra* $\{\overline{F}, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ *of* $\mathfrak{L}$. *Moreover,*

$$\nu''(|\varrho(\tau_1 \ldots \tau_m)|) = |\varrho(\nu' \tau_1 \ldots \nu' \tau_m)| \tag{11}$$

for every elementary formula $\varrho(\tau_1 \ldots \tau_m)$ *in* $\mathfrak{L}_I$, *and*

$$\nu''(|a|) = |a| \quad \textit{for every formula } a \textit{ in } \mathfrak{L}. \tag{12}$$

The proof is by an easy verification. Note that the hypothesis that ν is a mapping from I *onto* T is essential. Thanks to this hypothesis sets admissible for $\bigcup$ and $\bigcap$ in the I-algebra $\overline{F}_I$ are transformed by ν'' onto sets admissible for $\bigcup$ and $\bigcap$ in the Q-algebra $\overline{F}$.

The mapping (10) is called the *natural homomorphism* (from $\overline{F}_I$ onto $\overline{F}$) *determined by* ν.

§ 6. Realizations of a formalized language of the first order. Continuing the considerations from § 3, we shall examine in this section a fixed formalized language $\mathfrak{L} = \{A, T, F\}$ of the first order, the set of all free individual variables being denoted by V. The letter $\boldsymbol{A}$ will always denote a complete algebra

$$\{\boldsymbol{A}, \cap, \cup, \Rightarrow, -, \bigcup, \bigcap\} \tag{1}$$

with three binary operations $\cup$, $\cap$, $\Rightarrow$, a unary operation $-$, and two infinite operations $\bigcup$, $\bigcap$, the class of all subsets of $\boldsymbol{A}$ being the common domain for $\bigcup$ and $\bigcap$.

By a *realization of the language* $\mathfrak{L}$ *in a non-void set* J *and in a complete algebra* $\boldsymbol{A}$ (see (1)) we shall understand any mapping R such that

a) to every m-argument functor φ in $\mathfrak{L}$, R assigns an m-argument operation φ_R in J ($m = 0, 1, 2, \ldots$), i.e. a mapping

$$\varphi_R\colon J^m \to J;$$

b) to every m-argument predicate ϱ in $\mathfrak{L}$, R assigns an m-argument function defined in J with values in A ($m = 1, 2, \ldots$), i.e. a mapping

$$\varrho_R\colon J^m \to A\,.$$

Thus a realization of $\mathfrak{L}$ is a mapping defined on the union of the set Φ of all functors and the set P of all predicates. Roughly speaking, the realization R is composed of two parts: a realization of terms, and a realization of predicates. The realization of terms is the mapping R restricted to Φ. It coincides with the notion of realization of terms, discussed in § 3. The realization of predicates is the mapping R restricted to P.

For simplicity, if R is a realization of $\mathfrak{L}$, and τ is a term in $\mathfrak{L}$, then τ_R will denote the mapping $\tau_{R'}$ (see § 3, (7) and (8)) where R' is the restriction of R to Φ.

Given any realization R of $\mathfrak{L}$ in a set $J \neq 0$ and a complete algebra A, we can consider every formula α in $\mathfrak{L}$ as a mapping [1]

$$\alpha_R\colon J \times \ldots \times J \to A;$$

more exactly, it can be considered as a mapping

$$\alpha_R\colon J^{V_\alpha} \to A\,, \tag{2}$$

where V_α is the finite set of all individual variables appearing in α (if α is a closed formula, i.e. the set V_α is empty, then α_R is a constant element in A). For this purpose it suffices:

a) to interpret all free or bound individual variables in α as variables ranging over the set J;

b) to interpret every functor φ in α as the mapping $\varphi_R\colon J \times \ldots \times J \to J$;

c) to interpret every predicate ϱ in α as the mapping $\varrho_R\colon J \times \ldots \times J \to A$;

d) to interpret the logical connectives $\cup$, $\cap$, $\Rightarrow$, $-$ as the signs of the corresponding operations $\cup$, $\cap$, $\Rightarrow$, $-$ in A;

e) to interpret the quantifiers $\bigcup_\xi$, $\bigcap_\xi$ as the signs of the corresponding infinite operations $\bigcup_{\xi \in J}$, $\bigcap_{\xi \in J}$ in A.

For instance, let α be the formula

$$(\bigcup_\xi \varrho(x\, \varphi(\varphi(\xi\, \xi)\, \xi)) \cap \bigcap_\eta (\varrho(x\, \varphi(\eta\, \eta)) \Rightarrow \bigcup_\xi \varrho(\eta\, \varphi(\xi\, \varphi(\xi\, \xi))))),$$

---

[1] The interpretation of formulas (in a language of the first order) as mappings was first used by Mostowski [4].

where ϱ is a two-argument predicate and φ is a two-argument functor. Suppose that $\boldsymbol{A}$ is a two-element Boolean algebra, considered as a complete algebra of the form (1), $\bigcup$ and $\bigcap$ being the infinite join and meet respectively. Let J be the set of all positive integers, and let R be any realization of the language such that φ_R is the addition in J, and ϱ_R is the characteristic of the relation of equality in J, i.e.

$$\varrho_R(j_1, j_2) = \begin{cases} V & \text{if} \quad j_1 = j_2\,, \\ \Lambda & \text{if} \quad j_1 \neq j_2 \end{cases} \qquad \text{for } j_1, j_2 \in J\,.$$

Then it is easy to verify that the function a_R: $J \to \boldsymbol{A}$ is the function

$$a_R(j) = \begin{cases} V & \text{if } j \text{ is divisible by 3,} \\ \Lambda & \text{otherwise.} \end{cases}$$

The exact definition of a_R is by induction on the length of a. If a is a formula and $V_a = (x_1, \ldots, x_m)$, then

$$(2') \qquad a_R(v_{x_1}, \ldots, v_{x_m})$$

will denote the value of a_R at the point $v = \{v_x\}_{x \in V_a} \in J^{V_a}$. If τ is a term, $V_\tau = (x_1, \ldots, x_m)$ is the set of all free individual variables appearing in τ (see § 3, p. 217), then $\tau_R(v_{x_1}, \ldots, v_{x_m})$ will denote the value of τ_R at the point $v = \{v_x\}_{x \in V}$, the function τ_R being considered as a mapping τ_R: $J^{V_\tau} \to J$ (see § 3, (7), and V, § 4).

We can now formulate the inductive definition of a_R as follows:

A) if a is an elementary formula of the form $\varrho(\tau_1, \ldots, \tau_k)$, where ϱ is a predicate and $\tau_1, \ldots, \tau_k$ are terms, then a_R is the superposition

$$a_R(v_{x_1}, \ldots, v_{x_m}) = \varrho_R\big(\tau_{1R}(v_{x_{11}}, \ldots, v_{x_{1m_1}}), \ldots, \tau_{kR}(v_{x_{k1}}, \ldots, v_{x_{km_k}})\big)\,,$$

where

$$V_a = (x_1, \ldots, x_m) \quad \text{and} \quad V_{\tau_i} = (x_{i1}, \ldots, x_{im_i})\,, \quad i = 1, \ldots, k;$$

B) if β and γ are formulas and a has one of the forms $(a \cup \beta)$, $(a \cap \beta)$, $(a \Rightarrow \beta)$, $-\beta$, then a_R is defined by one of the following equations, respectively:

$$a_R(v_{x_1}, \ldots, v_{x_m}) = \beta_R(v_{x'_1}, \ldots, v_{x'_{m'}}) \cup \gamma_R(v_{x''_1}, \ldots, v_{x''_{m''}})\,,$$

$$a_R(v_{x_1}, \ldots, v_{x_m}) = \beta_R(v_{x'_1}, \ldots, v_{x'_{m'}}) \cap \gamma_R(v_{x''_1}, \ldots, v_{x''_{m''}})\,,$$

$$a_R(v_{x_1}, \ldots, v_{x_m}) = \beta_R(v_{x'_1}, \ldots, v_{x'_{m'}}) \Rightarrow \gamma_R(v_{x''_1}, \ldots, v_{x''_{m''}})\,,$$

$$a_R(v_{x_1}, \ldots, v_{x_m}) = -\beta_R(v_{x_1}, \ldots, v_{x_m})\,,$$

where $V_a = (x_1, \ldots, x_m)$, and in the three first equations $V_\beta = (x'_1, \ldots, x'_{m'})$, $V_\gamma = (x''_1, \ldots, x''_{m''})$;

C) if β is a formula, $V_\beta = (x_0, x_1, \ldots, x_m)$, and α is of one of the forms

$$\bigcup_\xi \beta(x_0/\xi), \quad \bigcap_\xi \beta(x_0/\xi),$$

then α_R is defined by one of the following equations, respectively:

$$\alpha_R(v_{x_1}, \ldots, v_{x_m}) = \bigcup_{j \in J} \beta_R(j, v_{x_1}, \ldots, v_{x_m}),$$

$$\alpha_R(v_{x_1}, \ldots, v_{x_m}) = \bigcap_{j \in J} \beta_R(j, v_{x_1}, \ldots, v_{x_m}).$$

One of the most important cases is that for which $\boldsymbol{A}$ is the two-element Boolean algebra. Any realization in a set $J \neq 0$ and the two-element Boolean algebra is called a *semantic realization of* $\mathfrak{L}$ *in* J. The notion of semantic realization is very close to the notion of interpretation examined in V, § 4. To formulate the connection between semantic realizations and interpretations of a language $\mathfrak{L}$ in a set J, let us assume the following notation. If $\mathfrak{I}$ is an interpretation, let $\mathfrak{I}^*$ denote the following semantic realization:

$$\varphi_{\mathfrak{I}^*} = \varphi_{\mathfrak{I}} \quad \text{for every functor } \varphi,$$

$$\varrho_{\mathfrak{I}^*} = \varrho^*_{\mathfrak{I}} \quad \text{for every predicate } \varrho.$$

Thus $\mathfrak{I}$ and $\mathfrak{I}^*$ yield the same realization of terms (see V, § 4, p. 162). For every predicate ϱ, $\varrho_{\mathfrak{I}^*}$ is the characteristic $\varrho^*_{\mathfrak{I}}$ of the relation $\varrho_{\mathfrak{I}}$ (for definition, see V, § 7). It is easy to see that the above equations establish a natural one-to-one correspondence between interpretations and semantic realizations.

The connection between interpretations and semantic realizations is given by the following theorem which follows immediately from the inductive definitions of α_R and the characteristic $\alpha^*_{\mathfrak{I}}$ of the propositional function $\alpha_{\mathfrak{I}}$ (see V, § 7).

6.1. *For every interpretation* $\mathfrak{I}$ *of a language* $\mathfrak{L}$ *in a set* J,

$$\alpha^*_{\mathfrak{I}} = \alpha_{\mathfrak{I}^*} \quad \textit{for every formula } \alpha,$$

$$\tau_{\mathfrak{I}} = \tau_{\mathfrak{I}^*} \quad \textit{for every term } \tau.$$

We come back to investigate realizations in an arbitrary complete algebra $\boldsymbol{A}$ of the type (1).

Every mapping $f\colon J^{V'} \to \boldsymbol{A}$, where $V' = (x_1, \ldots, x_m) \subset V$, can be interpreted as a mapping $f\colon J^V \to \boldsymbol{A}$ if we assume

$$f(v) = f(v_{x_1}, \ldots, v_{x_m})$$

for every valuation $v = \{v_x\}_{x \in V}$. The element $f(v)$ depends, of course, only on coordinates $v_{x_1}, \ldots, v_{x_m}$ of the point $v \in J^V$.

In particular, for every formula α in $\mathfrak{L}$, and a fixed realization R of $\mathfrak{L}$ in J and $\boldsymbol{A}$, the mapping α_R (see (2)) can be interpreted as a mapping

$$\alpha_R\colon J^V \to \boldsymbol{A}\,. \tag{3}$$

Then the value of α_R at a point $v \in J^V$ will be denoted by

$$\alpha_R(v)\,. \tag{3'}$$

Analogously to § 2 and § 3, for purely technical reasons it is often more convenient to interpret α_R as a mapping (3) since in this way we omit, some difficulties connected with the fact that, for different α, α', the mappings α_R, α'_R are defined on distinct sets, in general, in interpretation (2).

Therefore we adopt the convention that, from now on, α_R will always be interpreted as a mapping (3), for every formula α, unless the contrary is explicitly stated. Similarly, for every term τ, the corresponding mapping τ_R will always be interpreted as a mapping $\tau_R\colon J^V \to J$ (see § 3, (8)) unless the contrary is explicitly stated.

In this interpretation, the rather long formulas giving the inductive definition of α_R can be formulated in a simpler way as follows (where $v = \{v_x\}_{x\in V}$ is any valuation in J):

$$\begin{aligned}
\varrho(\tau_1 \dots \tau_k)_R(v) &= \varrho_R\big(\tau_{1R}(v), \dots, \tau_{kR}(v)\big)\,,\\
(\beta \cup \gamma)_R(v) &= \beta_R(v) \cup \gamma_R(v)\,,\\
(\beta \cap \gamma)_R(v) &= \beta_R(v) \cap \gamma_R(v)\,,\\
(\beta \Rightarrow \gamma)_R(v) &= \beta_R(v) \Rightarrow \gamma_R(v)\,,\\
(-\beta)_R(v) &= -\big(\beta_R(v)\big)\,,\\
\big(\bigcup_\xi \beta(x_0/\xi)\big)_R(v) &= \bigcup_{j\in J} \beta_R(w_j)\,,\\
\big(\bigcap_\xi \beta(x_0/\xi)\big)_R(v) &= \bigcap_{j\in J} \beta_R(w_j)\,,
\end{aligned} \tag{4}$$

where $w_j = \{w_{jx}\}_{x\in V} \in J^V$ is the valuation

$$w_{jx} = v_x \quad \text{for} \quad x \neq x_0\,, \qquad w_{jx_0} = j\,. \tag{4'}$$

The last two identities can also be written in the form

$$\begin{aligned}
\big(\bigcup_\xi \beta(x_0/\xi)\big)_R(v) &= \bigcup_{w\in W} \beta_R(w)\,,\\
\big(\bigcap_\xi \beta(x_0/\xi)\big)_R(v) &= \bigcap_{w\in W} \beta_R(w)\,,
\end{aligned}$$

where W is the set of all valuations $w = \{w_x\}_{x\in V} \in J^V$ such that

$$w_{x_1} = v_{x_1}, \dots, w_{x_m} = v_{x_m}\,,$$

$x_1, \ldots, x_m$ being all the free individual variables which appear in β besides x_0.

It follows immediately from the definition of a_R that

6.2. *If formulas α and α' are congruent, then $a_R = a'_R$, i.e.*

$$a_R(v) = a'_R(v) \quad \text{for every valuation } v.$$

6.3. *The element $a_R(v)$ depends only on $\varphi_{1R}, \ldots, \varphi_{nR}$, on $\varrho_{1R}, \ldots, \varrho_{kR}$ and on $v_{x_1}, \ldots, v_{x_m}$, where $\varphi_1, \ldots, \varphi_n$, $\varrho_1, \ldots, \varrho_k$ and $x_1, \ldots, x_m$ are, respectively, all functors, all predicates and all free individual variables which appear in the formula α.*

Suppose that $\mathfrak{L}'$ is an extension of the language $\mathfrak{L}$. A realization R' of the language $\mathfrak{L}'$ in J and A is said to be an *extension* of a realization R of $\mathfrak{L}$ in J and A provided

$$\varphi_{R'} = \varphi_R \quad \text{for every functor } \varphi \text{ in } \mathfrak{L},$$

$$\varrho_{R'} = \varrho_R \quad \text{for every predicate } \varrho \text{ in } \mathfrak{L}.$$

It follows immediately from 6.3 that

6.4. *If $\mathfrak{L}'$ is an extension of $\mathfrak{L}$, R' is an extension of R, and α is a formula in $\mathfrak{L}$, then*

$$a_{R'}(v') = a_R(v) \quad \text{for every valuation } v' \text{ of } \mathfrak{L}',$$

where v denotes the restriction of v' to the set of all free individual variables in $\mathfrak{L}$. In particular, if $\mathfrak{L}$ and $\mathfrak{L}'$ have the same set of free individual variables, then

$$a_{R'}(v) = a_R(v)$$

for every formula α in $\mathfrak{L}$ and for every valuation v.

The definition of the mapping a_R quoted above is an analogue to the descriptive definitions of a_A and τ_R given in § 2 and § 3 on pp. 212 and 218, respectively. In §§ 2 and 3 we also formulated other algebraic definitions of a_A and τ_R by means of the notion of homomorphism (see 2.3 and 3.4). We can also formulate an analogous algebraic definition of a_R and this definition is useful to prove theorems concerning a_R.

For this purpose let us add a set I of new free individual constants to the language $\mathfrak{L}$ such that the cardinal of I is not less than the cardinal of J. Thus there exists a one-to-one mapping ν from I onto J. The mapping ν will be fixed in the sequel. If $\nu(\iota) = j \in J$ for $\iota \in I$, then the individual constant ι is sometimes called the *name* of j. Thus the extension $\mathfrak{L}_I = \{A_I, T_I, F_I\}$ of $\mathfrak{L}$ is obtained from $\mathfrak{L}$ by adding a set of names of elements of J.

Every realization R of $\mathfrak{L}$ in J and $\mathfrak{A}$ can be extended, in a natural way, to a realization (denoted by the same letter R) of $\mathfrak{L}_I$ in J and $\mathfrak{A}$ by defining

$$\iota_R = v(\iota) \in J \quad \text{for every } \iota \text{ in } I. \tag{5}$$

Thus for every formula α in $\mathfrak{L}_I$, the mapping

$$\alpha_R\colon J^V \to \mathfrak{A}$$

is well defined by equations (4).

For every fixed valuation v, the expression $\alpha_R(v)$ can be considered as a function of α, i.e. as a mapping from F_I into $\mathfrak{A}$. It follows from 6.2 that $\alpha_R(v)$ can also be considered as function of $|\alpha| \in \overline{F}_I$, i.e. as a mapping from $\overline{F}_I = F_I/\sim$ into $\mathfrak{A}$ (viz. as the corresponding mapping in the sense defined in I, § 3, (3), p. 21). The last interpretation will often be used in what follows. Every valuation $v \in J^V$ uniquely determines a mapping

$$v_R\colon E_I \to \mathfrak{A}\,,$$

where E_I is the set of all elementary formulas in $\mathfrak{L}_I$, viz.

$$v_R\big(\varrho(\tau_1 \dots \tau_m)\big) = \varrho_R\big(\tau_{1R}(v), \dots, \tau_{mR}(v)\big)$$

for every m-argument predicate ϱ ($m = 1, 2, \dots$) and any terms $\tau_1, \dots, \tau_m$.

By 5.1 the mapping v_R can be uniquely extended to a homomorphism of $\overline{F}_I$ into $\mathfrak{A}$. Denote this homomorphism by the same symbol v_R, and its value at an element $|\alpha| \in \overline{F}_I$ by $v_R(|\alpha|)$. If α is a fixed formula, then the element $v_R(|\alpha|) \in \mathfrak{A}$ is a function of $v \in J^V$. Denote this function by α_R. By definition

$$\alpha_R(v) = v_R(|\alpha|)\,. \tag{6}$$

The mapping $\alpha_R\colon J^V \to \mathfrak{A}$ just defined satisfies identities (4). The proof is by a simple verification. The hypothesis that v maps I onto J plays an essential part in the verification of the last two identities of (4).

Since the mapping α_R defined by (6) satisfies identities (4), it coincides with the mapping α_R defined previously on p. 229. Thus the main result of the above consideration can be formulated as follows:

6.5. *For every fixed valuation v of $\mathfrak{L}_I$, the expression $\alpha_R(v)$ considered as a function of $|\alpha| \in \overline{F}_I$ is a homomorphism from $\overline{F}_I$ into $\mathfrak{A}$ such that*

$$\varrho(\tau_1 \dots \tau_m)_R(v) = \varrho_R\big(\tau_{1R}(v), \dots, \tau_{mR}(v)\big) \tag{7}$$

for every m-argument predicate ϱ ($m = 1, 2, \dots$) and for any terms $\tau_1, \dots, \tau_m$ in $\mathfrak{L}_I$ [1].

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[1] Sikorski [11].

Identity (6) is the algebraic definition of a_R mentioned above. The only difference between definition (6) and the analogous algebraic definitions of a_A and τ_R in §§ 2 and 3 is that we had to introduce an auxiliary set I of names. However the final result, definition (6) applied to formulas α in $\mathfrak{L}$, does not depend on the auxiliary set I and the mapping v (see 6.4).

The fundamental properties of mappings a_R are expressed in theorems 6.6, 6.7, 6.8 below, the last two of which are analogous to theorems 3.6 and 3.7 on realizations of terms, respectively. Theorems 6.6, 6.7, 6.8 can be proved directly by induction on the length of α. However, we shall quote here other algebraic proofs.

Let $\{\boldsymbol{B}, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ be a complete algebra similar to the algebra $\boldsymbol{A}$ (see (1)), and let $h\colon \boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism. If R is a realization of $\mathfrak{L}$ in J and $\boldsymbol{A}$, then the following formulas define another realization (denoted by hR) of $\mathfrak{L}$ in J and $\boldsymbol{B}$:

$$\begin{aligned} \varphi_{hR} &= \varphi_R \quad \text{for every functor} \quad \varphi, \\ \varrho_{hR} &= h\varrho_R \quad \text{for every predicate} \quad \varrho. \end{aligned} \tag{8}$$

Thus R and hR yield the same realization of terms. If ϱ is an m-argument predicate, then ϱ_{hR} is the superposition of h and ϱ_R:

$$\varrho_{hR}(j_1, \ldots, j_m) = h\big(\varrho_R(j_1, \ldots, j_m)\big) .$$

6.6. *For every homomorphism $h\colon \boldsymbol{A} \to \boldsymbol{B}$ and any realization R of $\mathfrak{L}$ in J and $\boldsymbol{A}$, we have*

$$a_{hR}(v) = h\big(a_R(v)\big) \tag{9}$$

for every formula α in $\mathfrak{L}$ and every valuation v.

We shall prove that (9) holds for every formula in $\mathfrak{L}_I$, the realization R being extended over $\mathfrak{L}_I$ by means of (5).

For every fixed valuation v, both sides of (9), considered as functions of $|\alpha|$, are homomorphisms from the generalized algebra $\overline{F}_I$ into $\boldsymbol{B}$. By (8), these homomorphisms coincide on the set of all elementary formulas in $\mathfrak{L}_I$ which generates $\overline{F}_I$. Thus both homomorphisms are equal.

For every substitution s in $\mathfrak{L}$, let $s'\alpha$ have the meaning explained in § 4, (6)-(8), and let $s_R v$ have the meaning explained in § 3, (10).

6.7. *For every substitution s and every valuation v*

$$a_R(s_R v) = s' a_R(v) . \tag{10}$$

Obviously the expression on the right-hand side of (10) is the value of the mapping $s'a_R$, determined by the formula $s'\alpha$, at the point v.

We shall prove, more generally, that identity (10) holds for any given substitution $s\colon V \to T_I$ and for every formula α in F_I. In fact, for

every fixed v, both sides of (10), when considered as functions of $|\alpha|$, are homomorphisms from the generalized algebra $\overline{F}_I$ into $\boldsymbol{A}$ (see 5.2 and 6.5). By 3.6 (applied to the language $\mathfrak{L}_I$), these homomorphisms coincide on the set of all elementary formulas which generates $\overline{F}_I$. Thus the two homomorphisms are equal.

Now let R be a realization of $\mathfrak{L}$ in J and $\boldsymbol{A}$, and let R_1 be a realization of $\mathfrak{L}$ in a set $J_1 \neq 0$ and in $\boldsymbol{A}$. Suppose that f is a mapping from J onto J_1 such that

$$f\big(\varphi_R(j_1, \ldots, j_m)\big) = \varphi_{R_1}\big(f(j_1), \ldots, f(j_m)\big)\,, \tag{11}$$

$$\varrho_R(j_1, \ldots, j_m) = \varrho_{R_1}\big(f(j_1), \ldots, f(j_m)\big) \tag{12}$$

for every m-argument functor φ ($m = 0, 1, 2, \ldots$), every m-argument predicate ϱ ($m = 1, 2, \ldots$) and for all $j_1, j_2, \ldots \in J$ (observe that (11) coincides with § 3, (12)). Similarly to § 3, p. 219, for every valuation $v = \{v_x\}_{x \in V} \in J^V$, the symbol fv will denote the valuation $fv = \{f(v_x)\}_{x \in V} \in J_1^V$.

Under the above hypotheses we have

6.8. *For every formula α and every valuation $v \in J^V$*

$$\alpha_{R_1}(fv) = \alpha_R(v)\,. \tag{13}$$

We shall prove, more generally, that identity (13) holds for all formulas α in L_I where I is a set of a power $\geqslant \overline{\overline{J}}$. The letter ν denotes, as previously, a mapping from I onto J, and the realization R is extended to a realization of $\mathfrak{L}_I$ by means of (5). Since f maps J onto J_1, the superposition $\nu_1 = f\nu$ maps I onto J_1. The mapping ν_1 will play the same role with regard to the set J_1 and the realization R_1, as the mapping ν does with regards to J and R. In particular, we extend the realization R_1 to a realization (denoted by the same letter R_1) of $\mathfrak{L}_I$ in J_1 and $\boldsymbol{A}$ by assuming

$$\iota_{R_1} = \nu_1(\iota) = f\big(\nu(\iota)\big)$$

for every ι in I. Hence $f(\iota_R) = \iota_{R_1}$, i.e. the extended realizations R and R_1 also satisfy (11) and (12). The further argument is the same as in the proofs of 6.6 and 6.7. For every fixed valuation v, both sides of (13), considered as functions of $|\alpha|$, are homomorphisms from the generalized algebra $\overline{F}_I$ into A. It follows from (11), (12) and 3.7 that equation (13) holds for all elementary formulas α. Since the set of all elementary formulas generates the generalized algebra $\overline{F}_I$, (13) holds for all α.

Now let $\mathfrak{L}_0 = \{A_0, F_0\}$ be a formalized language of zero order, V_0 being the set of all propositional variables in $\mathfrak{L}_0$. Let s: $V_0 \to F$ be a substitution from $\mathfrak{L}_0$ into $\mathfrak{L}$ (see § 2, (2)). We recall that, for every formula α in $\mathfrak{L}_0$, $s\alpha$ denotes the result of the substitution s in α, i.e. a formula in $\mathfrak{L}$ defined by § 2, (3).

Under the above hypotheses,

6.9. *For every valuation v and for every formula α in $\mathfrak{L}_0$,*

$$sa_R(v) = a_A(sv)\,, \tag{14}$$

where sv is the valuation $sv = \{sa_R(v)\}_{a\in V_0} \in A^{V_0}$.

Clearly, sa_R and sa_R here denote the mapping determined by the formulas sa and sa in $\mathfrak{L}$, respectively, and a_A denotes the mapping determined by the formula a in $\mathfrak{L}_0$ (see § 2, (9)).

To prove (14) let us observe that both sides of (14), when interpreted as functions of a, are homomorphisms from the algebra F_0 of all formulas in $\mathfrak{L}_0$ into the algebra A (see 2.2 and 2.3). These homomorphisms coincide on the set V_0 of all propositional variables which generates the algebra F_0. Thus they are equal, i.e. (14) holds.

§ 7. Canonical realizations of a formalized language of the first order. We shall now discuss some special realizations R of a language $\mathfrak{L} = \{A, T, F\}$ of the first order which will play an important part in the applications to Chapters VIII, X and XI.

No restriction was made on the set J, in which a realization R of $\mathfrak{L}$ is given, except that J is non-empty. In particular we can take as J the set T of all terms in $\mathfrak{L}$. In fact, T is never empty since T contains the infinite set V of all free individual variables.

We have seen in 3.1 that every m-argument functor φ in $\mathfrak{L}$ can be interpreted as an m-argument operation in T ($m = 0, 1, 2, \ldots$). This operation, defined by § 3, (1), is denoted by the same letter φ.

A realization R of $\mathfrak{L}$ is said to be a *canonical realization of terms* provided R is a realization in the set T of all terms such that, to every m-argument functor φ, R assigns the corresponding m-argument operation φ in T; in symbols:

$$\varphi_R = \varphi\,. \tag{1}$$

More precisely (see § 3, (1)), R is a canonical realization of terms, if for every m-argument functor φ in $\mathfrak{L}$ ($m = 0, 1, 2, \ldots$)

$$\varphi_R(\tau_1, \ldots, \tau_m) = \varphi(\tau_1 \ldots \tau_m)\,, \tag{1'}$$

where $\tau_1, \ldots, \tau_m$ are any terms.

By the *identity valuation* $\mathfrak{i}$: $V \to T$ we mean the identity mapping from V into T:

$$\mathfrak{i} = \{x\}_{x\in V}\,.$$

Any valuation $v \in T^V$ is a mapping v: $V \to T$, i.e. it is also a substitution. Consequently, for every formula a and for every term τ, the

symbols $v'a$ and $v\tau$ are defined, viz. they are the results of the substitution v in a or τ respectively (see § 4, p. 222, and § 3, p. 216). In particular,

$$\mathfrak{i}'a = a \quad \text{and} \quad \mathfrak{i}\tau = \tau$$

for every formula a and every term τ, $\mathfrak{i}$ being the identity valuation.

7.1. *If R is the canonical realization of terms, and v: $V \to T$, then, for every term τ,*

$$\tau_R(v) = v\tau . \tag{2}$$

In particular, for the identity valuation $\mathfrak{i}$,

$$\tau_R(\mathfrak{i}) = \tau . \tag{2'}$$

For every formula $\beta(x_0)$,

$$\big(\bigcup_\xi \beta(\xi)\big)_R(\mathfrak{i}) = \bigcup_{\tau \in T} \beta(\tau)_R(\mathfrak{i}) , \tag{3}$$

$$\big(\bigcap_\xi \beta(\xi)\big)_R(\mathfrak{i}) = \bigcap_{\tau \in T} \beta(\tau)_R(\mathfrak{i}) . \tag{3'}$$

In fact, both sides of (2), considered as functions of τ, are homomorphisms from the algebra $\{T, \{\varphi\}_{\varphi \in \Phi}\}$ into itself (see 3.4 and 3.3). It is easy to verify that (2) holds if τ is a free individual variable, i.e. (2) holds on a set of generators of T. Thus (2) holds for all terms τ in T. (2') is a direct consequence of (2).

By § 6, (4),

$$\big(\bigcup_\xi \beta(\xi)\big)_R(\mathfrak{i}) = \bigcup_{\tau \in T} \beta(x_0)_R(w_\tau),$$

where (see § 6, (4')) $w_\tau = \{w_{\tau x}\}_{x \in V}$ is defined by the equations

$$w_{\tau x} = \mathfrak{i}_x = x \quad \text{for} \quad x \neq x_0 , \quad w_{\tau x_0} = \tau .$$

Interpret the valuation w_τ as a substitution in $\mathfrak{L}$. By definition,

$$w'_\tau \beta(x_0) = \beta(\tau) .$$

Moreover, using the notation from § 3, p. 218,

$$(w_\tau)_R \mathfrak{i} = \{(w_\tau x)_R(\mathfrak{i})\}_{x \in V} = w_\tau$$

by (2'). Hence, by 6.7,

$$\beta(x_0)_R(w_\tau) = \beta(x_0)_R\big((w_\tau)_R \mathfrak{i}\big) = \big(w'\beta(x_0)\big)_R(\mathfrak{i}) = \beta(\tau)_R(\mathfrak{i})$$

which implies (3). The proof of (3') is analogous.

In § 6 we introduced the notion of realization of a formalized language $\mathfrak{L} = \{A, T, F\}$ of the first order in a set $J \neq 0$ and in a complete algebra $\boldsymbol{A}$ of the form § 6, (1). The hypothesis that $\boldsymbol{A}$ is complete was assumed only in order to assure that all infinite operations appearing

in the inductive definition of $a_R(v)$ (see § 6, (4)) are feasible, i.e. the homomorphism mentioned in 6.5 exists. However, it may happen that, for a given mapping R satisfying conditions § 6 a), b) (p. 226), where A is a generalized algebra of the type § 6, (1), all infinite operations appearing in the definition of $a_R(v)$ (by means of § 6, (4) or 6.5) are feasible, in spite of the fact that the generalized algebra A (see § 6, (1)) is incomplete. In this case, the mapping R will also be called a *realization of* $\mathfrak{L}$ in the set J and in the incomplete generalized algebra A.

The above extension of the notion of realization is not essential. Indeed, any generalized algebra A can be extended to a complete similar algebra, say A' (see I, § 4, p. 29). In this way, any realization R in the incomplete algebra A can be considered as a realization in the complete algebra A'. In other words, every realization in the sense just defined can be interpreted as a realization in the sense defined in § 6. However, if only elements of A are values of $a_R(v)$, and only infinite operations in A are used to calculate $a_R(v)$, then there is no reason to introduce any extension A' of A, and it is more natural to consider R as a realization in A.

The fact that any realization in an incomplete algebra A can be conceived as a realization in a complete extension A' of A has important consequences because it enables us to apply, to realizations in incomplete generalized algebras A, all theorems proved in § 6 for realizations in complete algebras.

Sometimes it is convenient to assume as A' the extension of A which was described in I, § 4, (22), p. 29. This extension is obtained from A by adding a new element a_0 to A (i.e. $A' = A \cup (a_0)$), and by assuming a_0 as the value of all finite and infinite operations in A', except when the operation in question is feasible in A. Hence it follows that if R is a realization of $\mathfrak{L}$ in J and A', and $a_R(v) \in A$ for every formula α in $\mathfrak{L}$ and for every valuation v, then R is a realization of $\mathfrak{L}$ in J and A. This remark will be used in the proofs of theorems 7.2 and 7.3 below. Without saying it explicitly, we shall first interpret the mapping R mentioned in 7.2 and the mapping R^0 mentioned in 7.3 as realizations in A', and we shall prove that $a_R(v)$ and $a_{R^0}(v)$ are always in A, which proves that R and R^0 are realizations in A.

We now quote examples of realizations in a generalized algebra A without the hypothesis that A is complete.

Besides the language $\mathfrak{L} = \{A, T, F\}$ we shall consider simultaneously an auxiliary language $\mathfrak{L}_I = \{A_I, T_I, F_I\}$ obtained from $\mathfrak{L}$ by adding a set I of individual constants (see § 5), the cardinal of I being not less than the cardinal of the set T of all terms in $\mathfrak{L}$. The letter ν will denote a fixed mapping from I onto T. According to the convention in § 6,

p. 230, if $\nu(\iota) = \tau$, then ι is often called the *name* of the term τ. Thus $\mathfrak{L}_I$ is obtained from $\mathfrak{L}$ by including a set I of names of terms.

Let g be a homomorphism of the I-algebra $\{\overline{F}_I, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ into a generalized algebra $\{\boldsymbol{A}, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ of the type described in § 6, (1). Let R be a mapping defined on the union of the set of all functors and the set of all predicates in $\mathfrak{L}_I$, such that

(a) R is a canonical realization of terms of the language $\mathfrak{L}_I$, i.e.

$$\varphi_R(\tau_1, \ldots, \tau_m) = \varphi(\tau_1 \ldots \tau_m)$$

for every m-argument functor φ in $\mathfrak{L}_I$ and for all terms $\tau_1, \ldots, \tau_m$ in $\mathfrak{L}_I$ ($m = 0, 1, 2, \ldots$).

(b) for every m-argument predicate ϱ ($m = 1, 2, \ldots$) and for all terms $\tau_1, \ldots, \tau_m$ in $\mathfrak{L}_I$,

$$\varrho_R(\tau_1, \ldots, \tau_m) = g(|\varrho(\tau_1 \ldots \tau_m)|) \in \boldsymbol{A}\,.$$

Using the above notation, we have the following theorem:

7.2. *R is a realization of the language $\mathfrak{L}_I$ in the set of all terms of $\mathfrak{L}_I$ and in the generalized algebra $\boldsymbol{A}$. Moreover, for every formula α in $\mathfrak{L}_I$ and for every valuation v,*

$$\alpha_R(v) = g(|v'\alpha|)\,. \tag{4}$$

In particular, for the identity valuation i,

$$\alpha_R(\mathrm{i}) = g(|\alpha|)\,. \tag{4'}$$

The expression on the right-hand side of (4) can be written in the form $g(v''(|\alpha|))$, where v'' is the homomorphism from $\overline{F}_I$ into itself, induced by the substitution v (see § 4, (9) and 5.2). Thus, for a fixed v, the right-hand side of (4), interpreted as a function of $|\alpha| \in \overline{F}_I$, is a homomorphism of the I-algebra $\overline{F}_T$ into $\boldsymbol{A}$. If α is an elementary formula $\varrho(\tau_1 \ldots \tau_m)$, where ϱ is an m-argument predicate and $\tau_1, \ldots, \tau_m$ are terms in $\mathfrak{L}_T$, then, by 7.1 and (b),

$$\begin{aligned} g(|v\alpha|) &= g(|v\varrho(\tau_1 \ldots \tau_m)|) = g(|\varrho(v\tau_1 \ldots v\tau_m)|) \\ &= g(|\varrho(\tau_1(v) \ldots \tau_m(v))|) = \varrho_R(\tau_1(v), \ldots, \tau_m(v))\,. \end{aligned}$$

Thus the homomorphism on the right-hand side of (4) satisfies the hypotheses of 6.5. Consequently it is equal to $\alpha_R(v)$, i.e. (4) holds. (4') follows immediately from (4).

The following modification of 7.2 is important for applications.

Let $\approx$ be an equivalence relation in the set F of all formulas in $\mathfrak{L} = \{A, T, F\}$, such that for any formulas α, β in $\mathfrak{L}$

(c) if $\alpha \sim \beta$, then $\alpha \approx \beta$.

Thus the equation

$$h_1(|\alpha|) = \|\alpha\| \in F/\approx \tag{5}$$

defines a mapping h_1 from $\overline{F} = F/\sim$ onto $F/\approx$. Suppose that in $F/\approx$ we define some binary operations $\cup, \cap, \Rightarrow$, a unary operation $-$ and two generalized operations $\bigcup, \bigcap$ such that

(d) the mapping h_1 is a homomorphism from the Q-algebra $\{\overline{F}, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ into the generalized algebra $\{F/\approx, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$.

Let h be a homomorphism from $\{F/\approx, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ into a generalized algebra A of the type § 6, (1). Let R^0 be a mapping defined on the set of all functors and all predicates in $\mathfrak{L}$ such that

A) R^0 is a canonical realization of terms of the language $\mathfrak{L}$;

B) for every m-argument predicate ϱ in $\mathfrak{L}$ ($m = 1, 2, \ldots$) and for arbitrary terms $\tau_1, \ldots, \tau_m$ in $\mathfrak{L}$,

$$\varrho_{R^0}(\tau_1, \ldots, \tau_m) = h(\|\varrho(\tau_1 \ldots \tau_m)\|) .$$

Under the above hypotheses we have

7.3. *R^0 is a realization of $\mathfrak{L} = \{A, T, F\}$ in the set T of all terms and in the generalized algebra A. Moreover, for every formula α in $\mathfrak{L}$ and for every valuation v,*

$$\alpha_{R^0}(v) = h(|v'\alpha|) . \tag{7}$$

In particular, for the identity valuation i,

$$\alpha_{R^0}(\mathrm{i}) = h(\|\alpha\|) . \tag{8}$$

Let v'' be the natural homomorphism from the I-algebra $\overline{F}_I$ onto the Q-algebra $\overline{F}$ (see § 5, (10) and 5.4). The mapping g,

$$g(|\alpha|) = h\big(h_1(v''(|\alpha|))\big) \quad \text{for} \quad \alpha \text{ in } \overline{F}_I,$$

is a homomorphism from $\overline{F}_I$ into A. Let R be the realization (of $\mathfrak{L}_I$ in the set of all terms of $\mathfrak{L}_I$ and in the algebra A) defined by (a) and (b). It follows from 7.2, (4), that

$$\alpha_R(v) = g(|v'\alpha|)$$

for every formula α in $\mathfrak{L}$ and for every valuation $v \in T^V$. However, $v''(|v'\alpha|) = |v'\alpha| \in \overline{F}$ by § 5, (12), since $v'\alpha$ belongs to F. Thus

$$\alpha_R(v) = h(|v'\alpha|)$$

by the definition of g and h_1.

To complete the proof of 7.3 it suffices to show that

$$a_{R^0}(v) = a_R(v) .$$

This follows from 6.8, where J is the set T_I of all terms in $\mathfrak{L}_I$, J_1 is the set T of all terms in $\mathfrak{L}$, and f is the natural homomorphism ν^* from T_I onto T (see § 5, (5)). As the language $\mathfrak{L}$ in 6.8 we take the language $\mathfrak{L}$ mentioned in 6.3. As the realization R in 6.8 we assume the realization R just defined, but restricted to the language $\mathfrak{L}$, and as the realization R_1 in 6.8 we assume the realization R^0.

The hypothesis § 6, (11), of 6.8 is satisfied by 5.3. The hypothesis § 6, (12), of 6.8 is also satisfied since, for any terms $\tau_1, \ldots, \tau_m$ in T_I and any m-argument predicate ϱ, we have

$$\varrho_{R^0}\big(f(\tau_1), \ldots, f(\tau_m)\big) = h\big(|\varrho(f(\tau_1) \ldots f(\tau_m))|\big)$$
$$= h\Big(h_1\big(\nu_0''(|\varrho(\tau_1 \ldots \tau_m)|)\big)\Big) = g(|\varrho(\tau_1 \ldots \tau_m)|) = \varrho_R(\tau_1, \ldots, \tau_m)$$

by § 5, (12). Hence, by 6.8,

$$a_{R^0}(fv) = a_R(v) .$$

Since $v \in T^V$, we have $fv = v$ by § 4, (14). Consequently

$$a_{R^0}(v) = a_R(v) = h(\|v\alpha\|) \in \boldsymbol{A} .$$

The realization R^0 defined by A), B) will be called the *canonical realization of* $\mathfrak{L}$ *determined by the homomorphism* h [1].

In the case where $\boldsymbol{A} = F/{\approx}$ and h is the identity mapping, R^0 is called the *canonical realization of* $\mathfrak{L}$ *in the algebra* $\boldsymbol{A} = F/{\approx}$. By definition, the canonical realization of $\mathfrak{L}$ in $\boldsymbol{A} = F/{\approx}$ is defined by equations

$$\varphi_{R^0}(\tau_1, \ldots, \tau_m) = \varphi(\tau_1 \ldots \tau_m) ,$$
$$\varrho_{R^0}(\tau_1, \ldots, \tau_m) = \|\varrho(\tau_1 \ldots \tau_m)\| , \tag{8}$$

where φ is any m-argument functor ($m = 0, 1, 2, \ldots$), ϱ is any m-argument predicate ($m = 1, 2, \ldots$) and $\tau_1, \ldots, \tau_m$ are any terms in $\mathfrak{L}$. The following theorem follows directly from 7.3:

7.4. *If an equivalence* $\approx$ *in* F *and a generalized algebra* $\boldsymbol{A} = F/{\approx}$ *satisfy hypotheses* (c) *and* (d), *and* R^0 *is the canonical realization* (8) *of a formalized language* $\mathfrak{L} = \{\boldsymbol{A}, T, F\}$ *in the set* T *and the algebra* $\boldsymbol{A}$, *then for every formula* α *in* $\mathfrak{L}$

$$a_{R^0}(v) = \|v'\alpha\| \tag{9}$$

---

[1] Canonical realizations and theorem 7.3, 7.4 were first used by Henkin [2] and independenty (in a slight different form) by Rasiowa [2]. Later they were used by Rasiowa and Sikorski [1], [2].

for every valuation v: $V \to T$. *In particular,*

$$a_{R_0}(\mathfrak{i}) = \|a\| \tag{10}$$

where $\mathfrak{i}$ *is the identity valuation.*

§ 8. Products of realizations. In this section we shall give another useful example of realizations of a language $\mathfrak{L}$ of the first order in an incomplete generalized algebra.

Let N be a non-empty set. Suppose that, for every $n \in N$, R_n is a realization of $\mathfrak{L}$ in a set J_n and in a complete algebra

$$\{A_n, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\} \tag{1}$$

with three binary operations $\cup, \cap, \Rightarrow$, a unary operation $-$, and two infinite operations $\bigcup, \bigcap$. Let

$$J = P_{n \in N} J_n \tag{2}$$

and

$$A = P_{n \in N} A_n . \tag{3}$$

According to I, § 4, p. 31, A will be considered as an algebra similar to (1), viz. as the product

$$\{A, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\} \tag{4}$$

of all the algebras A_n.

Let

$$j_1 = \{j_{1,n}\}_{n \in N}, \ \ldots, \ j_m = \{j_{m,n}\}_{n \in N}$$

be arbitrary points in J.

The realizations R_n uniquely determine a mapping R_0 defined on the union of the set of all functors and the set of all predicates, such that

1° R_0 assigns to every m-argument functor φ ($m = 0, 1, 2, \ldots$) the function φ_{R_0}: $J^m \to J$ defined by the equation

$$\varphi_{R_0}(j_1, \ldots, j_m) = \{\varphi_{R_n}(j_{1,n}, \ldots, j_{m,n})\}_{n \in N};$$

2° R_0 assigns to every m-argument predicate ϱ ($m = 1, 2, \ldots$) the function ϱ_{R_0}: $J^m \to A$ defined by the equation

$$\varrho_{R_0}(j_1, \ldots, j_m) = \{\varrho_{R_n}(j_{1,n}, \ldots, j_{m,n})\}_{n \in N} .$$

R_0 is said to be the *product of realizations* R_n [1].

Let V denote, as usual, the set of all free individual variables in $\mathfrak{L}$. Valuations $v = \{v_x\}_{x \in V} \in J^V$ in J are mappings which, to every free indi-

[1] Products of semantic realizations modulo prime filters were first used by Łoś [7] (see VIII, § 18, p. 338). The exposition of the theory of products of realizations in this section is taken from the paper of Sikorski [15].

vidual variable x, assign a point $v_x = \{v_{x,n}\}_{n \in N} \in J$. By definition, $v_{x,n} \in J_n$. Thus, for every fixed $n \in N$, $v^n = \{v_{x,n}\}_{x \in V} \in J_n^V$ is a valuation in J_n. Conversely if, for every $n \in N$, $v^n = \{v_{x,n}\}_{x \in V}$ is any valuation in J_n, then putting $v_x = \{v_{x,n}\}_{n \in N}$ we define a valuation $v = \{v_x\}_{x \in V}$ in J.

With the above notation, the following theorems hold.

8.1. *The product R_0 of the realization R_n is a realization of terms in the set J and for every term τ in* $\mathfrak{L}$

$$\tau_{R_0}(v) = \{\tau_{R_n}(v^n)\}_{n \in N} . \tag{5}$$

Consider the sets J and J_n ($n \in N$) as abstract algebras J_R and J_{nR_n} respectively, i.e. as algebras (see § 3, p. 218)

$$\left[J, \{\varphi_{R_0}\}_{\varphi \in \Phi}\right], \quad \left[J_n, \{\varphi_{R_n}\}_{\varphi \in \Phi}\right],$$

where Φ is the set of all functors in $\mathfrak{L}$. It follows from 1° that the algebra J_{R_0} is the product of algebras J_{nR_n}, $n \in N$ (see I, § 4, p. 27). For every fixed v, the left-hand side of (5), considered as a function of τ, is a homomorphism from the algebra T of all terms into the algebra J_{R_0} on account of 3.4. Similarly, the right-hand side of (5), considered as a function of τ, is a homomorphism from the algebra T of terms into the algebra J_{R_0} because of 3.4 and I, 4.10. The two homomorphisms coincide on the set V which generates the algebra T. Thus they are equal by I, 4.4.

8.2. *The product R_0 of realizations R_n of $\mathfrak{L}$ in J_n and A_n is a realization of $\mathfrak{L}$ in the product J of the sets J_n and in the product A of the algebras A_n. Moreover, for every formula α in* $\mathfrak{L}$,

$$\alpha_{R_0}(v) = \{\alpha_{R_n}(v^n)\}_{n \in N} . \tag{6}$$

According to the remark in § 7, p. 236, it will be convenient to replace, for a moment, the incomplete algebra A by its extension A' defined in I, § 4, (22), p. 29, in order to avoid the difficulties caused by the incompleteness of A. In other words, R_0 will be conceived as a realization of $\mathfrak{L}$ in J and A'.

We shall prove that (6) holds for every formula α in the extended language $\mathfrak{L}_I$ described in § 5. We recall that $\mathfrak{L}_I$ is obtained from $\mathfrak{L}$ by adding a set I of individual constants ι, such that $\overline{\overline{J}} \leqslant \overline{\overline{I}}$. Let ν be, as previously, a mapping from I onto J. By definition,

$$\nu(\iota) = \{\nu_n(\iota)\}_{n \in N} \quad \text{for} \quad \iota \in I, \tag{7}$$

where $\nu_n(\iota) \in J_n$ is the n-th coordinate of $\nu(\iota)$. Of course, ν_n is a mapping from I onto J_n.

We extend the realizations R_0 and R_n to realizations (denoted by the same letters R_0 and R_n) of $\mathfrak{L}_I$ by assuming

$$\iota_{R_0} = \nu(\iota), \quad \iota_{R_n} = \nu_n(\iota) \quad \text{for every} \quad \iota \in I \tag{8}$$

according to the convention in § 6, (5), p. 231. It follows from (7) and (8) that the extended realization R is the product of the extended realizations R_n, $n \in N$.

For every fixed valuation v, the left-hand side of (6), considered as a function of $|\alpha|$ (α in $\mathfrak{L}_I$), is a homomorphism from the I-algebra $\overline{F}_I$ of the formalized language $\mathfrak{L}$ into the algebra A' by 6.5. Similarly the right-hand side of (6), considered as a function of $|\alpha|$, is a homomorphism from $\overline{F}_I$ into A, and consequently into A', by 6.5 and I, 4.10. It follows from 2^0 and theorem 8.1 (applied to the extended realizations R, R_n) that these homomorphisms coincide on the set of all elementary formulas in $\mathfrak{L}_I$ which generates the algebra $\overline{F}_I$. Thus the two homomorphisms are equal.

Since the right-hand side of (6) always belongs to the product A, the left-hand side does also. This proves that R_0 is a realization of $\mathfrak{L}$ in J and the product A and completes the proof of 8.2.

Now let

$$\{B, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$$

be an algebra similar to (1) and (4). Let h be a homomorphism from the product A (see (4)) of all algebras A_n into the algebra B. Let R_0 be, as previously, the product of realizations R_n of $\mathfrak{L}$ in J_n and A_n respectively. By 8.2, R_0 is a realization of $\mathfrak{L}$ in J and A.

In accordance with § 6, p. 232, by hR_0 we shall denote the following realization in J and B:

$$(9) \qquad \begin{aligned} \varphi_{hR_0} &= \varphi_{R_0} \quad \text{for every functor } \varphi, \\ \varrho_{hR_0} &= h\varrho_{R_0} \quad \text{for every predicate } \varrho. \end{aligned}$$

Under the above hypotheses, the following theorem holds:

8.3. *For every formula* α,

$$\alpha_{hR_0}(v) = h\left(\alpha_{R_0}(v)\right).$$

This is an immediate consequence of theorem 6.6. The only difference is that 6.6 was proved under the hypotheses that the domain A of h is complete. However, in § 7 we have observed that all theorems in § 6, in particular theorem 6.6, remain true for realizations in incomplete algebras provided that all infinite operations appearing in the inductive definition of $\alpha_{R_0}(v)$ are feasible. This hypothesis is satisfied in the case examined in 8.3, by 8.2.

All remarks and theorems proved in § 8 remain true if the product of all algebras A_n is replaced everywhere by the complete product of all A_n. The proofs of 8.2 and 8.3 are then much simpler because the

complete product of complete algebras $\boldsymbol{A}_n$ is a complete algebra. However, the case of the product of algebras $\boldsymbol{A}_n$ is more important in applications than the case of the complete product of algebras $\boldsymbol{A}_n$ (see e.g. VIII, § 18).

§ 9. The algebra of open formulas. Let $\mathfrak{L} = \{A, T, F\}$ be a formalized language of the first order. We recall that a formula in $\mathfrak{L}$ is said to be open if it does not contain any sign of quantifiers. Let A^0 denote the alphabet obtained from the alphabet A by omitting quantifiers and bound individual variables. Let F^0 denote the set of all open formulas in $\mathfrak{L}$. According to V, § 14, p. 205, the triple $\mathfrak{L}^0 = \{A^0, T, F^0\}$ is called the language of open formulas of the language $\mathfrak{L}$.

9.1. *The set F^0 of all open formulas in $\mathfrak{L}$ is a subalgebra of the algebra F of all formulas in $\mathfrak{L}$, viz. it is the subalgebra generated by the set E of all elementary formulas.*

This follows immediately from the inductive definition of F^0 quoted in V, § 14, p. 205.

Observe that two open formulas are congruent if and only if they are equal. Thus F^0 can also be considered as a subalgebra of $\{\overline{F}, \cup, \cap, \Rightarrow, -\}$.

9.2. *The algebra $\{F^0, \cup, \cap, \Rightarrow, -\}$ of open formulas is free in the class of all algebras $\{A, o_1, o_2, o_3, o_4\}$ with three binary operations o_1, o_2, o_3 and one unary operation o_4. The set E of all elementary formulas is a set of free generators for F^0.*

The proof of 9.2 is the same as that of 2.1.

9.3. *Let R be a realization of $\mathfrak{L}$ in a set $J \neq 0$ and in an algebra A. For every fixed valuation v, the expression $a_R(v)$, considered as a function of an open formula a in F^0, is a homomorphism from the algebra F^0 into the algebra A, such that*

$$\varrho(\tau_1 \dots \tau_m)_R(v) = \varrho_R\big(\tau_{1R}(v), \dots, \tau_{nR}(v)\big) \tag{1}$$

for every elementary formula $\varrho(\tau_1 \dots \tau_m)$.

Theorem 9.3 can be proved directly, in the same way as the analogous theorem 2.3. It can be also deduced from 6.5 by interpreting F^0 as a subalgebra of $\{\overline{F}_I, \cup, \cap, \Rightarrow, -\}$.

Let $\mathfrak{L}_0 = \{A_0, F_0\}$ be a formalized language of the zero order such that the set V_0 of all propositional variables in $\mathfrak{L}_0$ has the same power as the set E of all elementary formulas in $\mathfrak{L}$. Let s_0 be a one-to-one mapping from V_0 onto E. By definition, s_0 is a substitution from $\mathfrak{L}_0$ into $\mathfrak{L}$ (see § 2, p. 210). By § 2, (2), (3), (4) (see also 2.2) the mapping s_0 can be uniquely extended to a homomorphism s from F_0 into F. For every formula a in F_0, sa denotes the image of a by the mapping s, i.e. the result of the substitution s_0 in a.

9.4. *If s_0 is a one-to-one mapping from V_0 onto E, then its homomorphic extension s is an isomorphism from F_0 onto F^0.*

It follows easily from the definition that sa contains no quantifiers. Thus the homomorphism s maps F_0 into F^0. On the other hand, s_0^{-1} can also be extended to a homomorphism from F^0 into F because of 9.2. By I, 4.5, s is an isomorphism from F_0 onto F^0.

Let $\mathfrak{L}_0 = \{A_0, F_0\}$ and s have the same meaning as before, and let $v = \{v_a\}_{a \in V_0}$ be a valuation of $\mathfrak{L}_0$ in a complete algebra $\boldsymbol{A}$ of the type § 6, (1). We define a realization R of $\mathfrak{L}$ in the set T (of all terms in $\mathfrak{L}$) and in the algebra $\boldsymbol{A}$ as follows:

R is the canonical realization of terms. For an m-argument predicate ϱ $(m = 1, 2, \ldots)$ and terms $\tau_1, \ldots, \tau_m$ let a be a propositional variable such that sa is the elementary formula $\varrho(\tau_1 \ldots \tau_m)$. Then let

$$\varrho_R(\tau_1, \ldots, \tau_m) = v_a \,. \tag{2}$$

Let $\mathfrak{i}$ be the identity valuation in $\mathfrak{L}$. Under the above hypotheses, the following theorem holds:

9.5. *For every formula a in $\mathfrak{L}_0$,*

$$a_{\boldsymbol{A}}(v) = sa_R(\mathfrak{i}) \,. \tag{3}$$

Theorem 9.5 follows immediately from 6.9 since $s\mathfrak{i} = v$, using the notation from 6.9.

§ 10. The algebra of a formalized theory. Let $\mathfrak{L}$ be a formalized language of zero order or of the first order, or the language of open formulas in a language of the first order.

Let $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ be a formalized theory based on the language $\mathfrak{L}$. Let F be the set of all formulas in $\mathfrak{L}$. The consequence operation C in $\mathfrak{L}$ can be completely arbitrary. The only hypothesis assumed in this section is that the set $C(\mathcal{A})$ of all theorems in $\mathfrak{T}$ is closed under modus ponens, i.e. that

(1) if a and $(a \Rightarrow \beta)$ are theorems in $\mathfrak{T}$, so is β.

In all theorems below hypothesis (1) is adopted.

For any formulas a, β in $\mathfrak{L}$ we shall write

$$a \leqslant \beta \tag{2}$$

provided the formula $(a \Rightarrow \beta)$ is a theorem in $\mathfrak{T}$.

10.1. *If all formulas of the form*

$$(a \Rightarrow a)\,, \tag{T_0}$$

$$((a \Rightarrow \beta) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow (a \Rightarrow \gamma)))\,, \tag{T_1}$$

where α, β are any formulas, are theorems in $\mathfrak{C}$, then relation (2) *is a quasi-ordering in F.*

It follows from (T_0) that

$$\alpha \leqslant \alpha \quad \text{for every formula } \alpha.$$

If $\alpha \leqslant \beta$ and $\beta \leqslant \gamma$, then $\alpha \leqslant \gamma$. In fact, then

$$(\alpha \Rightarrow \beta), \tag{3}$$

$$(\beta \Rightarrow \gamma) \tag{4}$$

are theorems. Applying *modus ponens* (1) to (3) and (T_1) we infer that

$$((\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma)) \tag{5}$$

is a theorem. Applying *modus ponens* (1) to (4) and (5) we infer that $(\alpha \Rightarrow \gamma)$ is a theorem, i.e. $\alpha \leqslant \gamma$.

We say that two formulas α, β are *equivalent in* $\mathfrak{C}$ if both formulas

$$(\alpha \Rightarrow \beta) \quad \text{and} \quad (\beta \Rightarrow \alpha)$$

are theorems in $\mathfrak{C}$, i.e.

$$\alpha \leqslant \beta \quad \text{and} \quad \beta \leqslant \alpha.$$

We then write

$$\alpha \approx \beta. \tag{6}$$

If $\leqslant$ is a quasi-ordering in F, then, by I, 5.2, relation (6) is an equivalence relation in F. The equivalence class containing α will be denoted by $\|\alpha\|$, according to I, § 3, p. 21. Sometimes we shall use the more precise notation $\|\alpha\|_{\mathfrak{C}}$ instead of $\|\alpha\|$.

By I, 5.2, the quasi-ordering (2) in F induces an ordering relation $\leqslant$ in $F/\approx$. We recall that $\leqslant$ is defined in $F/\approx$ as follows: for any formulas α, β in F,

(7) $\|\alpha\| \leqslant \|\beta\|$ if and only if $\alpha \leqslant \beta$, i.e. if $(\alpha \Rightarrow \beta)$ is a theorem.

The set $F/\approx$ will be denoted by $\mathfrak{A}(\mathrm{R})$ [1].

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[1] By theorems 10.5, 10.6 proved below, $\mathfrak{A}(\mathfrak{C})$ can be conceived as an abstract algebra with four operations corresponding to logical connectives. The algebra $\mathfrak{A}(\mathfrak{C})$ was first introduced by A. Lindenbaum and (in a slight different form) by A. Tarski. In recent years it was extensively applied and examined by many authors. See e.g. Henkin [2], Łoś, Rasiowa and Mostowski [1], [2], Mostowski [1], McKinsey [1], McKinsey and Tarski [3], Rasiowa [2] [4]-[10], Rasiowa and Sikorski [1]-[7], [9], Rieger [1], [3], Sikorski [6]-[8] and [10]-[15].

A. Lindenbaum was a prominent Polish mathematician who died precosiously (was killed by Nazis during the Second World War) and whose various results were

We have observed the following theorem:

10.2. *If $\mathfrak{C}$ satisfies the hypotheses mentioned in* 10.1, *then $\mathfrak{A}(\mathfrak{C})$ is ordered by the relation $\leqslant$, and* (7) *holds.*

We now prove the following theorem:

10.3. *If all the formulas of the form* (T_0), (T_1) *and all the formulas of the form*

$$(T_2) \qquad (\alpha \Rightarrow (\alpha \cup \beta)),$$

$$(T_3) \qquad (\beta \Rightarrow (\alpha \cup \beta)),$$

$$(T_4) \qquad ((\alpha \Rightarrow \gamma) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow ((\alpha \cup \beta) \Rightarrow \gamma))),$$

$$(T_5) \qquad ((\alpha \cap \beta) \Rightarrow \alpha),$$

$$(T_6) \qquad ((\alpha \cap \beta) \Rightarrow \beta),$$

$$(T_7) \qquad ((\gamma \Rightarrow \alpha) \Rightarrow ((\gamma \Rightarrow \beta) \Rightarrow (\gamma \Rightarrow (\alpha \cap \beta))))$$

are theorems in $\mathfrak{C}$, then the relation $\leqslant$ in $\mathfrak{A}(\mathfrak{C})$ is a lattice ordering and, consequently, $\mathfrak{A}(\mathfrak{C})$ is a lattice. Moreover, for any formulas α, β,

$$(8) \qquad \|\alpha\| \cup \|\beta\| = \|(\alpha \cup \beta)\|, \quad \|\alpha\| \cap \|\beta\| = \|(\alpha \cap \beta)\|,$$

i.e. relation (6) *is a congruence with respect to the operations $\cup$ and $\cap$ in the algebra of formulas F.*

Observe that $\cup$, $\cap$ on the left-hand side of (8) denote the join and meet in $\mathfrak{A}(\mathfrak{C})$, induced by the ordering $\leqslant$. On the right-hand side $\cup$, $\cap$ denote the disjunction sign and the conjunction sign, respectively.

To prove 10.3, let $\|\alpha\|$, $\|\beta\|$ be any elements in $\mathfrak{A}(\mathfrak{C})$. By (T_2), (T_3) and (7),

$$\|\alpha\| \leqslant \|(\alpha \cup \beta)\| \quad \text{and} \quad \|\beta\| \leqslant \|(\alpha \cup \beta)\|.$$

Suppose now that $\|\gamma\|$ is another element in $\mathfrak{A}(\mathfrak{C})$ such that

$$\|\alpha\| \leqslant \|\gamma\| \quad \text{and} \quad \|\beta\| \leqslant \|\gamma\|,$$

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not published. To honour Lindenbaum, Polish logicians call usually $\mathfrak{A}(\mathfrak{C})$ the *Lindenbaum algebra*. This name for $\mathfrak{A}(\mathfrak{C})$ was also used by the authors of this book. We were also suggested by the fact that McKinsey [1], a close collaborator of Tarski, called the method of treating the set of all formulas of a formalized language of the zero order "an unpublished method of Lindenbaum" (p. 222, line 12) "explained to me by Professor Tarski" (footnote 7). We have the duty to quote here also the following remark of Henkin and Tarski [2] (p. 85, footnote 4): "Apart from a different treatment on the notion of equality the quotient algebras $F/\approx$ were explicitly introduced in Tarski [5], p. 510", "the historical justification given in Rasiowa and Sikorski [6], p. 143, footnote 1, for calling these algebras Lindenbaum algebras seems to be incorrect". (The references in this text are numbered according to the numeration adopted in this book).

i.e. that the formulas

(9) $$(\alpha \Rightarrow \gamma),$$

(10) $$(\beta \Rightarrow \gamma)$$

are theorems. Applying *modus ponens* (1) to (9) and (T_4) we infer that the formula

(11) $$((\beta \Rightarrow \gamma) \Rightarrow ((\alpha \cup \beta) \Rightarrow \gamma))$$

is a theorem. Applying *modus ponens* (1) to (10) and (11) we infer that the formula $((\alpha \cup \beta) \Rightarrow \gamma)$ is a theorem, i.e. that

$$\|(\alpha \cup \beta)\| \leqslant \|\gamma\|$$

by (7). This proves that $\|(\alpha \cup \beta)\|$ is the join of $\|\alpha\|$ and $\|\beta\|$ in $\mathfrak{A}(\mathfrak{C})$.

Using (T_5), (T_6), (T_7) instead of (T_2), (T_3), (T_4) we prove, in a similar way, that $\|(\alpha \cap \beta)\|$ is the meet of $\|\alpha\|$ and $\|\beta\|$ in $\mathfrak{A}(\mathfrak{C})$.

Thus (8) holds. Since the join and meet always exist, $\mathfrak{A}(\mathfrak{C})$ is a lattice.

10.4. *If all the formulas of the form* (T_1)-(T_7) *and all the formulas of the form*

(T_8) $$((\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \cap \beta) \Rightarrow \gamma)),$$

(T_9) $$((\alpha \cap \beta) \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow (\beta \Rightarrow \gamma)))$$

are theorems in $\mathfrak{C}$, *then all formulas of the form* (T_0) *are also theorems. Consequently,* $\mathfrak{A}(\mathfrak{C})$ *is a lattice, and* (7) *and* (8) *hold. Moreover,* $\mathfrak{A}(\mathfrak{C})$ *is then a relatively pseudo-complemented lattice, viz.*

(12) $$\|\alpha\| \Rightarrow \|\beta\| = \|(\alpha \Rightarrow \beta)\|$$

for all formulas α, β, *i.e. relation* (6) *is a congruence in* F *with respect to* $\Rightarrow$. *An element* $\|\alpha\|$ *is the unit element of the lattice* $\mathfrak{A}(\mathfrak{C})$ *if and only if* α *is a theorem in* $\mathfrak{C}$.

Observe that $\Rightarrow$ on the left-hand side denotes the relative pseudo-complement in the lattice $\mathfrak{A}(\mathfrak{C})$. On the right-hand side it denotes the implication sign.

To prove 10.4 let us recall that (T_0) is derivable from formulas of the form (T_5), (T_6), (T_9) by means of *modus ponens* only (see V, 9.1). Since $\mathfrak{C}$ is closed under *modus ponens*, all formulas in such a formal proof of (T_0) are theorems in $\mathfrak{C}$. Thus, in particular, (T_0) is a theorem in $\mathfrak{C}$.

The hypotheses of 10.3 being satisfied, $\mathfrak{A}(\mathfrak{C})$ is a lattice, and (7) and (8) hold.

Now let $\|\alpha\|$, $\|\beta\|$ be any elements in $\mathfrak{A}(\mathfrak{C})$. The formulas

$$((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \beta)),$$

$$(((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \beta)) \Rightarrow (((\alpha \Rightarrow \beta) \cap \alpha) \Rightarrow \beta))$$

are theorems because they are of the form (T_8) and (T_3). By *modus ponens* (1), the formula

$$(((\alpha \Rightarrow \beta) \cap \alpha) \Rightarrow \beta)$$

is also a theorem, i.e.

$$\|((\alpha \Rightarrow \beta) \cap \alpha)\| \leqslant \|\beta\| .$$

By (8) we can write this inequality in the form

$$\|\alpha\| \cap \|(\alpha \Rightarrow \beta)\| \leqslant \|\beta\| . \tag{13}$$

Now we shall prove that, for every element $\|\gamma\|$ in $\mathfrak{A}(\mathfrak{C})$,

$$\text{if} \quad \|\alpha\| \cap \|\gamma\| \leqslant \|\beta\| , \quad \text{then} \quad \|\gamma\| \leqslant \|(\alpha \Rightarrow \beta)\| . \tag{14}$$

In fact, if $\|\alpha\| \cap \|\gamma\| \leqslant \|\beta\|$, then, by (7) and (8), the formula

$$((\gamma \cap \alpha) \Rightarrow \beta)$$

is a theorem. The formula

$$((\gamma \cap \alpha) \Rightarrow \beta) \Rightarrow (\gamma \Rightarrow (\alpha \Rightarrow \beta)))$$

is also a theorem since it is of the form (T_9). By *modus ponens* (1), the formula

$$(\gamma \Rightarrow (\alpha \Rightarrow \beta))$$

is a theorem, i.e. $\|\gamma\| \leqslant \|(\alpha \Rightarrow \beta)\|$.

Identity (12) follows directly from (13) and (14) and the definition of the operation $\Rightarrow$ in lattices (see I, § 11, (1)). Thus $\mathfrak{A}(\mathfrak{C})$ is a relatively pseudo-complemented lattice and, consequently, has the unit element V. It follows from (12) and I, § 11, (3), that

$$V = \|\alpha\| \Rightarrow \|\alpha\| = \|(\alpha \Rightarrow \alpha)\| \tag{15}$$

for every formula α.

We have proved in V, 9.1, that every formula of the form

$$(\alpha \Rightarrow (\beta \Rightarrow \alpha)) \tag{16}$$

is derivable from the set of all formulas of the form (T_5), (T_6) and (T_9) by means of *modus ponens* only. Since $C(\mathcal{A})$ is closed under *modus ponens*, all formulas in such a formal proof of (16) belong to $C(\mathcal{A})$. Thus, in particular, (16) belong to $C(\mathcal{A})$, i.e. all formulas of the form (16) are theorems in $\mathfrak{C}$. Thus any formula of the form

$$(\alpha \Rightarrow ((\alpha \Rightarrow \alpha) \Rightarrow \alpha))$$

is also a theorem. If a is a theorem in $\mathfrak{C}$, then by ***modus ponens*** (1), the formula $((a \Rightarrow a) \Rightarrow a)$ is a theorem. Thus, by (7) and (15),

$$V = \|(a \Rightarrow a)\| \leqslant \|a\|,$$

i.e. $\|a\| = V$. Conversely, if $\|a\| = V$, then $\|(a \Rightarrow a)\| \leqslant \|a\|$ by (15), i.e. the formula

$$((a \Rightarrow a) \Rightarrow a)$$

is a theorem. Since $(a \Rightarrow a)$ is also a theorem, the formula a is a theorem by modus ponens (1). This proves the last part of 10.4.

10.5. *If all formulas of the form* (T_1)-(T_9) *and all formulas of the form*

$$(T_{10}) \qquad ((a \cap -a) \Rightarrow \beta),$$

$$(T_{11}) \qquad ((a \Rightarrow (a \cap -a)) \Rightarrow -a)$$

are theorems in $\mathfrak{C}$, *then* $\mathfrak{A}(\mathfrak{C})$ *is a pseudo-Boolean algebra and* (7), (8) *and* (12) *hold. Moreover, for every formula* a,

$$(17) \qquad -\|a\| = \|-a\|,$$

i.e. the equivalence relation (2) *is a congruence in the algebra* $\{F, \cup, \cap, \Rightarrow, -\}$ *of formulas of the theory* $\mathfrak{C}$.

Observe that, on the left-hand side of (17), the symbol $-$ denotes the complement in $\mathfrak{A}(\mathfrak{C})$. On the right-hand side it denotes the negation sign.

By 10.4, $\mathfrak{A}(\mathfrak{C})$ is a pseudo-complemented lattice, and (7), (8), (12) hold. It follows from (T_{10}) and (7) that

$$\|(a \cap -a)\| \leqslant \|\beta\|$$

for every formula β, i.e. that, for every formula a, the element $\|(a \cap -a)\|$ is the zero element Λ of $\mathfrak{A}(\mathfrak{C})$:

$$\|(a \cap -a)\| = \Lambda.$$

Consequently, by (8),

$$\|a\| \cap \|-a\| = \Lambda.$$

This implies, by the definition of pseudo-complement in a lattice (see I, § 10, p. 53) that

$$\|-a\| \leqslant -\|a\|.$$

On the other hand,

$$-\|a\| = \|a\| \Rightarrow \Lambda = \|a\| \Rightarrow \|(a \cap -a)\| = \|(a \Rightarrow (a \cap -a))\| \leqslant \|-a\|$$

by I, § 11, (6), and by (12) and (T_{11}). This proves (17).

10.6. *If all formulas of the form* (T_1)-(T_{11}) *and all formulas of the form*

$$(T_{12}) \qquad (a \cup -a)$$

are theorems in $\mathfrak{T}$, *then* $\mathfrak{A}(\mathfrak{T})$ *is a Boolean algebra, and* (7), (8), (12), (17) *hold.*

By (T_{12}), and the last part of 10.4, we have

$$\|a\| \cup \dot{-}\|a\| = V$$

by (8) and (17). This proves that the pseudo-complement $-\|a\|$ of $\|a\|$ (see 10.5) is the complement of $\|a\|$ by I, 10.1, and I, 12.1. By II, 1.2, $\mathfrak{A}(\mathfrak{T})$ is a Boolean algebra.

We shall finish this section with the following simple applications of 10.4, 10.5, and 10.6:

10.7. *Suppose that all formulas of the form* (T_1)-(T_9) *are theorems in* $\mathfrak{T}$. *Then the theory* $\mathfrak{T}$ *is consistent if and only if the lattice* $\mathfrak{A}(\mathfrak{T})$ *is not degenerate, i.e. if it contains at least two elements.*

In fact, by the last part of 10.4, $\|a\| = V$ if and only if a is a theorem. Thus $\mathfrak{A}(\mathfrak{T})$ has an element different from V if and only if there exists a formula a which is not any theorem.

10.8. *Suppose that all the formulas* (T_1)-(T_6) *are theorems in* $\mathfrak{T}$. *If* a *and* β *are theorems in* $\mathfrak{T}$, *then* $(a \cap \beta)$ *is also a theorem. If* β *is a theorem in* $\mathfrak{T}$ *and* a *is any formula, then* $(a \cup \beta)$, $(\beta \cup a)$, *and* $(a \Rightarrow \beta)$ *are theorems in* $\mathfrak{T}$.

For if $\|a\| = \|\beta\| = V$, then also $\|(a \cap \beta)\| = \|a\| \cap \|\beta\| = V$ which proves the first part of 10.8. If $\|\beta\| = V$, then also

$$\|(a \cup \beta)\| = \|(\beta \cup a)\| = \|a\| \cup \|\beta\| = V,$$

$$\|(a \Rightarrow \beta)\| = \|a\| \Rightarrow \|\beta\| = \|a\| \Rightarrow V = V,$$

which proves the second part of 10.8.

Of course, theorem 10.8 can also be proved by purely logical means, using *modus ponens* and theorems (T_1)-(T_9) only.

10.9. *If all formulas* (T_1)-(T_{11}) *are theorems in* $\mathfrak{T}$, *and* a *is a theorem in* $\mathfrak{T}$, *then* $--a$ *is also a theorem in* $\mathfrak{T}$.

If all formulas (T_1)-(T_{12}) *are theorems in* $\mathfrak{T}$, *then a formula* a *is a theorem in* $\mathfrak{T}$ *if and only if* $--a$ *is a theorem in* $\mathfrak{T}$.

Under the hypothesis of the first part of 10.9, $\mathfrak{A}(\mathfrak{T})$ is a pseudo-Boolean algebra. Thus $\|a\| \leqslant --\|a\| = \|--a\|$ by I, 12.3, (29), and by (17). If a is a theorem, then $\|a\| = V$. Consequently $\|--a\| = V$, i.e. $--a$ is a theorem.

Under the hypothesis of the second part of 10.9, $\mathfrak{A}(\mathfrak{T})$ is a Boolean algebra. Thus $\|--a\| = --\|a\| = \|a\|$ by (17) and II, 1.3, (5). Consequently,

$\|a\| = V$ if and only if $\|--a\| = V$, i.e. a is a theorem if and only if $--a$ is a theorem.

10.10. *If all formulas* (T_1)-(T_{11}) *are theorems in* $\mathfrak{T}$, *then a formula* a *is refutable (irrefutable) if and only if* $\|a\| = \Lambda$ $(\|a\| \neq \Lambda)$.

a is refutable, i.e. $-a$ is a theorem, if and only if $\|-a\| = V$. By (17) and I, 12.3, (26), this is equivalent to $\|a\| = \Lambda$.

The remaining part of 10.10 follows from the part just proved.

10.11. *If all formulas* (T_1)-(T_{12}) *are theorems in* $\mathfrak{T}$, *then a formula* a *(a formula* $-a$*) is irrefutable if and only if the formula* $-a$ *(the formula* a*) is not a theorem in* $\mathfrak{T}$.

Since $\mathfrak{A}(\mathfrak{T})$ is a Boolean algebra, $\|a\| \neq \Lambda$ is equivalent to $-\|a\| \neq V$, i.e. to $\|-a\| \neq V$ which proves 10.11 by 10.4 (the last sentence) and 10.10. Similarly we prove the remaining part of 10.11.

§ 11. The Q-algebra of a formalized theory of the first order. In this section we shall examine a fixed formalized language $\mathfrak{L} = \{S, T, F\}$ of the first order and a fixed theory $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$. We shall always suppose that the set $C(\mathcal{A})$ of all theorems in $\mathfrak{T}$ is closed under all the rules of inference (r_1), (r_2), (r_3), (r_4), (r_5), (r_6) described in V, § 8. We adopt the notation from § 10.

11.1. *If all formulas of the form* (T_0) *and* (T_1) *are theorems in* $\mathfrak{T}$, *and* T_0 *is any set of terms such that* T_0 *contains infinitely many free individual variables, then for every formula* $\beta(x)$

$$\|\bigcup_\xi \beta(\xi)\| = \text{l. u. b.}_{\tau \in T_0} \|\beta(\tau)\|, \tag{1}$$

$$\|\bigcap_\xi \beta(\xi)\| = \text{g. l. b.}_{\tau \in T_0} \|\beta(\tau)\| \tag{2}$$

in the ordered set $\mathfrak{A}(\mathfrak{T})$ [1].

The formula

$$(\bigcup_\xi \beta(\xi) \Rightarrow \bigcup_\xi \beta(\xi))$$

is a theorem since it is of the form (T_0). By the rule (r_5) of elimination of an existential quantifier, the formula

$$(\beta(x) \Rightarrow \bigcup_\xi \beta(\xi)),$$

where x does not appear in $\beta(\xi)$, is also a theorem. Hence, by the rule (r_2) of substitution for free individual variables, the formula

$$(\beta(\tau) \Rightarrow \bigcup_\xi \beta(\xi))$$

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[1] Rasiowa [2], [4], and Henkin [2]. Cf. also Chandrasekharan [1].

is a theorem, i.e.

$$\|\beta(\tau)\| \leqslant \|\bigcup_\xi \beta(\xi)\| \quad \text{for every term } \tau \text{ in } T_0.$$

Suppose now that $\|a\|$ is another element in $\mathfrak{A}(\mathfrak{C})$ such that

$$\|\beta(\tau)\| \leqslant \|a\| \quad \text{for every term } \tau \text{ in } T_0.$$

Since a contains only a finite number of free individual variables and T_0 contains infinitely many free individual variables, there exists an individual variable x in T_0 such that x does not appear in a. Since $\|\beta(x)\| \leqslant \|a\|$, i.e.

$$(\beta(x) \Rightarrow a)$$

is a theorem, the formula

$$(\bigcup_\xi \beta(\xi) \Rightarrow a)$$

is a theorem by the rule (r_3) of introduction of an existential quantifier. Thus

$$\|\bigcup_\xi \beta(\xi)\| \leqslant \|a\|,$$

which proves (1).

The proof of (2) is by duality. It suffices to apply (r_6) and (r_4) instead of (r_5) and (r_3), respectively, and to write everywhere $\geqslant$ instead of $\leqslant$.

As a simple corollary from 11.1 and 10.3 we get the theorem

11.2. *If all formulas of the form* (T_1)-(T_7) *are theorems in* $\mathfrak{C}$, *then, for every formula* $\beta(x)$,

$$\text{(Q)} \qquad \|\bigcup_\xi \beta(\xi)\| = \bigcup_{\tau \in T} \|\beta(\tau)\|,$$

$$\|\bigcap_\xi \beta(\xi)\| = \bigcap_{\tau \in T} \|\beta(\tau)\|$$

in the lattice $\mathfrak{A}(\mathfrak{C})$.

In this book we shall mainly investigate the case in which either all formulas of the form (T_1)-(T_{12}) are theorems in $\mathfrak{C}$ (Chapter VIII) or all formulas of the form (T_1)-(T_{11}) are theorems in $\mathfrak{C}$ (Chapter X). Then $\mathfrak{A}(\mathfrak{C})$ is a Boolean algebra or a pseudo-Boolean algebra and the identities (Q) hold. We shall always conceive of $\mathfrak{A}(\mathfrak{C})$ as a generalized algebra

$$\text{(3)} \qquad \{\mathfrak{A}(\mathfrak{C}), \cup, \cap, \Rightarrow, -, \bigcup|\mathfrak{Q}, \bigcap|\mathfrak{Q}\},$$

where $\bigcup$ and $\bigcap$ are respectively the infinite join and meet. As the common domain $\mathfrak{Q}$ of $\bigcup$ and $\bigcap$ in $\mathfrak{A}(\mathfrak{C})$ we assume only the class of all indexed sets

$$\text{(4)} \qquad \{\|\beta(\tau)\|\}_{\tau \in T},$$

where $\beta(x)$ is any formula. In other words, $\bigcup$ and $\bigcap$ are restricted in $\mathfrak{A}(\mathfrak{C})$ only to the infinite joins and meets (Q), i.e. to infinite joins and

meets corresponding to quantifiers. Other infinite joins and meets existing in $\mathfrak{A}(\mathfrak{C})$ will not be taken into consideration.

The generalized algebra (3) will often be called the *quantifier algebra of the theory* $\mathfrak{C}$, or briefly: the Q-*algebra* of $\mathfrak{C}$.

11.3. *If all formulas of the form* (T_1)-(T_{11}) *are theorems in* $\mathfrak{C}$, *and formulas* α, α' *in* $\mathfrak{L}$ *are congruent, then*

$$\|\alpha\| = \|\alpha'\| . \tag{5}$$

Consequently, the equation

$$\boldsymbol{h}(|\alpha|) = \|\alpha\| \tag{6}$$

defines a mapping $\boldsymbol{h}$: $\overline{F} \to \mathfrak{A}(\mathfrak{C})$. *The mapping* $\boldsymbol{h}$ *is a homomorphism of the* Q-*algebra* $\overline{F}$ *into the* Q-*algebra* $\mathfrak{A}(\mathfrak{C})$.

The proof of (5) is by induction with respect to the length of α. If α is an elementary formula and $\alpha \sim \alpha'$, then α' is equal to α, and, consequently, (5) holds. If α is a disjunction $(\beta \cup \gamma)$ and $\alpha \sim \alpha'$, then α' is also a disjunction $(\beta' \cup \gamma')$ and $\beta \sim \beta'$, $\gamma \sim \gamma'$. If (5) holds for β and γ, it holds also for α, since

$$\|\alpha\| = \|\beta\| \cup \|\gamma\| = \|\beta'\| \cup \|\gamma'\| = \|\alpha'\| .$$

Similarly we verify the case in which α is the conjunction or implication of two formulas or a negation of a formula. If, for a formula $\beta(x)$, α is of the form $\bigcup_\xi \beta(\xi)$ and $\alpha \sim \alpha'$, then, for a formula $\beta'(x) \sim \beta(x)$, α' is of the form $\bigcup_\xi \beta'(\xi)$. If (5) holds for $\beta(x)$ it also holds for α since

$$\|\alpha\| = \|\bigcup_\xi \beta(\xi)\| = \bigcup_{\tau \in T} \|\beta(\tau)\| = \bigcup_{\tau \in T} \|\beta'(\tau)\| = \|\bigcup_\xi \beta'(\xi)\| = \|\alpha'\| .$$

Similarly we verify the case in which α is of the form $\bigcap_\xi \beta(\xi)$.

The proof that (6) is a homomorphism can be obtained by a simple verification.

Homomorphisms and isomorphisms from the Q-algebra $\mathfrak{A}(\mathfrak{C})$ into another generalized algebra $\boldsymbol{B}$ similar to (3) will be called Q-*homomorphisms* and Q-*isomorphism*, respectively, to indicate that they also preserve the infinite operations (Q) corresponding to quantifiers. In applications $\boldsymbol{B}$ will always be a lattice, and $\bigcup$, $\bigcap$ in $\boldsymbol{B}$ will be the infinite join and meet, respectively. Thus this terminology will be consistent with that introduced in I, § 7, p. 43.

The following two theorems are a supplement to 10.8.

11.4. *Suppose that all formulas of the form* (T_1)-(T_9) *are theorems in* $\mathfrak{C}$. *If a formula* $\beta(x)$ *is a theorem in* $\mathfrak{C}$ *and* ξ *does not appear in* $\beta(x)$, *then also the formula* $\bigcup_\xi \beta(\xi)$ *is a theorem in* $\mathfrak{C}$.

In other words:

The set of all theorems in $\mathfrak{C}$ is closed under the rule of inference

$$\frac{\beta(x)}{\bigcap_{\xi} \beta(\xi)}\,.$$

This rule of inference is called the *rule of generalization.*

To prove 11.4, let us observe that if $\beta(x)$ is a theorem, then $\beta(\tau)$ is also a theorem for every term τ, i.e. $\|\beta(\tau)\| = V$ for every $\tau \in T$. Hence

$$\|\bigcap_{\xi} \beta(\xi)\| = \bigcap_{\tau \in T} \|\beta(\tau)\| = V$$

by (2), which proves that $\bigcap_{\xi} \beta(\xi)$ is a theorem.

Observe that the set of all theorems in $\mathfrak{C}$ is also closed under the converse rule of elimination of an existential quantifier

$$\frac{\bigcap_{\xi} \beta(\xi)}{\beta(x)}\,.$$

In fact, the formula

$$(\bigcap_{\xi} \beta(\xi) \Rightarrow \bigcap_{\xi} \beta(\xi))$$

is a theorem, since it is of the form (T_0) (see 10.4). By the rule (r_6) of elimination of a universal quantifier (V, § 8), the formula

$$(\bigcap_{\xi} \beta(\xi) \Rightarrow \beta(x))$$

is also a theorem. Hence, by *modus ponens*, we infer that $\beta(x)$ is a theorem.

We recall that a formula is said to be *closed* if it does not contain any free individual variable. A formula $\overline{\alpha}$ is said to be the *closure* of a formula α provided it is of the form

$$\bigcap_{\xi_1} \bigcap_{\xi_2} \dots \bigcap_{\xi_n} \alpha(x_1/\xi_1, x_2/\xi_2, \dots, x_n/\xi_n)\,,$$

where $x_1, \dots, x_n$ are all free individual variables appearing in α, i.e. it is a closed formula obtained from α by a multiple application of the rule of generalization.

The following theorem follows immediately from the above considerations:

11.5. *Suppose that all formulas of the form* (T_1)-(T_9) *are theorems in* $\mathfrak{C}$. *A formula* α *is a theorem in* $\mathfrak{C}$ *if and only if its closure* $\overline{\alpha}$ *is a theorem in* $\mathfrak{C}$.

It follows from the first identity (Q) in 11.2 that

$$\|\alpha(x)\| \leqslant \|\bigcup_{\xi} \alpha(\xi)\|\,.$$

Consequently, if the element on the left-hand side is not the zero of $\mathfrak{A}(\mathfrak{C})$, then the element on the right-hand side is not the zero of $\mathfrak{A}(\mathfrak{C})$. By 10.10 this remark can also be formulated in the following form:

11.6. *Suppose that all formulas of the form* (T_1)-(T_{11}) *are theorems in* $\mathfrak{C}$. *If a formula* $\alpha(x)$ *is irrefutable, then the formula* $\bigcup_\xi \alpha(\xi)$ *is also irrefutable.*

A little more generally: *if a formula* $\alpha(x_1, \ldots, x_n)$ *is irrefutable, then the formula* $\bigcup_{\xi_1} \ldots \bigcup_{\xi_n} \alpha(\xi_1, \ldots, \xi_n)$ *is also irrefutable.*

Theorem 11.7 below follows directly from (1) and (2) (where $T_0 = T$) by induction on the length of formulas:

11.7. *If all formulas of the form* (T_0), (T_1) *are theorems in* $\mathfrak{C}$ *(in particular, if all formulas of the form* (T_1)-(T_9) *are theorems in* $\mathfrak{C}$*) and* β, γ *are congruent formulas, then*

$$\|\beta\| = \|\gamma\| . \tag{7}$$

Consequently, if β *and* γ *are closures of the same formula* α, *then* (7) *holds.*

CHAPTER VII

CLASSICAL PROPOSITIONAL CALCULI

§ 1. Preliminaries. In this Chapter we shall deal with a fixed formalized language $\mathfrak{L}_0 = \{A_0, F_0\}$ of zero order. The alphabet A_0 of $\mathfrak{L}_0$ and the set F_0 of all formulas in $\mathfrak{L}_0$ were described in V § 6. The letter V_0 will always denote the set of all propositional variables $a, b, c, \ldots$ in $\mathfrak{L}_0$. By a formula we shall always mean, in this Chapter, a formula from $\mathfrak{L}_0$, unless the contrary is explicitly stated.

The letter C will exclusively denote the classical (or: two-valued) consequence operation defined in V, § 11, p. 188.

In this Chapter we shall examine, in a detailed way, the classical (i.e. two-valued) propositional calculus based on $\mathfrak{L}_0$, i.e. the deductive system

$$\mathfrak{S}_0 = \{\mathfrak{L}_0, C\}$$

defined in V, § 11.

We know the reason for introducing the deductive system $\mathfrak{S}_0$: it enables us to make precise the notion of propositional tautology, i.e. a sentence true by its connective syntax only. It can also be used to investigate when a sentence a can be recognized as a logical consequence of certain sentences $a_1, a_2, \ldots$ by virtue of the connective syntax of $a, a_1, a_2, \ldots$ only.

A finite sequence $a_1, \ldots, a_n$ is said to be a *formal proof of a formula* a *in* $\mathfrak{S}_0$ *from a set* S of formulas provided a_n is the formula a, and for every $j \leqslant n$, either a_j is a logical axiom or it is a formula in S, or it is derivable from some formulas a_{j_1}, a_{j_2} with $j_1, j_2 < j$ by means of *modus ponens*. A formula a is said to be *derivable from* S *in* $\mathfrak{S}_0$ if there exists a formal proof of a from S in $\mathfrak{S}_0$. According to V § 11 the symbol $C(S)$ or, more precisely, $C_{\mathfrak{L}_0}(S)$, will denote the set of all formulas derivable from S in $\mathfrak{S}_0$. A formula a is said to be *derivable in* $\mathfrak{S}_0$ provided it is in $C(0)$.

Sometimes in this Chapter we shall consider formalized theories $\mathfrak{T} = \{\mathfrak{L}_0, C, \mathcal{A}\}$ of zero order where $\mathcal{A}$ is a set of formulas (mathematical axioms of the theory $\mathfrak{T}$). Formulas in $C(\mathcal{A})$ are called *theorems of* $\mathfrak{T}$. According to the convention assumed on p. 187, the formalized theory $\{\mathfrak{L}_0, C, 0\}$ with the empty set of mathematical axioms will be identified with the deductive system $\mathfrak{S}_0 = \{\mathfrak{L}_0, C\}$.

By a *theory* we shall always mean, in this Chapter, a theory of zero order based on the language $\mathfrak{L}_0$, unless the contrary is explicitly stated.

We recall (see VI, § 10) that the algebra $\mathfrak{A}(\mathfrak{T})$ of a theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ is the algebra $F_0/\approx$ where $\approx$ is the congruence relation in $\{F_0, \cup, \cap, \Rightarrow, -\}$ defined as follows:

(1) $\alpha \approx \beta$ if and only if both $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are theorems in $\mathfrak{T}$.

In particular, $\mathfrak{A}(\mathfrak{S}_0)$ is the algebra $F_0/\approx$ where $\approx$ is the congruence in $\{F_0, \cup, \cap, \Rightarrow, -\}$ defined as follows:

(2) $\alpha \approx \beta$ if and only if both the formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are derivable in $\mathfrak{S}_0$.

From VI, 10.4-10.7, 10.10 follows

1.1. *The algebra $\mathfrak{A}(\mathfrak{T})$ is a Boolean algebra. Moreover, for any formulas α, β*

$$\begin{gathered} \|\alpha\| \cup \|\beta\| = \|(\alpha \cup \beta)\|, \quad \|\alpha\| \cap \|\beta\| = \|(\alpha \cap \beta)\|, \\ \|\alpha\| \Rightarrow \|\beta\| = \|(\alpha \Rightarrow \beta)\|, \quad -\|\alpha\| = \|-\alpha\|. \end{gathered} \tag{3}$$

The relation

$$\|\alpha\| \leqslant \|\beta\| \tag{4}$$

holds if and only if $(\alpha \Rightarrow \beta)$ is a theorem in $\mathfrak{T}$. A formula is a theorem in $\mathfrak{T}$ if and only if the element $\|\alpha\|$ is the unit V of $\mathfrak{A}(\mathfrak{T})$. A formula α is irrefutable in $\mathfrak{T}$ if and only if $\|\alpha\| \neq \Lambda$. The theory $\mathfrak{T}$ is consistent if and only if the Boolean algebra $\mathfrak{A}(\mathfrak{T})$ is non-degenerate.

Let $\{A, \cup, \cap, \Rightarrow, -\}$ be an abstract algebra with three binary operations $\cup, \cap, \Rightarrow$ and one unary operation $-$. As was explained in VI, § 2, every formula α in F_0 uniquely determines a mapping

$$\alpha_A : A^{V_0} \to A \tag{5}$$

where V_0 is the set of all propositional variables in $\mathfrak{S}_0$. Every element $v = \{v_a\}_{a \in V_0}$ of the Cartesian product A^{V_0}, i.e. every mapping

$$v : V_0 \to A \tag{6}$$

is called a *valuation* in A.

In this Chapter we shall always suppose that

$$\{A, \cup, \cap, \Rightarrow, -\} \text{ is a non-degenerate Boolean algebra.}$$

In particular, we shall sometimes assume as $\boldsymbol{A}$ the algebra $\mathfrak{A}(\mathfrak{T})$ of a formalized consistent theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$. Then the valuation

$$v^0 = \{\|a\|\}_{a \in V_0} \in \mathfrak{A}(\mathfrak{T})^{V_0} \tag{7}$$

is called the *canonical valuation for* $\mathfrak{T}$ (see VI, § 2). From VI, 2.6 follows

1.2. *If v^0 is the canonical valuation for $\mathfrak{T}$, then for every formula a,*

$$a_{\mathfrak{A}(\mathfrak{T})}(v^0) = \|a\| \in \mathfrak{A}(\mathfrak{T}) . \tag{8}$$

A valuation $v \in \boldsymbol{A}^{V_0}$ in a non-degenerate Boolean algebra $\boldsymbol{A}$ is said to be a *model for a set* S of formulas if

$$a_A(v) = \mathit{V} \quad \text{for every formula } a \text{ in } S.$$

In particular (when S consists of one formula only), the valuation v is said to be a *model for a formula* a if $a_A(v) = \mathit{V}$. A valuation v is said to be a *model for a theory* $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ provided v is a model for the set $\mathcal{A}$ of axioms of $\mathfrak{T}$.

A formula a is said to be *valid in a Boolean algebra* $\boldsymbol{A}$ if every valuation $v \in \boldsymbol{A}^{V_0}$ is a model for a in $\boldsymbol{A}$, i.e. if the mapping a_A is identically equal to the unit element V of $\boldsymbol{A}$. We then write

$$a_A \equiv \mathit{V} . \tag{9}$$

A formula is said to be *valid* if it is valid in every Boolean algebra.

1.3. *Every formula derivable in $\mathfrak{S}_0$ is valid.*

By a simple calculation we verify that each of the logical axioms (T_1)-(T_{12}) is valid. As an example, we shall verify that every formula of the form (T_2) is valid. For any valuation v in a Boolean algebra $\boldsymbol{A}$, we have

$$\begin{aligned}(a \Rightarrow (a \cup \beta))_A(v) &= a_A(v) \Rightarrow \big(a_A(v) \cup \beta_A(v)\big) \\ &= -a_A(v) \cup a_A(v) \cup \beta_A(v) = \mathit{V} \cup \beta_A(v) = \mathit{V}\end{aligned}$$

by VI, § 2, (10).

If $a_A \equiv \mathit{V}$ and $(a \Rightarrow \beta)_A \equiv \mathit{V}$, then also $\beta_A \equiv \mathit{V}$ since, for every valuation v,

$$\beta_A(v) = \mathit{V} \Rightarrow \beta_A(v) = a_A(v) \Rightarrow \beta_A(v) = (a \Rightarrow \beta)_A(v) = \mathit{V}$$

by I, § 12, (8) and VI, § 2, (10). Thus the class of all valid formulas is closed under *modus ponens*. Since it contains all the logical axioms (T_1)-(T_{12}), it contains all derivable formulas. This proves 1.3.

A theorem converse to 1.3 will be proved in § 2. Now we shall mention briefly the case of the two-element Boolean algebra $\boldsymbol{A}_0$ which will play a special role in our investigation. By V, § 6 (see p. 168), a formula

α in $\mathcal{S}_0$ is a tautology if and only if it is valid in the two-element Boolean algebra $\boldsymbol{A}_0$, i.e. if

$$\alpha_{\boldsymbol{A}_0} \equiv V \,. \tag{10}$$

Any model for a set of formulas (or: for a formula, or: for a theory) in the two-element Boolean algebra $\boldsymbol{A}_0$ is called a *semantic model.*

The following statement results from 1.3:

1.4. *The propositional calculus $\mathcal{S}_0$ is consistent.*

For if not, then there would exist a formula α in $\mathfrak{L}_0$ such that both formulas α and $-\alpha$ would be derivable in $\mathcal{S}_0$. By 1.3 they are tautologies. Let v be a valuation in the two-element Boolean algebra $\boldsymbol{A}_0$. Since α is a tautology, we have $\alpha_{\boldsymbol{A}_0}(v) = V$. Hence $\Lambda = -\alpha_{\boldsymbol{A}_0}(v) = (-\alpha)_{\boldsymbol{A}_0}(v)$. Since $-\alpha$ is a tautology, we infer that $(-\alpha)_{\boldsymbol{A}_0}(v) = V$. Thus $\Lambda = V$ which does not hold in $\boldsymbol{A}_0$.

§ 2. The completeness of the propositional calculi. Theorems 1.3 and 1.4 imply that every derivable formula is a tautology. This statement was proved directly in theorem V, 11.1. The converse statement is also true. In fact, we can prove the following theorem, called the *completeness theorem for the* (classical) *propositional calculi*:

2.1. *A formula α in $\mathcal{S}_0 = \{\mathfrak{L}_0, C\}$ is a propositional tautology if and only if it is derivable in $\mathcal{S}_0$* [1].

Theorem 2.1 is a part of the following theorem:

2.2. *For every formula α in $\mathcal{S}_0$ the following five conditions are equivalent*:

(i) α *is derivable in* $\mathcal{S}_0$;

(ii) α *is valid*;

(iii) α *is valid in every field of sets* $\boldsymbol{B}$;

(iv) $\alpha_{\mathfrak{A}(\mathcal{S}_0)}(v^0) = V$, *where* v^0 *is the canonical valuation for* $\mathcal{S}_0$;

(v) α *is a tautology.*

Condition (i) implies (ii) by 1.3. It is obvious that (ii) implies (iii) and (iv). By 1.2 and 1.1, (iv) implies (i). (iii) implies (v) since the two-element Boolean algebra can be interpreted as the field of all subsets of a one-element space.

To complete the proof it suffices to show that (v) implies (i).

Suppose that (i) is not true, i.e. α is not derivable in $\mathcal{S}_0$. By 1.1, $\|\alpha\| \neq V$. By I, 8.5, there exists a maximal ideal Δ in $\mathfrak{A}(\mathcal{S}_0)$ such that

---

[1] This theorem is due to Łukasiewicz [2]. For another proof see e.g. Hilbert and Ackermann [1]. The first proof by the Boolean method was known to the authors (see Rasiowa and Sikorski [1], p. 200). See also Łoś [2].

$\|a\| \in \Delta$. Let h be the natural homomorphism from $\mathfrak{A}(\mathfrak{S}_0)$ onto the two-element Boolean algebra $\boldsymbol{A}_0 = \mathfrak{A}(\mathfrak{S}_0)/\Delta$ (see II, 5.2). The valuation $v = hv^0$, where v^0 is the canonical valuation in $\mathfrak{A}(\mathfrak{S}_0)$, i.e. the valuation $v = \{h(\|a\|)\}_{a \in V_0}$, has the property

$$a_{\boldsymbol{A}_0}(v) = h\big(a_{\mathfrak{A}(\mathfrak{S}_0)}(v^0)\big) = h(\|a\|) = 1_{\boldsymbol{A}_0}$$

by 1.2 and VI, 2.4. Thus a is not any tautology, i.e. (v) is not true.

It follows from 2.1 that in order to prove that a formula in F_0 is derivable in $\mathfrak{S}_0$ it suffices to show that it is a tautology, i.e. $a_{\boldsymbol{A}_0} \equiv 1$. Thus we have a general decision procedure for determining whether or not a formula in $\mathfrak{S}_0$ is derivable. This method consists in verifying whether $a_{\boldsymbol{A}_0}$ is identically equal to 1. This method is the truth-table method described in V, § 5.

§ 3. Examples of propositional tautologies. By 2.1 the meaning of the phrases "propositional tautology" and "formula derivable in $\mathfrak{S}_0$" is the same. In practice, we shall use the first more frequently.

3.1. *The following formulas, where α, β, γ are any formulas in $\mathfrak{L}_0$, are tautologies*:

(T_0) $(\alpha \Rightarrow \alpha)$,

(T_{13}) $(\alpha \Rightarrow (\beta \Rightarrow \alpha))$,

(T_{14}) $(\alpha \Rightarrow (\beta \Rightarrow (\alpha \cap \beta)))$,

(T_{15}) $((\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow (\beta \Rightarrow (\alpha \Rightarrow \gamma)))$,

(T_{16}) $((\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma)))$,

(T_{17}) $(\alpha \Rightarrow --\alpha)$,

(T_{18}) $(--\alpha \Rightarrow \alpha)$,

(T_{19}) $-(\alpha \cap -\alpha)$,

(T_{20}) $((\alpha \Rightarrow \beta) \Rightarrow (-\alpha \cup \beta))$,

(T_{21}) $((-\alpha \cup \beta) \Rightarrow (\alpha \Rightarrow \beta))$,

(T_{22}) $(-(\alpha \cup \beta) \Rightarrow (-\alpha \cap -\beta))$,

(T_{23}) $((-\alpha \cap -\beta) \Rightarrow -(\alpha \cup \beta))$,

(T_{24}) $(-(\alpha \cap \beta) \Rightarrow (-\alpha \cup -\beta))$,

(T_{25}) $((-\alpha \cup -\beta) \Rightarrow -(\alpha \cap \beta))$,

(T_{26}) $((\alpha \Rightarrow \beta) \Rightarrow (-\beta \Rightarrow -\alpha))$,

(T_{27}) $((\alpha \Rightarrow -\beta) \Rightarrow (\beta \Rightarrow -\alpha))$,

(T_{28}) $((-\alpha \Rightarrow \beta) \Rightarrow (-\beta \Rightarrow \alpha))$,

(T_{29}) $((-\alpha \Rightarrow -\beta) \Rightarrow (\beta \Rightarrow \alpha))$,

(T_{30}) $(((\alpha \Rightarrow \beta) \Rightarrow \alpha) \Rightarrow \alpha)$.

The tautology (T_{13}) is called the *simplification law*. The tautology (T_{15}) is the *law of the interchange of premises*. The formula (T_{16}) is known as the *Frege law*. Formulas (T_{17}) and (T_{18}) are the *laws of double negation*. Formula (T_{19}) is known as the *denial of contradiction*. Formulas (T_{22})-(T_{25}) are *De Morgan laws*. Formulas (T_{26})-(T_{29}) are the *contraposition laws*. Formula (T_{30}) is known as the *Peirce law*.

Theorem 3.1 can be proved by the truth-table method. The easy verification is left to the reader. We can also verify the validity of each of the formulas (T_0), (T_{13})-(T_{30}) in any Boolean algebra, using a corresponding law from the theory of Boolean algebras. We indicate below the number of statements in Chapters I and II from which the validity of the formulas follows:

(T_0) I, § 12, (6); (T_{13}) I, § 12, (13); (T_{14}) I, § 12, (22); (T_{15}) I, § 12, (18); (T_{16}) I, § 12, (23); (T_{17}) and (T_{18}) II, § 1, (5); (T_{19}) I, § 12, (28); (T_{20}) and (T_{21}) II, § 1, (l_6); (T_{22}) and (T_{23}) II, § 1, (9), (5); (T_{24}) and (T_{25}) II, § 1, (8); (T_{26}) and (T_{29}) II, § 1, (6); (T_{27}) I, § 12, (35); (T_{28}) II, § 1, (7); (T_{30}) II, § 1, (12).

By 2.1 all the formulas (T_0), (T_{13})-(T_{30}) have formal proofs in S_0, i.e. they can be derived from (T_1)-(T_{12}) by means of *modus ponens*. We shall not quote here any formal proofs for them. In § 6 we shall explain a general method of finding a formal proof for any tautology.

§ 4. The algebra of the two-valued propositional calculus. The following theorem characterizes the algebra $\mathfrak{A}(S_0)$ in the class of all Boolean algebras:

4.1. *$\mathfrak{A}(S_0)$ is a free algebra in the class of all Boolean algebras, the set of all elements $\|a\|$ for a in V_0 being the set of free generators in $\mathfrak{A}(S_0)$.*

It follows directly from the structure of formulas in S_0 that all elements $\|a\|$, where a is any propositional variable, generate $\mathfrak{A}(S_0)$.

Let f be an arbitrary mapping defined on the set of all elements $\|a\|$, where a is in V_0, with values in a Boolean algebra $\boldsymbol{A}$. The mapping f induces a mapping v: $V_0 \to \boldsymbol{A}$, viz. v is defined as follows:

$$v_a = f(\|a\|) \quad \text{for any } a \text{ in } V_0 .$$

The mapping $v = \{v_a\}_{a \in V_0}$ is a valuation in the algebra $\boldsymbol{A}$. By theorem VI, 2.1, v can be extended to a homomorphism v_A from F_0 into $\boldsymbol{A}$. Moreover, by VI, § 2, (11),

$$v_A(\gamma) = \gamma_A(v)$$

for every formula γ.

Let α, β be any formulas in S_0 such that $\|\alpha\| = \|\beta\|$. Then the formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are derivable in S_0. By 1.3 they are valid. In particular, they are valid in the algebra $\boldsymbol{A}$.

Consequently, for the valuation v,

$$v_A(\alpha) \Rightarrow v_A(\beta) = \alpha_A(v) \Rightarrow \beta_A(v) = (\alpha \Rightarrow \beta)_A(v) = V ,$$
$$v_A(\beta) \Rightarrow v_A(\alpha) = \beta_A(v) \Rightarrow \alpha_A(v) = (\beta \Rightarrow \alpha)_A(v) = V .$$

Thus the condition $\|\alpha\| = \|\beta\|$ implies that $v_A(\alpha) = v_A(\beta)$ (see I, 12.2, (5)). Consequently, the equation

$$h(\|\alpha\|) = v_A(\alpha)$$

defines a mapping h: $\mathfrak{A}(\mathcal{S}_0) \to \boldsymbol{A}$. The mapping h is an extension of the mapping f because

$$h(\|\alpha\|) = v_A(\alpha) = f(\|\alpha\|)$$

for every propositional variable a. Since v_A is a homomorphism from F_0 into $\boldsymbol{A}$, the mapping h is a homomorphism from $\mathfrak{A}(\mathcal{S}_0) = F_0/\approx$ into $\boldsymbol{A}$, by I, 4.9.

The following simple remark will be useful in the proof of the next theorem on the representation of $\mathfrak{A}(\mathcal{S}_0)$:

4.2. *For any distinct propositional variables a, b*

$$\|a\| \neq \|b\| .$$

This follows from the fact that $(a \Rightarrow b)$ is not a tautology.

Let $\mathfrak{D}$ be the Cartesian product,

$$\mathfrak{D} = U^{V_0},$$

where U is the set composed only of the integers 0 and 1. In other words, $\mathfrak{D}$ is the set of all $u = \{u_a\}_{a \in V_0}$ such that $u_a = 0$ or $u_a = 1$ for every a in V_0.

According to II, § 11, let D_a be the set of all $u \in \mathfrak{D}$ such that $u_a = 1$. Let $\boldsymbol{D}_0$ be the class of all sets D_a (a in V_0) and let $\boldsymbol{D}$ be the field of subsets of $\mathfrak{D}$ generated by $\boldsymbol{D}_0$. The set $\mathfrak{D}$ will be considered as a topological space, $\boldsymbol{D}_0$ being assumed as a subbasis. By the definition in II, § 11, $\mathfrak{D}$ is the Cantor discontinuum over V_0. It is known (see II, 11.2) that the field $\boldsymbol{D}$ consists of all both open and closed subsets of $\mathfrak{D}$ and is equivalent to the Stone field $\boldsymbol{P}(\boldsymbol{D})$ of $\boldsymbol{D}$.

From the above remark and from 4.1, 4.2 and I, 4.6, II, 11.1, we obtain the following representation theorem for $\mathfrak{A}(\mathcal{S}_0)$:

4.3. *The algebra $\mathfrak{A}(\mathcal{S}_0)$ of the two-valued propositional calculus $\mathcal{S}_0$ is isomorphic to the field $\boldsymbol{D}$ of all both open and closed subsets of the Cantor discontinuum over V_0.*

The Stone field $\boldsymbol{P}(\mathfrak{A}(\mathcal{S}_0))$ is equivalent to the field $\boldsymbol{D}$.

§ 5. Normal forms. A formula β is said to be in the *disjunctive normal form* if it is of the form

$$\bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} \beta_{ij}\,, \tag{1}$$

where β_{ij} is either a propositional variable or a negation of a propositional variable ($i = 1, \ldots, m$, $j = 1, \ldots, n_i$).

A formula β is said to be a *disjunctive normal form of a formula* α provided β is in the disjunctive normal form and both the implications $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are propositional tautologies.

A formula γ is said to be in the *conjunctive normal form* if it is of the form

$$\bigcap_{i=1}^{m} \bigcup_{j=1}^{n_i} \gamma_{ij}\,, \tag{2}$$

where γ_{ij} is either a propositional variable or a negation of a propositional variable ($i = 1, \ldots, m$, $j = 1, \ldots, n_i$).

A formula γ is said to be a *conjunctive normal form of a formula* α provided γ is in the conjunctive normal form and both the implications $(\alpha \Rightarrow \gamma)$ and $(\gamma \Rightarrow \alpha)$ are propositional tautologies.

5.1. *Every formula α has a disjunctive normal form and a conjunctive normal form.*

Consider the Boolean algebra $\mathfrak{A}(\mathcal{S}_0)$ of the two-valued propositional calculus $\mathcal{S}_0 = \{\mathfrak{L}_0, \mathfrak{C}\}$. By 4.1, the set of all elements $\|a\|$, where a is any propositional variable, generates $\mathfrak{A}(\mathcal{S}_0)$. By II 2.1, every element $\|a\| \in \mathfrak{A}(\mathcal{S}_0)$ is of the form

$$\|a\| = \bigcup_{i=1}^{m} \bigcap_{j=1}^{n_i} \|\beta_{ij}\|\,,$$

where β_{ij} is either a propositional variable or the negation of a propositional variable. Hence it follows that formula (1) is a disjunctive normal form of α.

The proof of the remaining part of 11.1 is analogous.

Observe that for any given formula α it is very easy to find a disjunctive normal form for α and a conjunctive normal form for α, by means of the De Morgan laws, the distributive laws and the identity II, § 1, (l_6) expressing $\Rightarrow$ by $\cup$ and $-$. We explain this using the example of the formula

$$(a \cap -((a \Rightarrow b) \cap -c))\,. \tag{3}$$

In $\mathfrak{A}(\mathcal{S}_0)$ we have

$$\begin{aligned} \|(a \cap -((a \Rightarrow b) \cap -c))\| &= \|a\| \cap -((-\|a\| \cup \|b\|) \cap -\|c\|) \\ &= \|a\| \cap ((\|a\| \cap \|-b\|) \cup \|c\|) = (\|a\| \cap \|-b\|) \cup (\|a\| \cap \|c\|) \\ &= \|a\| \cap (\|-b\| \cup \|a\|) \cap (\|a\| \cup \|c\|) \cap (\|-b\| \cup \|c\|)\,. \end{aligned}$$

Thus the formula

$$((a \cap -b) \cup (a \cap c))$$

is a disjunctive normal form of (3), and the formula

$$(((a \cap (-b \cup a)) \cap (a \cup c)) \cap (-b \cup c))$$

is a conjunctive normal form of (3).

If we know a conjunctive normal form (2) of a formula α, then we can easily verify whether α is a propositional tautology or not. In fact, then α is a propositional tautology if and only if (2) is a propositional tautology. On the other hand, (2) is a propositional tautology if and only if each of the formulas

$$\bigcup_{j=1}^{n_i} \gamma_{ij} \quad (i = 1, \ldots, m)$$

is a propositional tautology. But it is easy to verify that a disjunction $\bigcup_{j=1}^{n} \gamma_j$, where γ_j $(j = 1, \ldots, n)$ is either a propositional variable or a negation of a propositional variable, is a propositional tautology if and only if there is a propositional variable a such that both a and $-a$ occur in the sequence $\gamma_1, \ldots, \gamma_n$.

§ 6. Diagrams of formulas. In this section we shall quote another method of verification of whether a given formula is a propositional tautology. This method consists in the decomposition of formulas into parts of which it is built. This method can be considered as another formalization of the method quoted at the end of § 5.

Let $\mathfrak{L}_0$ be a language of zero order.

The letter Γ (with indices, if necessary) will exclusively denote finite sequences

(1) $$\alpha_1, \ldots, \alpha_m$$

of formulas in $\mathfrak{L}_0$. The case of the empty sequence (i.e. the case of $m = 0$) is admitted.

If Γ is a sequence (1), Γ' is a sequence

$$\beta_1, \ldots, \beta_n$$

and α, β are any formulas, then Γ, α, Γ' and $\Gamma, \alpha, \beta, \Gamma'$ will denote the sequences

$$\alpha_1, \ldots, \alpha_m, \alpha, \beta_1, \ldots, \beta_n$$

and

$$\alpha_1, \ldots, \alpha_m, \alpha, \beta, \beta_1, \ldots, \beta_n,$$

respectively.

A formula in $\mathfrak{L}_0$ is said to be *indecomposable* if it is either a propositional variable or the negation of a propositional variable.

A sequence (1) is said to be *indecomposable* provided it is formed only of indecomposable formulas.

A sequence (1) is said to be *fundamental* if it simultaneously contains a formula a and its negation $-a$.

By a *schema* we shall understand a pair $\{\Gamma, \Gamma^0\}$ or a triple $\{\Gamma, \Gamma^0, \Gamma^1\}$ (of non-empty sequences of formulas) usually written in the form

(2) $$\frac{\Gamma}{\Gamma_0}$$

or

(3) $$\frac{\Gamma}{\Gamma^0; \Gamma^1},$$

respectively. Γ is called the *conclusion* of the schema, and Γ^0, Γ^1 are called its *premises*. If the schema is of the form (3), then Γ^0 and Γ^1 are called the *left* and the *right premise*, respectively.

In the sequel we shall consider only the following schemas, where Γ' always denotes an indecomposable sequence (which may be empty):

(D) $$\frac{\Gamma', (a \cup \beta), \Gamma''}{\Gamma', a, \beta, \Gamma''},$$

(−D) $$\frac{\Gamma', -(a \cup \beta), \Gamma''}{\Gamma', -a, \Gamma''; \Gamma', -\beta, \Gamma''},$$

(C) $$\frac{\Gamma', (a \cap \beta), \Gamma''}{\Gamma', a, \Gamma''; \Gamma', \beta, \Gamma''},$$

(−C) $$\frac{\Gamma', -(a \cap \beta), \Gamma''}{\Gamma', -a, -\beta, \Gamma''},$$

(I) $$\frac{\Gamma', (a \Rightarrow \beta), \Gamma''}{\Gamma', -a, \beta, \Gamma''},$$

(−I) $$\frac{\Gamma', -(a \Rightarrow \beta), \Gamma''}{\Gamma', a, \Gamma''; \Gamma', -\beta, \Gamma''},$$

(−N) $$\frac{\Gamma', --a, \Gamma''}{\Gamma', a, \Gamma''}.$$

Observe that the schemas

(4) (D), (−C), (I), (−N)

are of type (2), i.e. they have only one premise. The schemas

(5) (−D), (C), (−I)

are of type (3), i.e. they have two premises. If a sequence Γ of formulas is not indecomposable, then Γ is the conclusion of one of the schemas (4) or (5). The hypothesis that Γ' denotes exclusively an indecomposable sequence implies that Γ is then the conclusion of exactly one of the schemas (4), (5).

The letters i, j will denote exclusively finite sequences

$$i_1, \ldots, i_n \tag{6}$$

of integers 0 and 1. We shall write $j \leqslant i$ if j is an initial (proper or improper) segment of i. If i is the sequence (6), then $i, 0$ and $i, 1$ will denote the sequences

$$i_1, \ldots, i_n, 0 \quad \text{and} \quad i_1, \ldots, i_n, 1$$

respectively. The empty sequence (the case of $n = 0$ in (6)) is admitted and will be denoted by O. By definition, $O \leqslant i$ for every i.

By the *diagram* of a formula α_0 in $\mathfrak{L}_0$ we shall mean any mapping which, with certain sequences i, associates non-empty finite sequences Γ_i of formulas, and which is defined by induction as follows:

1) Γ_O is the sequence formed only of the formula α_0.

2) If Γ_i is defined but is either fundamental or indecomposable, then $\Gamma_{i,0}$ and $\Gamma_{i,1}$ are not defined.

3) If Γ_i is defined and is neither fundamental nor indecomposable, then Γ_i is the conclusion of exactly one of the schemas (4), (5). If Γ_i is the conclusion of a schema (4), then $\Gamma_{i,0}$ is the only premise of this schema and $\Gamma_{i,1}$ is not defined. If Γ_i is the conclusion of a schema (5), then $\Gamma_{i,0}$ and $\Gamma_{i,1}$ are the left and the right premises of this schema.

4. If Γ_i is not defined, then $\Gamma_{i,0}$ and $\Gamma_{i,1}$ are not defined.

Note that the diagram $\{\Gamma_i\}$ of α is uniquely determined by the formula α. The diagram $\{\Gamma_i\}$ of any formula is always finite, i.e. Γ_i is defined only for a finite number of sequences i.

Γ_i is said to be an *end sequence* of the diagram of α if Γ_i is fundamental or indecomposable, i.e. if $\Gamma_{i,0}$ and $\Gamma_{i,1}$ are not defined.

6.1. *A formula α_0 is a propositional tautology if and only if all end sequences in the diagram of α_0 are fundamental.*

Consider the Boolean algebra $\mathfrak{A}(\mathfrak{S}_0)$ of the two-valued propositional calculus $\mathfrak{S}_0 = \{\mathfrak{L}_0, \mathfrak{C}\}$.

For any non-empty sequence Γ of formulas (1), let δ_Γ denote the disjunction $\bigcup_{i=1}^{m} \alpha_i$ of all formulas in Γ. If Γ is fundamental, then

$$\|\delta_\Gamma\| = V.$$

On the other hand, if a non-empty sequence Γ is indecomposable but not fundamental, then

$$\|\delta_\Gamma\| \neq V$$

since δ_Γ is then not a tautology. In fact, δ_Γ is a disjunction of some $a_1, \ldots, a_p, -a_{p+1}, \ldots, -a_{p+q}$ where all the propositional variables $a_1, \ldots, a_{p+q}$ are different. Any valuation $v = \{v_a\}$ in the two-element Boolean algebra $\mathfrak{A}_0$, such that

$$v_{a_i} = \begin{cases} \Lambda & \text{for} \quad i = 1, \ldots, p, \\ V & \text{for} \quad i = p+1, \ldots, p+q, \end{cases}$$

yields $a_{\mathfrak{A}_0}(v) = \Lambda$.

Let $\{\Gamma_i\}$ be the diagram of a_0. It is easy to verify that if Γ_i is the conclusion of one of the schemas (4), then

$$\|\delta_{\Gamma_i}\| = \|\delta_{\Gamma_{i,0}}\| .$$

If Γ_i is the conclusion of one of the schemas (5), then

$$\|\delta_{\Gamma_i}\| = \|\delta_{\Gamma_{i,0}}\| \cap \|\delta_{\Gamma_{i,1}}\| .$$

Hence it follows that a_0 is a tautology, i.e. $\|a_0\| = \|\delta_{\Gamma_0}\| = V$ if and only if $\|\delta_{\Gamma_i}\| = V$ for every end sequence Γ_i, i.e. if all end sequences are fundamental.

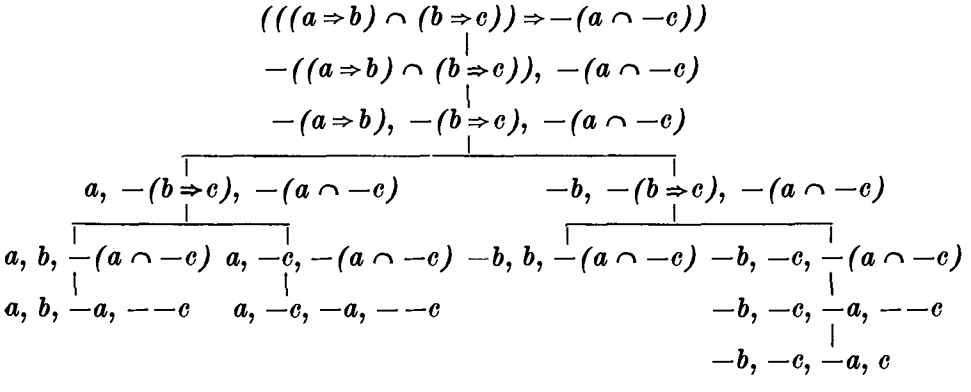
As examples we shall quote here the diagrams of the formulas

$$(((a \Rightarrow b) \cap (b \Rightarrow c)) \Rightarrow -(a \cap -c)) ,$$

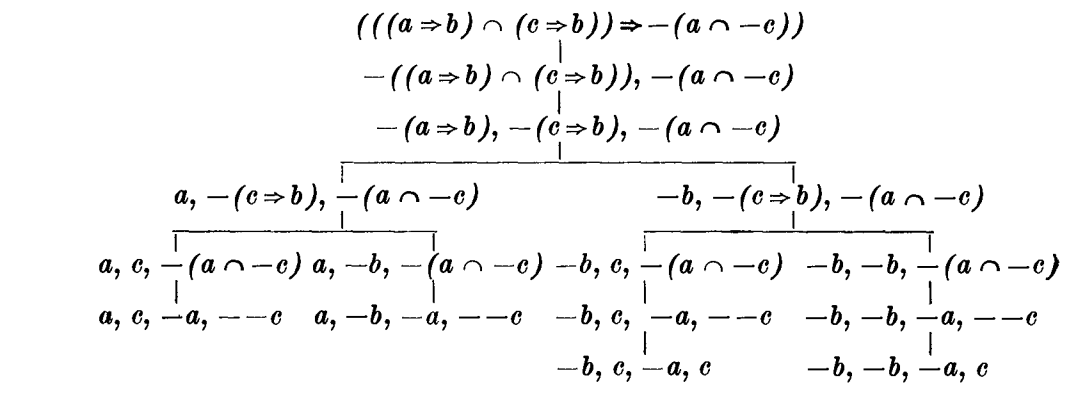
$$(((a \Rightarrow b) \cap (c \Rightarrow b)) \Rightarrow -(a \cap -c)) .$$

Sequences $\Gamma_{i,0}$ and $\Gamma_{i,1}$ are written below Γ_i and joined to Γ_i by a line.

The first formula is a tautology because each of its end sequences is fundamental:



The second formula is not a tautology because two of the end sequences of its diagram are not fundamental:



It is easy to verify that if

$$(7) \qquad \frac{\Gamma}{\overline{\Gamma^0}} \quad \text{or} \quad \frac{\Gamma}{\overline{\Gamma^0}\ ;\ \overline{\Gamma^1}}$$

is one of the schemas (4) or (5) respectively, then

$$(8) \qquad \frac{\delta_{\Gamma^0}}{\delta_\Gamma} \quad \text{or} \quad \frac{\delta_{\Gamma^0},\ \delta_{\Gamma^1}}{\delta_\Gamma}$$

is a rule of inference in the sense defined in V, § 8, p. 173. We will denote by symbols

$$(9) \qquad (\mathrm{D}^*),\ (-\mathrm{D}^*),\ (\mathrm{C}^*),\ (-\mathrm{C}^*),\ (\mathrm{I}^*),\ (-\mathrm{I}^*),\ (-\mathrm{N}^*)$$

the rules of inference corresponding to the schemas (D), (−D), (C), (−C), (I), (−I), (−N) respectively.

A formula in $\mathfrak{L}_0$ is said to be *fundamental* if it is of the form δ_Γ, where Γ is a fundamental sequence.

6.2. *The set of all propositional tautologies is the least set containing all fundamental formulas and closed under the rules of inference* (9) [1].

In fact, every fundamental formula is a propositional tautology. Since rules of inference lead from tautologies to tautologies (see V, 8.1), every formula in the least set containing all fundamental formulas and closed under the rules of inference (9) is a propositional tautology.

---

[1] Theorems 6.1-6.2 are a modification of a theorem of Gentzen [1]. This modification is due to Schütte [1]. See also Kanger [1]. The quoted proof is due to Rasiowa and Sikorski [8]. For another modification see Beth [2] and [4].

On the other hand, if α_0 is a propositional tautology, then all end sequences in the diagram $\{\Gamma_i\}$ of α_0 are fundamental. We can arrange all indexes i occurring in the diagram in a sequence

$$i_1, \ldots, i_k \tag{10}$$

such that if $i \leqslant j$ $(i \neq j)$ then j precedes i in the sequence (10) (in other words, if $i_p \leqslant i_q$, then $q \leqslant p$). By definition, i_k is the empty sequence O. It follows from the arrangement of (10) that the sequence of formulas

$$\delta_{\Gamma_{i_1}}, \ldots, \delta_{\Gamma_{i_k}} \tag{11}$$

is a formal proof of α_0 from the set of fundamental formulas by means of the rules of inference (9). This proves that α_0 is in the least set containing all fundamental formulas and closed under (9).

We see from the second part of the proof of 6.2 that, for any given propositional tautology α_0, we can effectively find a formal proof of α_0 from the set of all fundamental formulas by means of the rules of inference (9). More precisely, the diagram of the formula α_0 is its formal proof (from fundamental formulas by means of (9)) but written in a different way than (11). We should like to emphasize the important property of the proof represented by the diagram: the process of deduction always leads here from simpler to more complicated formulas, and all formulas in the formal proof are parts or disjunctions of parts of the formula α_0.

In 2.2 we have proved that if a formula α_0 is a propositional tautology, then it is derivable in $\mathfrak{S}_0 = \{\mathfrak{L}_0, \mathfrak{C}\}$, i.e. there exists a formal proof of α_0 from axioms (T_1)-(T_{12}) (see p. 260) by means of *modus ponens*. However, we did not give any method of construction of such a proof. The diagram of α_0 can also be used to yield such a construction.

The construction is based on the fact that for every fundamental formula β we can effectively give a formal proof of β from (T_1)-(T_{12}) by means of *modus ponens*. Moreover, if δ_Γ is obtained from δ_{Γ^0} or from δ_{Γ^0} and δ_{Γ^1} (see (7) and (8)) by means of one of the rules of (9), then we can also give an effective formal proof of δ_Γ from (T_1)-(T_{12}) and δ_{Γ^0} or from (T_1)-(T_{12}) and δ_{Γ^0}, δ_{Γ_1} respectively, by means of *modus ponens*. We shall not describe these formal proofs in detail because the description is rather long.

If α_0 is a tautology, we can write its diagram $\{\Gamma_i\}$. For every end sequence Γ_i we can write a formal proof of δ_{Γ_i} from (T_1)-(T_{12}) by means of *modus ponens*. If Γ_i is not an end sequence, we can write a formal proof of δ_{Γ_i} from (T_1)-(T_{12}) and $\delta_{\Gamma_{i,0}}$ or from (T_1)-(T_{12}) and $\delta_{\Gamma_{i,0}}$, $\delta_{\Gamma_{i,1}}$ respectively, by means of *modus ponens*. By a suitable arrangement of all formulas occurring in these formal proofs we obtain, with the help of (11), a formal proof of α_0 from (T_1)-(T_{12}) by means of *modus ponens*.

§ 7. Consistency and existence of models. We shall now examine theories of zero order.

7.1. *If a is a theorem of a theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$, then every model for $\mathfrak{T}$ is a model for a.*

Let $v \in \boldsymbol{A}^{V_0}$ be a model for $\mathfrak{T}$ in a Boolean algebra $\boldsymbol{A}$. By 1.3, every logical axiom is valid; in particular, it is valid in $\boldsymbol{A}$. Hence v is a model for every logical axiom. By the definition of a model for a theory, v is a model for every formula a in $\mathcal{A}$. Moreover, if v is a model for a formula a and for a formula $(a \Rightarrow \beta)$, then v is a model for the formula β. In fact, if $a_A(v) = V$ and $a_A(v) \Rightarrow \beta_A(v) = (a \Rightarrow \beta)_A(v) = V$, then $\beta_A(v) = V$ by I, § 12, (8) or (4). Consequently, the set of all formulas for which v is a model contains all logical axioms, all formulas in $\mathcal{A}$ and is closed under *modus ponens*. Thus, it contains $\mathfrak{C}(\mathcal{A})$, i.e. the set of all theorems of $\mathfrak{T}$. This proves 7.1.

7.2. *If there exists a model $v \in \boldsymbol{A}^{V_0}$ in a non-degenerate Boolean algebra $\boldsymbol{A}$ for a theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$, then $\mathfrak{T}$ is consistent.*

The proof is the same as that of 1.4.

If both formulas a and $-a$ are theorems of $\mathfrak{T}_0$, then $a_A(v) = V$ and $(-a)_A(v) = V$ by 7.1. Hence, $a_A(v) = V$ and $a_A(v) = -V = \Lambda$. Thus $\Lambda = V$, which does not hold in a non-degenerate Boolean algebra $\boldsymbol{A}$.

A model $v \in \boldsymbol{A}^{V_0}$ of a theory $\mathfrak{T}$ in a Boolean algebra $\boldsymbol{A}$ is said to be *adequate* for $\mathfrak{T}$ provided that, for every formula a in $\mathfrak{T}$, a is a theorem of $\mathfrak{T}$ if and only if v is a model for a.

7.3. *Every consistent theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ has an adequate model. Viz. the canonical valuation is an adequate model.*

If $\mathfrak{T}$ is consistent, then by 1.1 its algebra $\mathfrak{A}(\mathfrak{T})$ is a non-degenerate Boolean algebra. Let $v^0 = \{\|a\|\}_{a \in V_0}$ be the canonical valuation in $\mathfrak{A}(\mathfrak{T})$. By 1.2 we have

$$a_{\mathfrak{A}(\mathfrak{T})}(v^0) = \|a\| \quad \text{for all formulas } a. \tag{1}$$

It follows from 1.1 and (1) that

$$a_{\mathfrak{A}(\mathfrak{T})}(v^0) = V \quad \text{if and only if} \quad a \in \mathfrak{C}(\mathcal{A}). \tag{2}$$

Since, for every a in $\mathcal{A}$, $a_{\mathfrak{A}(\mathfrak{T})}(v^0) = V$ (see (2)), the valuation v^0 is a model for $\mathfrak{T}$. At the same time (2) proves that the model v^0 is adequate for $\mathfrak{T}$.

7.4. *If a formula β is irrefutable in a theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$, then there exists a semantic model for $\mathfrak{T}$ which is also a model for β.*

Since β is irrefutable, we have $\|\beta\| \neq \Lambda_{\mathfrak{A}(\mathfrak{T})}$ by 1.1. By I, 8.5, there exists a maximal filter $\mathfrak{V}$ in $\mathfrak{A}(\mathfrak{T})$, such that $\|\beta\| \in \mathfrak{V}$. Let h be the natural

homomorphism from $\mathfrak{A}(\mathfrak{T})$ onto the two-element Boolean algebra $\boldsymbol{A}_0 = \mathfrak{A}(\mathfrak{T})/\nabla$ (see II, 5.2). The valuation $v = hv^0$, where v^0 is the canonical valuation in $\mathfrak{A}(\mathfrak{T})$, i.e. the valuation

$$v = \{h(\|a\|)\}_{a \in V_0}$$

is the required model. Indeed, for every formula a,

$$a_{\boldsymbol{A}_0}(v) = h\big(a_{\mathfrak{A}(\mathfrak{T})}(v^0)\big) = h(\|a\|)$$

by 1.2 and VI, 2.4. Hence by 1.1,

$$a_{\boldsymbol{A}_0}(v) = h(\|a\|) = h(V_{\mathfrak{A}(\mathfrak{T})}) = V \in \boldsymbol{A}_0$$

for every formula a in $\mathcal{A}$. Similarly

$$\beta_{\boldsymbol{A}_0}(v) = h(\|\beta\|) = V \in \boldsymbol{A}_0$$

since $\|\beta\| \in \nabla$. This means that v is a model for $\mathfrak{T}$ and for β.

7.5. *If a formula a is not a theorem in $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$, then there exists a semantic model for $\mathfrak{T}$ which is not a model for a.*

Theorem 7.5 can be proved analogously to 7.4 (instead of $\|\beta\| \in \nabla$ we shall now require $\|a\| \notin \nabla$). It can also be deduced from 7.4 by assuming that β is the formula $-a$ (this follows from VI, 10.11, and the identity $(-a)_{\boldsymbol{A}}(v) = -\big(a_{\boldsymbol{A}}(v)\big)$). Note that similarly 7.4 can be deduced directly from 7.5 by assuming that a is the formula $-\beta$.

7.6. *Every consistent theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ has a semantic model* [1].

Theorem 7.6 follows directly from 7.4 or 7.5.

We summarize the main results of this section in the following two theorems. The first explains the connection between the consistency and existence of models for a theory.

7.7. *For every theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ the following conditions are equivalent*:

(i) *$\mathfrak{T}$ is consistent*;

(ii) *there exists a model for $\mathfrak{T}$*;

(iii) *there exists an adequate model for $\mathfrak{T}$*;

(iv) *there exists a semantic model for $\mathfrak{T}$*.

Indeed, (i) implies (iii) by 7.3 and (iii) trivially implies (ii). (ii) implies (i) by 7.2. (i) is equivalent to (iv) by 7.2 and 7.6.

The next theorem gives necessary and sufficient conditions for a formula to be a theorem of a theory.

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[1] Malcev [1].

7.8. *For every formula α in a consistent theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ the following conditions are equivalent:*

(i) *α is a theorem of $\mathfrak{T}$;*

(ii) *every model for $\mathfrak{T}$ is a model for α;*

(iii) $\alpha_{\mathfrak{A}(\mathfrak{T})}(v^0) = V$, *for the canonical valuation v^0 in the algebra $\mathfrak{A}(\mathfrak{T})$ of the theory $\mathfrak{T}$;*

(iv) *every semantic model for $\mathfrak{T}$ is a model for α.*

Condition (i) implies (ii) by 7.1. Condition (ii) implies (iii), since by 7.3 the canonical valuation v^0 in $\mathfrak{A}(\mathfrak{T})$ is a model for $\mathfrak{T}$. It follows from 7.3 that (iii) implies (i), because the canonical valuation in $\mathfrak{A}(\mathfrak{T})$ is an adequate model for $\mathfrak{T}$. Thus conditions (i), (ii), (iii) are equivalent. On the other hand, conditions (i) and (iv) are equivalent by 7.1 and 7.5.

7.9. *Every linguistic extension of a theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ is inessential.*

Let a language $\mathfrak{L}_1$ be an extension of $\mathfrak{L}_0$, and let $\mathfrak{T}_1 = \{\mathfrak{L}_1, \mathfrak{C}, \mathcal{A}\}$. We have to prove that if a formula α in $\mathfrak{L}_0$ is not a theorem of $\mathfrak{T}$, it is not a theorem of $\mathfrak{T}_1$.

By 7.5 there exists a valuation v: $V_0 \to A$ which is a model for $\mathfrak{T}$ but $\alpha_A(v) \neq V$. Let V_1 be the set of all propositional variables in $\mathfrak{L}_1$. Extend the mapping v: $V_0 \to A$ to a mapping v_1: $V_1 \to A$. We have

$$\beta_A(v_1) = \beta_A(v)$$

for every formula β in $\mathfrak{L}_0$ (see the second part of VI, 2.3). Hence it follows that v_1 is a model for $\mathfrak{T}_1$ but not for α. By 7.1, α is not any theorem in $\mathfrak{T}_1$.

§ 8. Deduction theorems. This name is given to the following two theorems [1].

8.1. *A formula β is a theorem in a theory $\mathfrak{T}' = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A} \cup [\alpha]\}$ if and only if the formula $(\alpha \Rightarrow \beta)$ is a theorem in the theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$.*

If $(\alpha \Rightarrow \beta)$ is a theorem in $\mathfrak{T}$, then it is also a theorem in $\mathfrak{T}'$. Since α is an axiom in $\mathfrak{T}'$, by *modus ponens* the formula β is a theorem in $\mathfrak{T}'$.

If $(\alpha \Rightarrow \beta)$ is not any theorem in $\mathfrak{T}$, then by 7.5 there is a semantic model v for $\mathfrak{T}$ in the two-element Boolean algebra A_0 such that v is not a model for $(\alpha \Rightarrow \beta)$. Thus $\alpha_{A_0}(v) \Rightarrow \beta_{A_0}(v) = (\alpha \Rightarrow \beta)_{A_0}(v) = \Lambda$, i.e. $\alpha_{A_0}(v) = V$ and $\beta_{A_0}(v) = \Lambda$. Thus v is a model for α but not for β. Since v is a model for $\mathfrak{T}'$ but not for β, by 7.1 β is not a theorem in $\mathfrak{T}'$.

8.2. *A formula β is a theorem in a theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ with a non-empty set $\mathcal{A}$ of axioms if and only if there exists a conjunction α of a finite number of axioms in $\mathcal{A}$ such that the implication $(\alpha \Rightarrow \beta)$ is a propositional tautology.*

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[1] Herbrand [1], [2], and Tarski [1].

Suppose that β is a theorem in $\mathfrak{T}$, i.e. β belongs to $C(\mathcal{A})$. Then there exists a finite non-empty set $\mathcal{A}_1 \subset \mathcal{A}$ such that β is in $C(\mathcal{A}_1)$ (see V, 10.1). Let α be a conjunction of all formulas in $\mathcal{A}_1$. By *modus ponens* and (T_6), $\mathcal{A}_1 \subset C([\alpha])$. Consequently β belongs to $C([\alpha])$, i.e. β is a theorem in the theory $\{\mathfrak{L}_0, \mathfrak{C}, [\alpha]\}$. By 8.1 (where $\mathcal{A}$ is empty) the implication $(\alpha \Rightarrow \beta)$ is a theorem in $\{\mathfrak{L}_0, \mathfrak{C}, 0\}$, i.e. it is a propositional tautology.

Conversely, if there exists a conjunction α of axioms in $\mathcal{A}$ such that $(\alpha \Rightarrow \beta)$ is a propositional tautology, then α is a theorem in $\mathfrak{T}$ (see VI, 10.8) and, by *modus ponens*, β is also a theorem in $\mathfrak{T}$.

§ 9. The connection between theories and filters [1]. In this section $\|\alpha\|$ denotes the element of $\mathfrak{A}(\mathfrak{S}_0)$ determined by a formula α.

For every set $\mathcal{A}$ of formulas in $\mathfrak{L}_0$, let $\nabla_{0\mathcal{A}}$ be the set of all elements $\|\alpha\|$ in $\mathfrak{A}(\mathfrak{S}_0)$ where α is in $\mathcal{A}$, and let $\nabla_{\mathcal{A}}$ be the set of all elements $\|\alpha\|$ in $\mathfrak{A}(\mathfrak{S}_0)$ where α is a theorem in the theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$. The last definition can be formulated as follows:

(1) $\|\alpha\| \in \nabla_{\mathcal{A}}$ if and only if α is a theorem in $\mathfrak{T}$.

9.1. *$\nabla_{\mathcal{A}}$ is a filter in the Boolean algebra $\mathfrak{A}(\mathfrak{S}_0)$ and $\nabla_{0\mathcal{A}}$ is a set of generators for the filter $\nabla_{\mathcal{A}}$.*

The theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ is consistent if and only if the filter $\nabla_{\mathcal{A}}$ is proper.

By the deduction theorem 8.2, α is a theorem in $\mathfrak{T}$ if and only if there exists a conjunction $\bigcap_{i=1}^{m} \alpha_i$ of formulas α_i in $\mathcal{A}$, such that $(\bigcap_{i=1}^{m} \alpha_i \Rightarrow \alpha)$ is a propositional tautology, i.e. if

$$\|(\bigcap_{i=1}^{m} \alpha_i \Rightarrow \alpha)\| = V \in \mathfrak{A}(\mathfrak{S}_0) .$$

In other words, α is a theorem in $\mathfrak{T}$ if and only if there exist formulas $\alpha_1, \ldots, \alpha_m$ in $\mathcal{A}$ such that

$$\bigcap_{i=1}^{m} \|\alpha_i\| \leqslant \|\alpha\| .$$

This proves the first part of 9.1 by I 8.1.

By V, 14.1, the theory $\mathfrak{T}$ is consistent if and only if there exists a formula which is not a theorem in $\mathfrak{T}$. By (1), this is equivalent to the statement that the filter $\nabla_{\mathcal{A}}$ is proper. This completes the proof of 9.1.

We have deduced the first part of 9.1 from 8.2. It is easy to see that, conversely, 8.2 follows directly from the first part of 9.1. Thus the first part of 9.1 is only another algebraic formulation of the deduction theorem 8.2. It also explains the algebraic contents of the notions of the set of axioms and the set of all theorems of a theory.

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[1] The results contained in this section are due to Tarski [5], [7].

9.2. *A formula a is irrefutable in a theory $\mathcal{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ if and only if the theory $\{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A} \cup [a]\}$ is consistent.*

A formula β is not a theorem in a theory $\mathcal{T}$ if and only if the theory $\{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A} \cup [-\beta]\}$ is consistent.

For the filter generated by $\nabla_{0\mathcal{A}}$ and $\|a\|$ (i.e. generated by $\nabla_{\mathcal{A}}$ and $\|a\|$) is proper if and only if $-\|a\| \notin \nabla_{\mathcal{A}}$ (see I, 13.9). Similarly, the filter generated by $\nabla_{0\mathcal{A}}$ and $-\|\beta\|$ (i.e. generated by $\nabla_{\mathcal{A}}$ and $-\|\beta\|$) is proper if and only if $\|\beta\| \notin \nabla_{\mathcal{A}}$ (see II, 5.3).

It is easy to see that the second part of 9.2 follows directly from the first (it suffices to replace a by $-\beta$) and similarly the first follows directly from the second. Theorem 9.2 can also be deduced immediately from 7.4, 7.5 and 7.2.

9.3. *A theory $\mathcal{T}' = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}'\}$ is a linguistically invariant extension of a theory $\mathcal{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ if and only if $\nabla_{\mathcal{A}} \subset \nabla_{\mathcal{A}'}$.*

By definition, $\mathcal{T}'$ is a linguistically invariant extension of $\mathcal{T}$ if and only if $\mathfrak{C}(\mathcal{A}) \subset \mathfrak{C}(\mathcal{A}')$. By (1), this is equivalent to the inclusion $\nabla_{\mathcal{A}} \subset \nabla_{\mathcal{A}'}$.

9.4. *Theories $\mathcal{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ and $\mathcal{T}' = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}'\}$ are equivalent if and only if $\nabla_{\mathcal{A}} = \nabla_{\mathcal{A}'}$.*

This follows immediately from the definition of equivalent theories and (1) (or from 9.3).

We shall complete the first part of 9.1 by the following remark:

9.5. *For every filter ∇ in $\mathfrak{A}(\mathcal{S}_0)$ there exists a theory $\mathcal{T}_0 = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ such that $\nabla = \nabla_{\mathcal{A}}$. Moreover, for every set ∇_0 of generators of ∇ we can choose the set $\mathcal{A}$ of axioms for $\mathcal{T}$ in such a way that $\nabla_0 = \nabla_{0\mathcal{A}}$.*

For every element $e \in \nabla_0$, let a_e be a formula such that $\|a_e\| = e$, and let $\mathcal{A}$ be the set of all formulas a_e, $e \in \nabla_0$. By definition, $\nabla_{0\mathcal{A}} = \nabla_0$. Consequently $\nabla_{\mathcal{A}} = \nabla$ by the first part of 9.1.

Theorems 9.1, 9.4 and 9.5 state that there is a natural one-to-one correspondence between filters in $\mathfrak{A}(\mathcal{S}_0)$ and formalized theories based on $\mathcal{S}_0$, two theories being identified if and only if they are equivalent.

To prove the next theorem we shall use the more precise notation introduced in VI, § 10, p. 245: for every formula a, the symbol $\|a\|_{\mathcal{S}_0}$ will denote the corresponding element in the algebra $\mathfrak{A}(\mathcal{S}_0)$ of the propositional calculus $\mathcal{S}_0 = \{\mathfrak{L}_0, \mathfrak{C}\}$ and the symbol $\|a\|_{\mathcal{T}}$ will denote the corresponding element of the algebra $\mathfrak{A}(\mathcal{T})$ of a theory $\mathcal{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$.

If formulas $(a \Rightarrow \beta)$ and $(\beta \Rightarrow a)$ are derivable in $\mathcal{S}_0$, then they are also theorems in $\mathcal{T}$. Hence it follows that the equation

$$h(\|a\|_{\mathcal{S}_0}) = \|a\|_{\mathcal{T}}$$

defines a mapping h from $\mathfrak{A}(\mathfrak{S}_0)$ onto $\mathfrak{A}(\mathfrak{T})$. It is easy to verify that h is a Boolean homomorphism (this follows from 1.1, (3)). By the definition of h and $\mathcal{V}_{\mathcal{A}}$, the element $h(\|a\|_{\mathfrak{S}_0})$ is the unit element of $\mathfrak{A}(\mathfrak{T})$ if and only if a is a theorem in $\mathfrak{T}$, i.e. if $\|a\|_{\mathfrak{S}_0} \in \mathcal{V}_{\mathcal{A}}$. Hence by I, 13.4 there follows:

9.6. *The algebra $\mathfrak{A}(\mathfrak{T})$ of a theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ is isomorphic to the algebra $\mathfrak{A}(\mathfrak{S}_0)/\mathcal{V}_{\mathcal{A}}$.*

Theorem 9.6 gives a method of construction of $\mathfrak{A}(\mathfrak{T})$ from $\mathfrak{A}(\mathfrak{S}_0)$.

9.7. *For every Boolean algebra A there exists a theory $\mathfrak{T}_0 = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ of zero order, such that A is isomorphic with $\mathfrak{A}(\mathfrak{T}_0)$.*

Let A_0 be a set of generators for A, and let $\mathfrak{m}$ be the cardinal of A_0. Take a formalized language $\mathfrak{L}_0$ of zero order, such that the cardinal of the set V_0 of all propositional variables in $\mathfrak{L}_0$ is $\mathfrak{m}$. Let f be a mapping from V_0 onto A_0. By theorem 4.1 (see also 4.2), there exists a homomorphism h: $\mathfrak{A}(\mathfrak{S}_0) \to A$ such that

$$h(\|a\|) = f(a) \quad \text{for every } a \text{ in } V_0.$$

Since A_0 generates A, the homomorphism h maps $\mathfrak{A}(\mathfrak{S}_0)$ onto A. By I, 13.4, the algebra A is isomorphic with $\mathfrak{A}(\mathfrak{S}_0)/\mathcal{V}$ where $\mathcal{V}$ is the filter of all $\|a\| \in \mathfrak{A}(\mathfrak{S}_0)$ such that $h(\|a\|) = V \in A$. By theorems 9.5 and 9.6, there exists a set $\mathcal{A}$ of formulas in $\mathfrak{L}_0$, such that $\mathfrak{A}(\mathfrak{S}_0)/\mathcal{V}$ is isomorphic with $\mathfrak{A}(\mathfrak{T}_0)$, where $\mathfrak{T}_0 = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$. Consequently A is isomorphic with $\mathfrak{A}(\mathfrak{T}_0)$.

§ 10. Maximal and prime theories [1]. We recall that a consistent theory $\mathfrak{T}$ is said to be maximal provided every linguistically invariant consistent extension of $\mathfrak{T}$ is equivalent to $\mathfrak{T}$. The following theorem characterizes maximal theories:

10.1. *The following conditions are equivalent for every theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$:*

(i) *$\mathfrak{T}$ is maximal;*

(ii) *$\mathcal{V}_{\mathcal{A}}$ is a maximal filter in $\mathfrak{A}(\mathfrak{S}_0)$;*

(iii) *for every formula a in $\mathfrak{L}_0$, exactly one of the formulas a and $-a$ is a theorem in $\mathfrak{T}$;*

(iv) *$\mathfrak{A}(\mathfrak{T})$ is the two-element Boolean algebra.*

(i) is equivalent to (ii). This follows from the natural one-to-one correspondence between filters in $\mathfrak{A}(\mathfrak{S}_0)$ and classes of equivalent theories (see § 9, p. 274) and from 9.3, 9.4.

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[1] The results of this section, except 10.4, are due to Tarski [5], [7].

Condition (iii) means that for every element $\|a\|$ in $\mathfrak{A}(\mathfrak{S}_0)$ exactly one of the elements $\|a\|$, $-\|a\|$ is in $\mathcal{V}_{\mathcal{A}}$. Thus, by II, 5.2, (ii) is equivalent to (iii).

(iv) is equivalent to (ii) by 9.6 and II, 5.2.

The next theorem gives another characterization by means of models.

10.2. *For every theory* $\mathcal{T} = \{\mathfrak{L}_0, \mathcal{C}, \mathcal{A}\}$, *the following conditions are equivalent*:

(i) $\mathcal{T}$ *is maximal*;

(ii) $\mathcal{T}$ *is consistent and every semantic model for* $\mathcal{T}$ *is adequate*;

(iii) $\mathcal{T}$ *has an adequate semantic model.*

Let $\boldsymbol{A}_0$ denote the two-element Boolean algebra.

Suppose that $\mathcal{T}$ is maximal and v is a semantic model for $\mathcal{T}$. If a is not a theorem in $\mathcal{T}$, then by 10.1, (iii), the formula $-a$ is a theorem, i.e. $(-a)_{\boldsymbol{A}_0}(v) = \mathcal{V}$, i.e. $a_{\boldsymbol{A}_0}(v) = \mathcal{\Lambda}$. This proves that v is adequate.

Thus (i) implies (ii). Since (ii) implies (iii) trivially (see 7.6), it suffices to show that (iii) implies (i). Suppose that v is an adequate model for $\mathcal{T}$ in $\boldsymbol{A}_0$. For every a, exactly one of the elements $a_{\boldsymbol{A}_0}(v)$, $(-a)_{\boldsymbol{A}_0}(v)$ is equal to the unit element in $\boldsymbol{A}_0$. Since v is adequate, we infer that exactly one of the formulas a and $-a$ is a theorem in $\mathcal{T}$. Thus $\mathcal{T}$ is maximal by 10.1.

10.3. *If* v *is a valuation in the two-element Boolean algebra* $\boldsymbol{A}_0$ *and* $\mathcal{A}_0$ *is the set of all formulas* a *such that* $a_{\boldsymbol{A}_0}(v) = \mathcal{V}$, *then every theory* $\mathcal{T}$ *equivalent to* $\{\mathfrak{L}_0, \mathcal{C}, \mathcal{A}_0\}$ *is maximal.*

In fact, v is then an adequate semantic model for $\mathcal{T}$.

10.4. *Every consistent theory* $\mathcal{T} = \{\mathfrak{L}_0, \mathcal{C}, \mathcal{A}\}$ *can be extended to a maximal theory* $\mathcal{T}_0 = \{\mathfrak{L}, \mathcal{C}, \mathcal{A}_0\}$ [1].

Since $\mathcal{T}$ is consistent, the filter $\mathcal{V}_{\mathcal{A}}$ in $\mathfrak{A}(\mathfrak{S}_0)$ is proper (see 9.1). By I, 8.4, the filter $\mathcal{V}_{\mathcal{A}}$ can be extended to a maximal filter $\mathcal{V}$ in $\mathfrak{A}(\mathfrak{S}_0)$. Let $\mathcal{A}_0$ be any set of formulas such that $\mathcal{V} = \mathcal{V}_{\mathcal{A}_0}$ (see 9.5). By 10.1, the theory $\mathcal{T}_0 = \{\mathfrak{L}_0, \mathcal{C}, \mathcal{A}_0\}$ is maximal. It is an extension of $\mathcal{T}$ by 9.3.

Another proof of 10.4 is as follows. Let v be a semantic model for $\mathcal{T}$ and let $\mathcal{A}_0$ be the set of all formulas a such that $a_{\boldsymbol{A}_0}(v) = \mathcal{V}$. The theory $\mathcal{T}_0 = \{\mathfrak{L}_0, \mathcal{C}, \mathcal{A}_0\}$ is an extension of $\mathcal{T}$ and is maximal, by 10.3.

We recall that a theory $\mathcal{T} = \{\mathfrak{L}_0, \mathcal{C}, \mathcal{A}\}$ is said to be prime provided that for any formulas a, β, if $(a \cup \beta)$ is a theorem in $\mathcal{T}$, then either a or β is a theorem in $\mathcal{T}$ (see V § 14).

10.5. *The following conditions are equivalent for every theory* $\mathcal{T} = \{\mathfrak{L}_0, \mathcal{C}, \mathcal{A}\}$:

(i) $\mathcal{T}$ *is prime*;

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[1] This theorem is due to A. Lindenbaum (see Tarski [1]).

(ii) $\nabla_{\mathcal{A}}$ *is a prime filter in* $\mathfrak{A}(\mathfrak{S}_0)$;

(iii) $\mathfrak{C}$ *is maximal.*

The equivalence of (i) and (ii) follows directly from the definition. Since prime ideals coincide with maximal ideals in Boolean algebras (see II, 5.2), (ii) is equivalent to (iii) by 10.1.

§ 11. Effectivity problems. Consider the following statements which are particular cases of theorems I, 8.4, I, 8.5 and I, 8.6:

(F_1) Every proper filter in a Boolean algebra can be extended to a maximal filter.

(F_2) Every element $e \neq \Lambda$ in a Boolean algebra belongs to a maximal filter.

(F) Every non-degenerate Boolean algebra has a maximal filter.

Consider also the following theorems proved in § 7, (7.6) and § 10, (10.4):

(M_0) Every consistent theory of zero order has a semantic model.

(M_0') Every consistent theory of zero order can be extended to a maximal theory.

Theorem (F_1) was obtained by means of the induction principle for ordered sets I, 5.1 (see the proof of I, 8.4) which is a consequence of the axiom of choice (more exactly, it is another formulation of the axiom of choice [1]). Thus our proof of (F_1) is not effective since it is based on the axiom of choice. It is known that (F_1) is not effectively equivalent to the axiom of choice [2]. However (F_1) effectively implies some weaker forms of the axiom of choice and some consequences of the axiom of choice, e.g. the existence of sets non-measurable in the sense of Lebesgue [3].

Obviously (F_1) effectively implies (F_2) and (F_2) effectively implies (F) (in this way we have deduced theorem I, 8.5 from I, 8.4 and theorem I, 8.6 from I, 8.5). We shall prove that (F) effectively implies (F_1). If ∇_0 is a proper filter in a Boolean algebra $\boldsymbol{A}$, then the algebra $\boldsymbol{A}/\nabla_0$ is not degenerate and, by (F), it contains a maximal filter ∇'. The set ∇ of all elements $e \in \boldsymbol{A}$ such that $|e| \in \nabla'$ is a maximal filter and $\nabla_0 \subset \nabla'$.

Thus the statements (F_1), (F_2), (F) are effectively equivalent [4].

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[1] See e.g. Birkhoff [3], p. 45. Note, that theorem I, 8.4 is effectively equivalent to the axiom of choice. See e.g. Scott [1], Mrówka [1], [2].

[2] This non-published result is due to J. Halpern. For the equivalence of (F_1) with other statements see e. g. Łoś and Ryll-Nardzewski [1], Henkin [3], [4], Łoś [3], [8], Rubin and Scott [1], Scott [1], Mrówka [1], [2], Tarski [11], [12], [13].

[3] Sierpiński [2].

[4] Stone [2].

We shall show that they are also effectively equivalent to each of the statements $(\mathbf{M}_0)$ and $(\mathbf{M}'_0)$ [1].

In fact, the statement $(\mathbf{M}_0)$, i.e. theorem 7.6, was deduced from a slightly stronger theorem 7.4 which was effectively deduced from $(\mathbf{F}_2)$ (observe also that 7.6 can be immediately deduced from (F) in the same way that 7.6 was deduced from $(\mathbf{F}_2)$). The statement $(\mathbf{M}'_0)$, i.e. theorem 10.4 was effectively deduced from 7.6, i.e. from $(\mathbf{M}_0)$ (see the second proof of 10.4 on p. 276). Thus it suffices to show that $(\mathbf{M}'_0)$ effectively implies (F).

By theorem 9.7 it suffices to prove that every Boolean algebra $\mathfrak{A}(\mathfrak{T})$ contains a maximal filter, $\mathfrak{T}$ being any consistent theory of zero order. By $(\mathbf{M}'_0)$, $\mathfrak{T}$ can be extended to a maximal theory $\mathfrak{T}'$. It is easy to verify that the set of all elements $\|\alpha\| \in \mathfrak{A}(\mathfrak{T})$, where α is a theorem in $\mathfrak{T}'$ is a maximal filter in $\mathfrak{A}(\mathfrak{T})$.

Observe that if the language $\mathfrak{L}_0$ of formalized theories examined in §§ 7-10 is enumerable (i.e. the set of all formulas in $\mathfrak{L}_0$ is enumerable), then our proofs of $(\mathbf{M}_0)$ and $(\mathbf{M}'_0)$ and of all other theorems are effective. This follows from the fact that the fundamental theorem I, 8.4 on existence of maximal filters can be effectively proved (by a slight modification of the proof given in I, § 8). Consequently theorems $(\mathbf{F}_1)$, $(\mathbf{F}_2)$, (F) are effective for enumerable Boolean algebras.

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[1] See Henkin [3] and Łoś [8].

CHAPTER VIII

CLASSICAL ELEMENTARY FORMALIZED THEORIES

§ 1. Preliminaries. In this Chapter we shall deal with a fixed formalized language $\mathfrak{L} = \{A, T, F\}$ of the first order. The alphabet A, the set T of all terms and the set F of all formulas in $\mathfrak{L}$ were described in V, § 3. The letter V will always denote the set of all free individual variables $x, y, z, \ldots$ in $\mathfrak{L}$. By a *formula* we shall mean, in this Chapter, a formula in $\mathfrak{L}$, unless the contrary is explicitly stated.

The letter C will denote exclusively the classical (i.e. two-valued) consequence operation defined in V, § 11, p. 188.

In this Chapter we shall examine the classical (i.e. two-valued) predicate calculus based on $\mathfrak{L}$, i.e. the deductive system

$$\mathfrak{S} = \{\mathfrak{L}, \mathrm{C}\}$$

and classical formalized elementary theories based on $\mathfrak{L}$

$$\mathfrak{T} = \{\mathfrak{L}, \mathrm{C}, \mathcal{A}\}$$

where $\mathcal{A}$ is any set of formulas.

A finite sequence $\alpha_1, \ldots, \alpha_n$ of formulas is said to be a *formal proof* of a formula α in the predicate calculus $\mathfrak{S} = \{\mathfrak{L}, \mathrm{C}\}$ from a set S of formulas provided α_n is the formula α and, for every $j \leqslant n$, either α_j is a logical axiom (i.e. a formula of a form (T_1)-(T_{12})—see V, § 11, p. 188), or it is in S, or it is an immediate consequence of certain formulas $\alpha_{j_1}, \alpha_{j_2}$ $(j_1, j_2 < j)$ by means of one of the rules (r_1)-(r_6) from V, § 8.

A formula α is in $\mathrm{C}(S)$ if and only if there exists a formal proof for α from S in $\mathfrak{S}$. In this case the formula α is said to be *derivable from S in* $\mathfrak{S}$ (see V, 10.2).

The set $\mathrm{C}(\mathcal{A})$ plays a special role for the theory $\mathfrak{T} = \{\mathfrak{L}, \mathrm{C}, \mathcal{A}\}$: formulas in $\mathrm{C}(\mathcal{A})$ are called *theorems* of the theory $\mathfrak{T}$.

We recall that when $\mathcal{A}$ is empty, the theory $\{\mathfrak{L}, \mathrm{C}, 0\}$ is identified with the predicate calculus $\mathfrak{S} = \{\mathfrak{L}, \mathrm{C}\}$ described in V, § 11. Theorems in $\{\mathfrak{L}, \mathrm{C}, 0\}$ are called *derivable formulas* or, more precisely, *formulas derivable in* $\mathfrak{S}$. By definition, a formula α is derivable if and only if there exists a formal proof for α in $\mathfrak{S}$ from the empty set of formulas.

Unless the contrary is explicitly stated, by *theory* we shall understand in this Chapter any formalized theory of the first order $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$.

We recall (see VI, § 10, p. 245) that the algebra $\mathfrak{A}(\mathfrak{T})$ of a theory $\mathfrak{T}$ is the algebra $F/\approx$, where $\approx$ is the congruence in the algebra of formulas $\{F, \cup, \cap, \Rightarrow, -\}$ defined as follows:

(1) $\alpha \approx \beta$ if and only if the formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are theorems in $\mathfrak{T}$.

In particular, $\mathfrak{A}(S)$ is the algebra $F/\approx$, where $\approx$ is the congruence in $\{F, \cup, \cap, \Rightarrow, -\}$ defined as follows:

(1′) $\alpha \approx \beta$ if and only if the formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are derivable in S.

From VI, 10.4, 10.6, 10.7, 10.10, 11.2 follows

1.1. *The algebra* $\{\mathfrak{A}(\mathfrak{T}), \cup, \cap, \Rightarrow, -\}$ *of a theory* $\mathfrak{T}$ *is a Boolean algebra. For all formulas* α, β

$$\begin{aligned} \|\alpha\| \cup \|\beta\| &= \|(\alpha \cup \beta)\|, \qquad \|\alpha\| \cap \|\beta\| = \|(\alpha \cap \beta)\|, \\ \|\alpha\| \Rightarrow \|\beta\| &= \|(\alpha \Rightarrow \beta)\|, \qquad -\|\alpha\| = \|-\alpha\|. \end{aligned} \tag{2}$$

Moreover

$$\|\alpha\| \leqslant \|\beta\| \text{ if and only if } (\alpha \Rightarrow \beta) \text{ is a theorem in } \mathfrak{T}. \tag{3}$$

For every formula α,

$$\|\alpha\| = V \text{ if and only if } \alpha \text{ is a theorem in } \mathfrak{T}, \tag{4}$$

$$\|\alpha\| \neq \Lambda \text{ if and only if } \alpha \text{ is irrefutable in } \mathfrak{T}. \tag{5}$$

Thus the Boolean algebra $\mathfrak{A}(\mathfrak{T})$ *is non-degenerate if and only if the theory* $\mathfrak{T}$ *is consistent.*

For every formula $\beta(x)$,

$$\begin{aligned} \|\bigcup_\xi \beta(\xi)\| &= \bigcup_{\tau \in T} \|\beta(\tau)\|, \\ \|\bigcap_\xi \beta(\xi)\| &= \bigcap_{\tau \in T} \|\beta(\tau)\|. \end{aligned} \tag{Q}$$

According to VI, § 11, we shall always conceive of $\mathfrak{A}(\mathfrak{T})$ as the generalized algebra

$$\{\mathfrak{A}(\mathfrak{T}), \cup, \cap, \Rightarrow, -, \bigcup|\mathfrak{Q}, \bigcap|\mathfrak{Q}\}, \tag{6}$$

where $\bigcup$ and $\bigcap$ are respectively the infinite join and meet. As the common domain $\mathfrak{Q}$ of $\bigcup$ and $\bigcap$ in $\mathfrak{A}(\mathfrak{T})$ we assume only the class of all indexed sets

$$\{\|\beta(\tau)\|\}_{\tau \in T}, \tag{7}$$

where $\beta(x)$ is any formula in $\mathfrak{L}$. In other words, $\bigcup$ and $\bigcap$ are restricted in $\mathfrak{A}(\mathfrak{T})$ only to the infinite joins and meets (Q), i.e. to the infinite joins

and meets corresponding to quantifiers. The generalized algebra $(\mathfrak{Q})$ is called the *quantifier algebra of the theory* $\mathfrak{T}$, or briefly the **Q**-*algebra of* $\mathfrak{T}$.

By a **Q**-homomorphism from $\mathfrak{A}(\mathfrak{T})$ into any Boolean algebra we shall always understand any Boolean homomorphism preserving all the infinite joins and meets (**Q**). A maximal (i.e. prime) filter V in $\mathfrak{A}(\mathfrak{T})$ is said to be a **Q**-*filter* provided the natural homomorphism from $\mathfrak{A}(\mathfrak{T})$ onto the two-element Boolean algebra $\mathfrak{A}(\mathfrak{T})/V$ is a **Q**-homomorphism. By II, § 9 (p. 86), a maximal filter V is a **Q**-filter provided

(a) for any formula $\alpha(x)$, if $\|\bigcup_\xi \alpha(\xi)\| \in V$, then there exists a term τ such that $\|\alpha(\tau)\| \in V$;

(b) for any formula $\alpha(x)$, if $\|\bigcap_\xi \alpha(\xi)\| \in V$, then there exists a term τ such that $\|\alpha(\tau)\| \notin V$.

It is easy to see that conditions (a) and (b) are equivalent (replace α by $-\alpha$ and apply the De Morgan laws and II, 5.2, (d)). Thus a maximal filter is a **Q**-filter if it satisfies one of the conditions (a), (b).

§ 2. Models. We remind the reader that by a realization of a formalized language $\mathfrak{L} = \{S, T, F\}$ of the first order in a set $J \neq 0$ and in a complete algebra $\{A, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ (with three binary operations $\cup$, $\cap$, $\Rightarrow$, one unary operation $-$ and two infinite operations $\bigcup$, $\bigcap$) we mean any mapping R defined on the set of all functors in $\mathfrak{L}$ and on the set of all predicates in $\mathfrak{L}$ and such that (see VI, § 6):

(a) R assigns to every m-argument functor φ in $\mathfrak{L}$ an m-argument function φ_R in J, i.e. a mapping $\varphi_R\colon J^m \to J$,

(b) R assigns to every m-argument predicate ϱ in $\mathfrak{L}$ an m-argument function defined in J with values in A, i.e. a mapping $\varrho_R\colon J^m \to A$.

As was explained in VI, § 6, given any realization R of $\mathfrak{L}$ in J and A, every formula α in $\mathfrak{L}$ uniquely determines a mapping

$$\alpha_R\colon J^V \to A\,, \tag{1}$$

where V is the set of all free individual variables in $\mathfrak{L}$. The value of α_R at a point $v = \{v_x\}_{x \in V}$ of the Cartesian product J^V is denoted by $\alpha_R(v)$.

In the definition of a realization R of $\mathfrak{L}$ in $J \neq 0$ and A from VI, § 6, we have assumed that A is a complete algebra only in order to ensure the feasibility of all infinite operations appearing in the inductive definition of $\alpha_R(v)$. However it may happen, for an incomplete generalized algebra A and for a given mapping R satisfying conditions (a) and (b), that all infinite operations appearing in the definitions of $\alpha_R(v)$ are feasible in A for every α in $\mathfrak{L}$ and for every $v \in J^V$. In this case the mapping R is also called a realization of $\mathfrak{L}$ in J and A according to VI, § 7, p. 236.

In this Chapter we shall always assume that

$\{A, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ *is a non-degenerate Boolean algebra*

and $\bigcup$, $\bigcap$ are respectively the infinite join and meet.

Observe that for every Boolean algebra A there exists an isomorphism h^* from A into a complete Boolean algebra, such that h^* preserves all infinite joins and meets in A (see II, 10.1). In other words, A can be extended to a complete Boolean algebra A^* with preservation of all infinite joins and meets. Therefore any realization R in $J \neq 0$ and in an incomplete Boolean algebra A can be considered a realization in J and in the corresponding complete Boolean algebra A^*. However, if only elements in A are values of $a_R(v)$ and only infinite operations in A are used to calculate $a_R(v)$, there is no reason to introduce any such extension A^* of A, and it is much more natural to consider R as a realization in A.

Let $\mathfrak{L}$ be a fixed formalized language and let R be a fixed realization of $\mathfrak{L}$ in $J \neq 0$ and in a Boolean algebra A. We recall that any element $v = \{v_x\}_{x \in V} \in J^V$, i.e. any mapping $v\colon V \to J$, is said to be a valuation of $\mathfrak{L}$ in J. Sometimes we shall also say that v is a *valuation of* $\mathfrak{L}$ *in* R, since R determines the set J.

A valuation $v\colon V \to J$ is said to *satisfy a formula* α *in the realization* R provided that

$$a_R(v) = V \,. \tag{2}$$

A formula α is said to be *satisfiable in* R if there exists a valuation v in R which satisfies α, i.e. for which the condition (2) is fulfilled. A formula α is said to be *valid* [1] *in* R if every valuation v in R satisfies α, i.e. if (2) holds for every $v\colon V \to J$. In this case we say that the realization R is a *model for the formula* α. Observe that in the case of a closed formula α (i.e. a formula without free individual variables) the notions of satisfiability and validity coincide since a_R is then a constant function, i.e. a fixed element of the algebra A. Observe also the following simple connection between the satisfiability and validity of any formula:

(3) α is satisfiable if and only if $-\alpha$ is not valid;

(4) $-\alpha$ is satisfiable if and only if α is not valid.

This follows immediately from the identity $(-\alpha)_R(v) = -(a_R(v))$.

A set $\mathcal{A}$ of formulas in $\mathfrak{L}$ is said to be *valid in* R provided every formula α in $\mathcal{A}$ is valid in R. In this case we shall also say that R is a *model for the set* $\mathcal{A}$ of formulas in *the set* J *and the algebra* A.

[1] In the case where A is the two-element Boolean algebra, the notion of satisfiability and validity just defined (Rasiowa-Sikorski [3]) are equivalent to the analogous notions introduced by Tarski [4].

Let $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ be a formalized theory. A realization R of $\mathfrak{L}$ in J and $\boldsymbol{A}$ will be said to be a *model* [1] *for* $\mathfrak{T}$ *in* J *and* $\boldsymbol{A}$ provided R is a model for the set $\mathcal{A}$ of mathematical axioms. In other words, R is a model for $\mathfrak{T}$ if and only if every mathematical axiom of $\mathfrak{T}$ is valid in R. In particular, every realization R of $\mathfrak{L}$ is a model for $\{\mathfrak{L}, C, 0\}$.

By the *cardinal of a model* R for $\mathfrak{T}$ in J and $\boldsymbol{A}$ we shall mean the cardinal of the set J. In particular any model in an enumerable (finite) set is called an *enumerable* (*finite*) *model*.

Consider a formalized language $\mathfrak{L}_0$ of zero order with a set V_0 of all propositional variables. Let s be an arbitrary substitution from $\mathfrak{L}_0$ into a formalized language $\mathfrak{L}$ of the first order. We recall (see VI, § 2) that for any formula α in $\mathfrak{L}_0$ the formula $s\alpha$ is in $\mathfrak{L}$.

2.1. *For every propositional tautology* α *in* $\mathfrak{L}_0$ *the formula* $s\alpha$ *in the language* $\mathfrak{L}$ *is valid in every realization* R *of* $\mathfrak{L}$.

This follows immediately from VI, 6.9 and VII, 2.2.

2.2. *Let* R *be a model for a theory* $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$. *Then every formula* α *in* $C(\mathcal{A})$ (*i. e. every theorem in* $\mathfrak{T}$) *is valid in* R.

Let $F(R)$ be the set of all formulas in $\mathfrak{L}$ which are valid in R. We shall show that $C(\mathcal{A}) \subset F(R)$.

Since every logical axiom of $\mathfrak{T}$ can be conceived to be the result of a substitution s from a language $\mathfrak{L}_0$ of zero order into $\mathfrak{L}$ performed on a propositional tautology of the form (t_1)-(t_{12}) quoted in V, § 6 (see V, § 11, p. 188), the set $F(R)$ contains the set $\mathcal{A}_1$ of all logical axioms by 2.1. From the definition of a model it follows that $F(R)$ contains the set $\mathcal{A}$. Thus in order to prove the inclusion $C(\mathcal{A}) \subset F(R)$ it suffices to show that $F(R)$ is closed under the rules of inference (r_1)-(r_6).

Rule (r_1) of detachment. Suppose that α and $(\alpha \Rightarrow \beta)$ are in $F(R)$. Let v be a valuation in J. By the hypothesis, for every valuation v we have $\alpha_R(v) = \mathsf{V}$ and $\alpha_R(v) \Rightarrow \beta_R(v) = (\alpha \Rightarrow \beta)_R(v) = \mathsf{V}$. Thus $\beta_R(v) = \mathsf{V}$ for every v, i.e. β is in $F(R)$.

Rule (r_2) of substitution for individual variables. Suppose α is in $F(R)$. Let $s\colon V \to T$ be an arbitrary substitution in $\mathfrak{L}$. Let $s'\alpha$ be the result of the substitution s in α (see VI, § 4, (7), (8)). Given an arbitrary valuation $v\colon V \to J$, let $s_R v$ be the valuation $\{s x_R(v)\}_{x \in V} \in J^V$ defined in VI, § 3, p. 218. By VI, 6.7,

$$s'\alpha_R(v) = \alpha_R(s_R v) = \mathsf{V}$$

since α is in $F(R)$. This proves that $s'\alpha$ is in $F(R)$.

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[1] This general notion of model has been introduced by Rasiowa [4] and called there algebraic model.

Rule (r_3) of introduction of the existential quantifier. Suppose that a formula $(\beta \Rightarrow \alpha)$ is in $F(R)$, x_0 is a free individual variable which occurs in β but not in α, and ξ is a bound individual variable which does not appear in β. Let $v = \{v_x\}_{x \in V}$ be any valuation in J and, for every $j \in J$, let $w_j = \{w_{jx}\}_{x \in V}$ be the valuation

$$w_{jx} = v_x \quad \text{for} \quad x \neq x_0 \quad \text{and} \quad w_{jx_0} = j$$

(see VI, § 6, (4')). Since x_0 does not appear in α, we have $\alpha_R(w_j) = \alpha_R(v)$. Hence

$$\beta_R(w_j) \Rightarrow \alpha_R(v) = \beta_R(w_j) \Rightarrow \alpha_R(w_j) = (\beta \Rightarrow \alpha)_R(w_j) = V$$

since $(\beta \Rightarrow \alpha)$ is in $F(R)$. Thus

$$\beta_R(w_j) \leqslant \alpha_R(v) \quad \text{for every} \quad j \in J$$

and consequently, by VI, § 6, (4),

$$\big(\bigcup_\xi \beta(x_0/\xi)\big)_R(v) = \bigcup_{j \in J} \beta_R(w_j) \leqslant \alpha_R(v)\,,$$

i.e.

$$\big(\bigcup_\xi \beta(x_0/\xi) \Rightarrow \alpha\big)_R(v) = V\,.$$

This proves that the formula $\bigcup_\xi \beta(x_0/\xi) \Rightarrow \alpha$ is in $F(R)$.

Rule (r_4) of introduction of the universal quantifier. The proof is analogous to the case of (r_3).

Rule (r_5) of elimination of the existential quantifier. Suppose a formula $\bigcup_\xi \beta(x_0/\xi) \Rightarrow \alpha$ is in $F(R)$, i.e., by VI, § 6, (4),

$$\big(\bigcup_{j \in J} \beta_R(w_j)\big) \Rightarrow \alpha_R(v) = \big(\bigcup_\xi \beta(x_0/\xi) \Rightarrow \alpha\big)_R(v) = V$$

for every valuation v, the valuation w_j being defined as above. For $j_0 = v_x$ we have $w_{j_0} = v$. Hence

$$\beta_R(v) = \beta_R(w_{j_0}) \leqslant \bigcup_{j \in J} \beta_R(w_j) \leqslant \alpha_R(v)\,,$$

i.e.

$$(\beta \Rightarrow \alpha)_R(v) = \beta_R(v) \Rightarrow \alpha_R(v) = V$$

for every valuation v in J. This proves that $(\beta \Rightarrow \alpha)$ is in $F(R)$.

Rule (r_6) of elimination of the universal quantifier. The proof is analogous to the case of (r_5).

2.3. *Every formula derivable in the predicate calculus* $S = \{\mathfrak{L}, \mathfrak{C}\}$ *is valid in every realization* R.

This follows directly from 2.2 (the case in which the set $\mathfrak{A}$ is empty).

2.4. *If a theory* $\mathfrak{T}$ *has a model* R *in a set* $J \neq 0$ *and a non-degenerate Boolean algebra* A, *then* $\mathfrak{T}$ *is consistent.*

In fact, suppose that a formula α is a theorem in $\mathfrak{T}$. By 2.2, $\alpha_R(v) = V$ for any valuation v in R. Thus $(-\alpha)_R(v) = \Lambda$. By 2.2, the formula $-\alpha$ is not a theorem in $\mathfrak{T}$.

2.5. *The predicate calculus* $\mathcal{S} = \{\mathfrak{L}, \mathfrak{C}\}$ *is consistent.*

This follows directly from 2.4 since every realization is a model for the theory $\mathcal{S} = \{\mathfrak{L}, \mathfrak{C}, 0\}$.

Let $\boldsymbol{B}$ be a non-degenerate complete Boolean algebra and let h: $\boldsymbol{A} \to \boldsymbol{B}$ be a homomorphism from a Boolean algebra $\boldsymbol{A}$ into $\boldsymbol{B}$ preserving all infinite joins and meets in $\boldsymbol{A}$. If R is a realization of a language $\mathfrak{L}$ in $J \neq 0$ and in $\boldsymbol{A}$, then the following equations define another realization (denoted by hR) of $\mathfrak{L}$ in J and $\boldsymbol{B}$ (see VI, § 6, (8), p. 232):

$$\begin{aligned} &\varphi_{hR} = \varphi_R \quad \text{for every functor} \quad \varphi \text{ in } \mathfrak{L}, \\ &\varrho_{hR} = h\varrho_R \quad \text{for every predicate} \quad \varrho \text{ in } \mathfrak{L}. \end{aligned} \tag{5}$$

Thus R and hR yield the same realization of terms. If ϱ is an m-argument predicate, then

$$\varrho_{hR}(j_1, \ldots, j_m) = h\big(\varrho_R(j_1, \ldots, j_m)\big).$$

Under the hypotheses assumed above the following theorem holds.

2.6. *If R is a model for a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$, then hR is also a model for $\mathfrak{T}$. Moreover, for any formula α in $\mathfrak{L}$*

$$\alpha_{hR}(v) = h\big(\alpha_R(v)\big) \quad \textit{for every valuation } v. \tag{6}$$

Identity (6) was proved in VI, 6.6. If α is in $\mathcal{A}$, then α is valid in R, i.e. $\alpha_R(v) = V$. Hence by (6) $\alpha_{hR}(v) = V$ for any valuation v in hR. Thus hR is a model for $\mathfrak{T}$.

Let a formalized language $\mathfrak{L}'$ be an extension of a language $\mathfrak{L}$ (see V, § 14) and let a realization R' of $\mathfrak{L}'$ in a set $J \neq 0$ and in a Boolean algebra $\boldsymbol{A}$ be an extension of a realization R of $\mathfrak{L}$ in the same set J and in the same Boolean algebra $\boldsymbol{A}$. Under the above hypotheses the following theorem holds.

2.7. *R is a model for a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ if and only if R' is a model for the linguistic extension $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A}\}$ of $\mathfrak{T}$. More exactly, for any formula α in $\mathfrak{L}$ the following conditions are fulfilled:*

(i) *α is satisfiable in R if and only if α is satisfiable in R',*

(ii) *α is valid in R if and only if α is valid in R'.*

(i) and (ii) follow directly from VI, 6.4. Hence, by the definition of a model, R is a model for $\mathfrak{T}$ if and only if R' is a model for $\mathfrak{T}$, since $\mathcal{A}$ is a set of formulas in $\mathfrak{L}$.

2.8. *Let R be a model for a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ in a set $J \neq 0$ and in a complete Boolean algebra $\boldsymbol{A}$. Let f be a mapping from J onto J' and let R' be the following realization of $\mathfrak{L}$ in J' and in $\boldsymbol{A}$:*

$$\varphi_{R'}(f(j_1), \ldots, f(j_m)) = f(\varphi_R(j_1, \ldots, j_m)),$$

for any m-argument functor φ in $\mathfrak{L}$ $(m = 0, 1, \ldots)$,

$$\varrho_{R'}(f(j_1), \ldots, f(j_m)) = \varrho_R(j_1, \ldots, j_m)$$

for any m-argument predicate ϱ in $\mathfrak{L}$ $(m = 1, 2, \ldots)$.

Then R' is a model for $\mathfrak{T}$ in J' and in $\boldsymbol{A}$. Moreover for any formula α in $\mathfrak{L}$, α is valid in R if and only if α is valid in R'. Viz.

$$\alpha_{R'}(fv) = \mathsf{V} \quad \text{if and only if} \quad \alpha_R(v) = \mathsf{V}.$$

This results immediately from VI, 6.8.

§ 3. Canonical models. The consistency and existence of models. Let $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ be a consistent theory and let $\mathfrak{A}(\mathfrak{T})$ be the **Q**-algebra of $\mathfrak{T}$. By 1.1, $\mathfrak{A}(\mathfrak{T})$ has at least two elements. Let h be a **Q**-homomorphism (see VI, § 11) from $\mathfrak{A}(\mathfrak{T})$ into a non-degenerate Boolean algebra $\boldsymbol{A}$. Let R^0 be the mapping defined on the set of all functors in $\mathfrak{L}$ and the set of all predicates in $\mathfrak{L}$ such that:

(a) R^0 restricted to the set of all functors in $\mathfrak{L}$ is the canonical realization of terms in $\mathfrak{L}$, i.e.

$$\varphi_{R^0} = \varphi \tag{1}$$

for any m-argument functor φ $(m = 0, 1, \ldots)$. More precisely (see VI, § 3, (1)) the mapping φ_{R^0}: $T^m \to T$ is defined by the equation

$$\varphi_{R^0}(\tau_1, \ldots, \tau_m) = \varphi(\tau_1 \ldots \tau_m)$$

where $\tau_1, \ldots, \tau_m$ are any terms in $\mathfrak{L}$.

(b) For every m-argument predicate ϱ in $\mathfrak{L}$ $(m = 1, 2, \ldots)$ and for arbitrary terms $\tau_1, \ldots, \tau_m$ in $\mathfrak{L}$

$$\varrho_{R^0}(\tau_1, \ldots, \tau_m) = h(\| \varrho(\tau_1 \ldots \tau_m) \|). \tag{2}$$

By definition, R^0 is a realization of $\mathfrak{L}$ in the set T of all terms and in the Boolean algebra $\boldsymbol{A}$. It follows from VI, 11.3 that R^0 is a realization of the type described in VI, § 7, p. 239 (in VI, § 7 take as $\approx$ the equivalence relation § 1, (1), i. e. VI, § 10, (6)). According to VI, § 7, p. 239, the realization R^0 will be called the *canonical realization* for the theory $\mathfrak{T}$, *determined by the* **Q**-*homomorphism h*. Thus we can apply theorem VI, 7.3, to R^0.

For this purpose we recall that any valuation $v\colon V \to T$ in the set T of all terms in $\mathfrak{L}$ can be considered as a substitution in $\mathfrak{L}$. Therefore, for any term τ in $\mathfrak{L}$, and for any formula α in $\mathfrak{L}$, the symbols $v\tau$ and $v'\alpha$ will denote respectively the results of the substitution v in τ and α respectively, according to VI, § 3, p. 216, and VI, § 4, p. 222 (see also VI, § 7, p. 235). In particular, for the identity valuation $\mathfrak{i} = \{x\}_{x \in V}$ (i.e. the identity mapping $\mathfrak{i}\colon V \to T$), we have

$$\mathfrak{i}\tau = \tau\,, \qquad \mathfrak{i}'\alpha = \alpha \tag{3}$$

for every term τ and every formula α in $\mathfrak{L}$.

The following theorem is a particular case of VI, 7.3 on account of VI, 11.3.

3.1. *If R^0 is the canonical realization for a consistent theory $\mathfrak{T}$, determined by a* Q*-homomorphism h from $\mathfrak{A}(\mathfrak{T})$ into a Boolean algebra $\boldsymbol{A}$, then*

$$a_{R^0}(v) = h(\|v'\alpha\|) \tag{4}$$

for every valuation $v\colon V \to T$ and for every formula α. In particular, for the identity valuation $\mathfrak{i}$

$$a_{R^0}(\mathfrak{i}) = h(\|\alpha\|)\,. \tag{5}$$

If α is a theorem in $\mathfrak{T}$, then the formula $v'\alpha$ is also a theorem in $\mathfrak{T}$ since the set of all theorems is closed under the rule of substitution (r_2). Thus $\|v'\alpha\| = V$ for every valuation v. By (4), this implies

3.2. *The canonical realization for a consistent theory $\mathfrak{T}$, determined by a* Q*-homomorphism h, is a model for $\mathfrak{T}$.*

In the definition of R^0 we may assume, in particular, that $\boldsymbol{A} = \mathfrak{A}(\mathfrak{T})$ and h is the identity mapping. Then R^0 will be called the *canonical model for $\mathfrak{T}$ in $\mathfrak{A}(\mathfrak{T})$* (it is a model by 3.2). We recall that the canonical model R^0 for $\mathfrak{T}$ in $\mathfrak{A}(\mathfrak{T})$ is defined by (a) and the following condition (b′) which is particular case of (b):

(b′) for every m-argument predicate ϱ in $\mathfrak{L}$ $(m = 1, 2, \ldots)$ and for any terms $\tau_1, \ldots, \tau_m$ in $\mathfrak{L}$,

$$\varrho_{R^0}(\tau_1, \ldots, \tau_m) = \|\varrho(\tau_1 \ldots \tau_m)\| \in \mathfrak{A}(\mathfrak{T})\,.$$

A model R for a theory is said to be *adequate* provided, for every formula α, α is a theorem in $\mathfrak{T}$ if and only if α is valid in R. Every consistent theory has adequate models, for instance:

3.3. *The canonical model for $\mathfrak{T}$ in $\mathfrak{A}(\mathfrak{T})$ is adequate.*

More generally,

If h is a Q*-isomorphism from $\mathfrak{A}(\mathfrak{T})$ into a Boolean algebra $\boldsymbol{A}$, then the canonical realization for $\mathfrak{T}$, determined by h, is an adequate model for $\mathfrak{T}$.*

It suffices to prove the second part of 3.3 because the first part is its particular case.

If a is valid in R^0, then $h(\|a\|) = V_A$ by (5). Since h is an isomorphism, we have $\|a\| = V_{\mathfrak{A}(\mathfrak{C})}$, i.e. a is a theorem by 1.1. Conversely if a is a theorem, then a is valid in R^0 by 2.2 and 3.2.

The following statements follow from 3.2, 3.3, 2.4 and 2.2.

3.4. *A theory $\mathfrak{C}$ is consistent if and only if there exists a model for $\mathfrak{C}$.*

3.5. *For any formula a in a consistent theory $\mathfrak{C} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$, the following conditions are equivalent*:

(i) *a is a theorem of $\mathfrak{C}$,*

(ii) *a is valid in every model for $\mathfrak{C}$,*

(iii) $a_{R^0}(i) = V$ *in the canonical model R^0 for $\mathfrak{C}$ in $\mathfrak{A}(\mathfrak{C})$.*

As an immediate consequence of 3.3 and 2.3, 2.5 we obtain the following corollary.

3.6. *For any formula a in a language $\mathfrak{L}$ the following conditions are equivalent*:

(i) *a is derivable in $\mathfrak{S} = \{\mathfrak{L}, \mathfrak{C}\}$*

(ii) *a is valid in every realization R of $\mathfrak{L}$*

(iii) $a_{R^0}(i) = V$ *in the canonical realization R^0 for $\mathfrak{S}$ in $\mathfrak{A}(\mathfrak{S})$.*

Let a formalized language $\mathfrak{L}'$ be an extension of a formalized language $\mathfrak{L}$. Under the above hypotheses the following theorem holds.

3.7. *Any formula a in $\mathfrak{L}$ is a theorem of a theory $\mathfrak{C} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ if and only if it is a theorem of $\mathfrak{C}' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A}\}$, i.e. any linguistic extension of $\mathfrak{C}$ is inessential.*

In consequence, $\mathfrak{C}$ is consistent if and only if $\mathfrak{C}'$ is consistent.

If a is a theorem in $\mathfrak{C}$, then clearly it is a theorem in $\mathfrak{C}'$. Suppose now that a is not a theorem in $\mathfrak{C}$. By 3.5, a is not valid in the canonical model R^0 for $\mathfrak{C}$ in $\mathfrak{A}(\mathfrak{C})$. Let R be an extension of the realization R^0 of $\mathfrak{L}$ to the realization of the language $\mathfrak{L}'$ in the same set T and in the same algebra $\mathfrak{A}(\mathfrak{C})$. By 2.7, R is a model for $\mathfrak{C}'$ and a is not valid in R. Thus a is not a theorem in $\mathfrak{C}'$ by 3 5

The second part of 3.7 results from the first.

§ 4. Semantic models. According to VI, § 6, a realization R of a language $\mathfrak{L}$ in a set $J \neq 0$ and in the two-element Boolean algebra A_0 is called a semantic realization of $\mathfrak{L}$ in J. If a semantic realization R of $\mathfrak{L}$ in J is a model for a theory $\mathfrak{C} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$, then R is said to be a *semantic model for $\mathfrak{C}$ in J.*

Semantic realizations of $\mathfrak{L}$ and semantic models for theories $\mathfrak{C}$ play an essential role in metamathematical investigations because, roughly

speaking, the notion of semantic realization coincides with the notion of interpretation of $\mathfrak{L}$, discussed in V, § 4. More precisely, there is a natural one-to-one correspondence between interpretations of $\mathfrak{L}$ in J and semantic realizations of $\mathfrak{L}$ in J. Namely, we have seen in VI, § 6, p. 228, that for every interpretation $\mathfrak{I}$ of $\mathfrak{L}$ in a set J the equalities:

$$\varphi_{\mathfrak{I}^*} = \varphi_{\mathfrak{I}} \quad \text{for every functor} \quad \varphi\,,$$

$$\varrho_{\mathfrak{I}^*} = \varrho^*_{\mathfrak{I}} \quad \text{for every predicate} \quad \varrho\,,$$

(where $\varrho^*_{\mathfrak{I}}$ is the characteristic of the propositional function $\varrho_{\mathfrak{I}}$) define a semantic realization $\mathfrak{I}^*$ of $\mathfrak{L}$ in J. On the other hand, for every semantic realization R of $\mathfrak{L}$ in J there exists exactly one interpretation $\mathfrak{I}$ of $\mathfrak{L}$ in J such that

$$R = \mathfrak{I}^*\,. \tag{1}$$

By VI, 6.1, if (1) holds, then for every term τ and for every formula α in $\mathfrak{L}$

$$\tau_R = \tau_{\mathfrak{I}}\,, \qquad \alpha_R = \alpha^*_{\mathfrak{I}}\,,$$

i.e. the function α_R is equal to the characteristic of the propositional function $\alpha_{\mathfrak{I}}$ associated by $\mathfrak{I}$ with α. In consequence (see V, § 7, p. 172) for any formula $\alpha(x_1, \ldots, x_n)$, where $x_1, \ldots, x_n$ are all free individual variables appearing in α, and for any valuation $v\colon V \to J$ the sentence

$$\alpha_{\mathfrak{I}}(v_{x_1}, \ldots, v_{x_n}) \quad \text{is true if and only if} \quad \alpha_R(v_{x_1}, \ldots, v_{x_n}) = \mathsf{V}\,.$$

Hence a valuation v satisfies α in the realization R if and only if the sentence $\alpha_{\mathfrak{I}}(v_{x_1}, \ldots, v_{x_n})$ is true, and α is valid in R if and only if the propositional function $\alpha_{\mathfrak{I}}$ is true in J. R is a semantic model for $\mathfrak{C}$ if and only if for every mathematical axiom α of $\mathfrak{C}$ (i.e. for every formula α in $\mathfrak{A}$) the propositional function $\alpha_{\mathfrak{I}}$ is true in J, i.e. if $\mathfrak{I}$ is a model for $\mathfrak{C}$ in the sense used in intuitive mathematics.

From V, § 7, p. 172, and VI, 6.1 follows

4.1. *A formula α in $\mathfrak{L}$ is a predicate tautology if and only if $\alpha_R = \mathsf{V}$ for every semantic realization R in every set $J \neq 0$, i.e. if and only if α is valid in every semantic realization.*

Semantic realizations have the following property which fails, in general, for non-semantic realizations.

4.2. *A formula $\alpha(x)$ is satisfiable in a semantic realization R if and only if the formula $\bigcup_\xi \alpha(\xi)$ is satisfiable in R.*

A little more generally: *a formula $\alpha(x_1, \ldots, x_n)$ is satisfiable in a semantic realization R if and only if the formula $\bigcup_{\xi_1} \ldots \bigcup_{\xi_n} \alpha(\xi_1, \ldots, \xi_n)$ is satisfiable in R.*

The first part of 4.2 follows easily from the inductive definitions of mappings β_R (see VI, § 6, (4)) and the fact that the join of elements in the two-element Boolean algebra is equal to V if and only if one of these elements is equal to V. The second part follows immediately from the first.

According to the convention in V, § 12, p. 192, we say, for brevity, that a theory $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ *contains a sign* $\mathfrak{e}$ *of equality* if $\mathfrak{e}$ is a two-argument predicate in $\mathfrak{L}$ and all the axioms $(\mathfrak{e}_1)$, $(\mathfrak{e}_2)$, $(\mathfrak{e}_3)$, $(\mathfrak{e}_4)$, $(\mathfrak{e}_5)$ from V, § 12 are included to the set $\mathcal{A}$ of mathematical axioms of $\mathfrak{T}$.

Suppose $\mathfrak{T}$ contains a sign $\mathfrak{e}$ of equality.

Let R be a semantic model for $\mathfrak{T}$ in a set $J \neq 0$. The realization R restricted to the set Φ of all functors in $\mathfrak{L}$ determines uniquely an abstract algebra $\{J, \{\varphi_R\}_{\varphi \in \Phi}\}$ (see VI, § 3). For any j_1, j_2 in J let us set

$$(2) \qquad j_1 \sim j_2 \quad \text{if and only if} \quad \mathfrak{e}_R(j_1, j_2) = V .$$

Thus $\mathfrak{e}_R$ is the characteristic of the relation $\sim$ in J. We shall prove

4.3. *The relation* $\sim$ *in* J *is a congruence relation in the algebra* $\{J, \{\varphi_R\}_{\varphi \in \Phi}\}$. *Moreover for any* m*-argument predicate* ϱ *in* $\mathfrak{L}$ $(m = 1, 2, \ldots)$ *the following condition holds*:

$$(3) \qquad \text{if } j_i \sim j_i' \text{ for } i = 1, \ldots, m, \text{ then } \varrho_R(j_1, \ldots, j_m) = \varrho_R(j_1', \ldots, j_m') .$$

Let j be an arbitrary element in J and let $v\colon V \to J$ be any valuation such that $v_x = j$. Since the formula $\mathfrak{e}(x\,x)$ is in $\mathcal{A}$,

$$\mathfrak{e}_R(j, j) = \mathfrak{e}(x\,x)_R(v) = V .$$

Hence $j \sim j$, for any j in J.

Suppose $j_1 \sim j_2$ for any j_1, j_2 in J. Let $v = V \to J$ be any valuation such that $v_x = j_1$ and $v_y = j_2$. Since the formula $(\mathfrak{e}(x\,y) \Rightarrow \mathfrak{e}(y\,x))$ is in $\mathcal{A}$, we have

$$\mathfrak{e}_R(j_1, j_2) \Rightarrow \mathfrak{e}_R(j_2, j_1) = (\mathfrak{e}(x\,y) \Rightarrow \mathfrak{e}(y\,x))_R(v) = V .$$

Since $j_1 \sim j_2$, $\mathfrak{e}_R(j_1, j_2) = V$. In consequence, $\mathfrak{e}_R(j_2, j_1) = V$, i.e. $j_2 \sim j_1$, which proves that $\sim$ is a symmetric relation.

Suppose that for any j_1, j_2, j_3 in J, $j_1 \sim j_2$ and $j_2 \sim j_3$. Thus

$$(4) \qquad \mathfrak{e}_R(j_1, j_2) = V \quad \text{and} \quad \mathfrak{e}_R(j_2, j_3) = V .$$

Let $v\colon V \to J$ be an arbitrary valuation such that $v_x = j_1$, $v_y = j_2$ and $v_z = j_3$. Since $(\mathfrak{e}_3)$ is in $\mathcal{A}$, i.e. the formula $(\mathfrak{e}(x\,y) \Rightarrow \mathfrak{e}(y\,z) \Rightarrow \mathfrak{e}(x\,z)))$, then

$$\mathfrak{e}_R(j_1, j_2) \Rightarrow \big(\mathfrak{e}_R(j_2, j_3) \Rightarrow \mathfrak{e}_R(j_1, j_3)\big) = ((\mathfrak{e}(x\,y) \Rightarrow (\mathfrak{e}(y\,z) \Rightarrow \mathfrak{e}(x\,z)))_R(v) = V .$$

Hence, by (4),

$$\mathfrak{e}_R(j_1, j_3) = V , \quad \text{i.e.} \quad j_1 \sim j_3 ;$$

which proves that $\sim$ is a transitive relation in J. Thus $\sim$ is an equivalence relation in J.

In order to show that $\sim$ is a congruence relation in the algebra $\{J, \{\varphi_R\}_{\varphi \in \Phi}\}$ it suffices to prove that for any m-argument functor φ in $\mathfrak{L}$ $(m = 0, 1, \ldots)$

(5) if $j_1 \sim j_1', \ \ldots, \ j_m \sim j_m'$ then $\varphi_R(j_1, \ldots, j_m) \sim \varphi_R(j_1', \ldots, j_m')$.

The proof of (5) and of (3) can be carried out in a quite analogous way to that used in the case of the statement that $\sim$ is an equivalence relation in J. Instead of (e_1), (e_2) and (e_3) we apply the axioms (e_4) and (e_5) respectively. The proof is left to the reader.

It follows from 4.3 and (2) that in a semantic model R in $J \neq 0$ for a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ with a sign $\mathfrak{e}$ of equality, the mapping $\mathfrak{e}_R$ associated with the sign $\mathfrak{e}$ by R is the characteristic of a congruence relation $\sim$ in $\{J, \{\varphi_R\}_{\varphi \in \Phi}\}$ satisfying condition (3). In general, the relation $\sim$ defined by (2) does not have to be the relation of equality. If in a semantic model R in $J \neq 0$ for a theory $\mathfrak{T}$ with equality, the function $\mathfrak{e}_R$ is the characteristic of the relation of equality (i.e. if $\sim$ defined by (2) is the relation of equality in J), then R will be called an *ordinary semantic model* for $\mathfrak{T}$ [1].

There is a close relationship between semantic models and ordinary semantic models. Namely, every semantic model R in J for a theory $\mathfrak{T}$ with equality determines in a natural way an ordinary semantic model R' in the set $J' = J/\sim$ such that any formula α is satisfiable in R if and only if it is satisfiable in R'. More exactly:

4.4. *If R is a semantic model in a set $J \neq 0$ for a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ with an equality sign $\mathfrak{e}$, let R' be the semantic realization in $J' = J/\sim$ (where $\sim$ is the relation (2)) defined by the equations*:

$$\varphi_{R'}(|j_1|, \ldots, |j_m|) = |\varphi_R(j_1, \ldots, j_m)| \tag{6}$$

for any m-argument functor φ in $\mathfrak{L}$ $(m = 0, 1, \ldots)$ and for any $j_1, \ldots, j_m$ in J;

$$\varrho_{R'}(|j_1|, \ldots, |j_m|) = \varrho_R(j_1, \ldots, j_m) \tag{7}$$

for any m-argument predicate ϱ in $\mathfrak{L}$ $(m = 1, 2, \ldots)$ and for any $j_1, \ldots, j_m$ in J. Then R' is an ordinary semantic model for $\mathfrak{T}$. Moreover, any formula β in $\mathfrak{L}$ is satisfiable (valid) in R if and only if it is satisfiable (valid) in R'.

According to the definition of R', for any j_1, j_2 in J

$$\mathfrak{e}_{R'}(|j_1|, |j_2|) = \mathfrak{e}_R(j_1, j_2).$$

[1] Other authors say "model" instead of "ordinary semantic model".

Observe that $|j_1| = |j_2|$ if and only if $j_1 \sim j_2$, i.e. if and only if $e_{R'}(|j_1|, |j_2|) = e_R(j_1, j_2) = V$. Thus the function $e_{R'}$ is the characteristic of the relation of equality in $J' = J/\sim$.

It follows from 4.3 that the definitions (6) and (7) are correct. Let f be the mapping from J onto $J/\sim$ defined as follows:

$$f(j) = |j| \in J/\sim \text{ for any } j \text{ in } J .$$

The mapping f satisfies the conditions VI, § 6, (11) and (12). Thus by VI, 6.8, for any formula α in $\mathfrak{L}$ and for every valuation v: $V \to J$ the following equation holds

$$a_{R'}(fv) = a_R(v)$$

where $fv = \{fv_x\}_{x \in V} = \{|v_x|\}_{x \in V}$. In consequence any formula α in $\mathfrak{L}$ is satisfiable in R if and only if it is satisfiable in R'. Analogously any formula α in $\mathfrak{L}$ is valid in R if and only if it is valid in R'. Thus R' is an ordinary semantic model for $\mathfrak{T}$.

Since $\overline{\overline{J'}} \leqslant \overline{\overline{J}}$, it follows from 4.4 that the existence of a semantic model R of a cardinal $\leqslant \mathfrak{m}$ for a theory $\mathfrak{T}$ with equality is equivalent to the existence of an ordinary semantic model R' of a cardinal $\leqslant \mathfrak{m}$ for $\mathfrak{T}$. Moreover, we may then assume that, for any formula α, α is satisfiable (valid) in R if and only if α is satisfiable (valid) in R'.

For our purpose it will be convenient to examine semantic models instead of ordinary semantic models since the use of the first notion leads to some simplifications of the proofs. However all the theorems on existence of semantic models of a cardinal $\mathfrak{m}$ or of a cardinal $\leqslant \mathfrak{m}$ to be proved later can also be formulated as theorems on the existence of ordinary semantic models of a cardinal $\leqslant \mathfrak{m}$.

§ 5. The existence of semantic enumerable models for enumerable theories. In this section we shall consider the case of *enumerable theories*, i.e. of theories $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ based on an at most enumerable set $\mathcal{A}$ of mathematical axioms. We shall prove that any consistent enumerable theory has a semantic (enumerable) model. The analogous theorem can be proved for every (even non-enumerable) theory. However, the proof of this theorem without any hypothesis concerning the cardinal of the set of mathematical axioms will be carried out by means of another, more complicated method. For this reason we treat the case of enumerable theories separately. Moreover, the theorem on the existence of semantic models for enumerable theories will be used in the proof of the general theorem on the existence of semantic models for arbitrary consistent theories.

5.1. *Let $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ be a formalized theory with an at most enumerable set $\mathcal{A}$ of mathematical axioms. If a formula α is irrefutable in $\mathfrak{T}$,*

then there exists an enumerable semantic model R for $\mathfrak{T}$, such that α is satisfiable in R.

First consider the case in which the language $\mathfrak{L}$ of $\mathfrak{T}$ is *enumerable*, i.e. the alphabet of $\mathfrak{L}$ contains enumerably many signs and consequently the set T of all terms and the set F of all formulas in $\mathfrak{T}$ are enumerable. Thus the set (Q) of all joins and meets corresponding to logical quantifiers in the Q-algebra $\mathfrak{A}(\mathfrak{T})$ is enumerable. Since α is irrefutable, we have $\|\alpha\| \neq \Lambda$. By II, 9.3, there exists a Q-filter ∇ in $\mathfrak{A}(\mathfrak{T})$ such that $\|\alpha\| \in \nabla$. By definition of Q-filters (see § 1, p. 281, and II, § 9, p. 86), the natural homomorphism h from $\mathfrak{A}(\mathfrak{T})$ onto the two-element Boolean algebra $\boldsymbol{A}_0 = \mathfrak{A}(\mathfrak{T})/\nabla$ is a Q-homomorphism. Let R^0 be the canonical realization of $\mathfrak{L}$ determined by the homomorphism h (see § 3, p. 286). By definition, R^0 is a realization in the enumerable set T of all terms in $\mathfrak{T}$. By 3.2, R^0 is a semantic model for $\mathfrak{T}$. By 3.1, (5), we have, for the identity valuation i,

$$\alpha_{R^0}(\mathrm{i}) = h(\|\alpha\|) = \nabla_{\boldsymbol{A}_0}$$

since $\|\alpha\| \in \nabla$. Thus α is satisfiable in R^0.

Consider now the case of an arbitrary language $\mathfrak{L}$ of the first order. Suppose that the set $\mathcal{A}$ of all mathematical axioms in the theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ is enumerable. Then there exists an enumerable sublanguage $\mathfrak{L}'$ of $\mathfrak{L}$ such that $\mathfrak{L}'$ contains all signs occurring in the formula α and in all formulas in $\mathcal{A}$. The theory $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A}\}$ is enumerable and α is irrefutable in $\mathfrak{T}'$. By the part of the theorem just proved, there exists an enumerable semantic model R^0 for $\mathfrak{T}'$ such that α is satisfiable in R^0. Let R be a realization of $\mathfrak{L}$ such that R is an extension of R^0. By 2.7, R is a semantic model for $\mathfrak{T}$ and α is satisfiable in R.

5.2. *Let $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ be a formalized theory with an at most enumerable set $\mathcal{A}$ of mathematical axioms. If a formula β in $\mathfrak{L}$ is not a theorem in $\mathfrak{T}$, then there exists an enumerable semantic model R for $\mathfrak{T}$ such that β is not valid in R.*

Theorem 5.2 follows immediately from 5.1 where α is the negation $-\beta$ (see § 2, (3), (4) and VI, 10.11). Observe that 5.1 also follows directly from 5.2 (substitute $-\alpha$ for β). Thus both theorems express the same mathematical contents.

5.3. *Every consistent theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ with an at most enumerable set $\mathcal{A}$ of mathematical axioms has an enumerable semantic model* (1).

(1) This theorem is due to Gödel [1]. Cf. also Herbrand [2], Hilbert and Bernays [1]. The above proof is due to Rasiowa and Sikorski [1], [2]. For proofs based on similar ideas see Henkin [1], Rieger [3], Łoś [5], Beth [1], Hasenjaeger [1], Reichbach [1].

This follows immediately from 5.1 or 5.2.

The next theorem explains in greater detail the connection between the consistency of a theory and the existence of models of various kinds.

5.4. *The following conditions are equivalent for any theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ *with an at most enumerable set* $\mathfrak{A}$ *of mathematical axioms*:

(i) $\mathfrak{T}$ *is consistent*;

(ii) *there exists a model for* $\mathfrak{T}$;

(iii) *there exists a model for* $\mathfrak{T}$ *in a set* $J \neq 0$ *and in the field* $\mathfrak{B}(X)$ *of all subsets of a set* $X \neq 0$;

(iv) *there exists an adequate model for* $\mathfrak{T}$;

(v) *there exists a semantic model for* $\mathfrak{T}$;

(vi) *there exists a semantic model for* $\mathfrak{T}$ *in an enumerable set* J [1].

In fact, (i) implies (vi) by 5.3, (vi) implies (v) and (v) implies (iii) since the two-element Boolean algebra is isomorphic with the field of all subsets of a one-element set X, (iii) implies (ii) in an obvious way and (ii) implies (i) by 2.4. Thus the conditions (i), (ii), (iii), (v), (vi) are equivalent. On the other hand, (iv) implies (i) by 2.4 and (i) implies (iv) by 3.2.

We shall formulate conditions necessary and sufficient for any formula to be a theorem of an enumerable theory.

5.5. *For every formula* α *in a consistent theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ *with an at most enumerable set* $\mathfrak{A}$ *of axioms the following conditions are equivalent*:

(i) α *is a theorem in* $\mathfrak{T}$;

(ii) α *is valid in every model for* $\mathfrak{T}$;

(iii) α *is valid in every model for* $\mathfrak{T}$ *in any complete Boolean algebra*;

(iv) α *is valid in every model for* $\mathfrak{T}$ *in the field* $\mathfrak{B}(X)$ *of all subsets of any set* $X \neq 0$;

(v) α *is valid in every semantic model for* $\mathfrak{T}$ *in any enumerable set*;

(vi) *there exists a non-degenerate Boolean algebra* $\boldsymbol{A}$ *and an infinite set* J *such that* α *is valid in every model for* $\mathfrak{T}$ *in* J *and* $\boldsymbol{A}$;

(vii) $a_{R^0}(\mathfrak{i}) = V$ *for the canonical model* R^0 *for* $\mathfrak{T}$ *in* $\mathfrak{A}(\mathfrak{T})$ *and the identity valuation* $\mathfrak{i}$.

(i) implies (ii) by theorem 2.2.

(ii) implies (iii) and (iii) implies (iv). This is obvious.

(iv) implies (v) since the two-element Boolean algebra can be considered as the field $\mathfrak{B}(X)$ where X is a one-point set.

---

[1] The theorem stating that (v) implies (vi) (for ordinary semantic models) was proved by Skolem [2] and is a generalization of a result of Löwenheim [1] (Löwenheim-Skolem theorem).

(v) implies (i). In fact, if (i) does not hold, then (v) does not hold by 5.2.

(ii) implies (vi). This is obvious.

(vi) implies (v). In fact, suppose that (v) does not hold, i.e. there exists a model R for $\mathfrak{T}$ in an enumerable set J_0 and the two-element Boolean algebra $\boldsymbol{A}_0$, such that a is not valid in R. We may assume that $\boldsymbol{A}_0$ consists of the unit element and the zero element of $\boldsymbol{A}$. Thus R can be considered as a model for $\mathfrak{T}$ in J_0 and $\boldsymbol{A}$. Since J is infinite, and J_0 is enumerable, there exists a mapping f from J onto J_0. Since f is a mapping onto, there exists a realization R' in J and $\boldsymbol{A}$ such that

$$f\big(\varphi_{R'}(j_1, \ldots, j_m)\big) = \varphi_R\big(f(j_1), \ldots, f(j_m)\big),$$

$$\varrho_{R'}(j_1, \ldots, j_m) = \varrho_R\big(f(j_1), \ldots, f(j_m)\big)$$

for every m-argument functor φ ($m = 0, 1, 2, \ldots$), for every m-argument predicate ϱ ($m = 1, 2, \ldots$) and for all $j_1, \ldots, j_m \in J$. By 2.8, R' is a model for $\mathfrak{T}$ in J and $\boldsymbol{A}$, and a is not valid in R', i.e. (vi) does not hold.

(i) is equivalent to (vii) since $a_{R^0}(\mathrm{i}) = \|a\| \in \mathfrak{A}(\mathfrak{T})$ for every formula a (see 3.1, (5), where h is the identity mapping) and since $\|a\| = \mathrm{V}$ if and only if a is a theorem (see 1.1).

§ 6. The completeness of the predicate calculi. Examples of tautologies. In this section we shall examine the predicate calculus $\mathfrak{S} = \{\mathfrak{L}, \mathrm{C}\}$. According to our remark in § 1 and in V, § 10, the predicate calculus $\mathfrak{S} = \{\mathfrak{L}, \mathrm{C}\}$ can be identified with the theory $\mathfrak{T} = \{\mathfrak{L}, \mathrm{C}, 0\}$. Therefore we can apply to $\mathfrak{S}$ the results obtained in previous sections concerning theories.

Since $\mathfrak{S}$ has the empty set of mathematical axioms, it follows from the definition of model for a theory that any realization R of the language $\mathfrak{L}$ is a model for $\mathfrak{S}$. Thus the notion of model coincides with the notion of realization in the case of predicate calculus $\mathfrak{S}$. On the other hand, the notion of theorem corresponds to the notion of derivable formula in $\mathfrak{S}$ (see § 1, p. 279). Note, moreover, that by 4.1, a formula a in $\mathfrak{L}$ is a predicate tautology if and only if it is valid in every semantic realization in every set $J \neq 0$.

The following theorem characterizes derivable formulas in $\mathfrak{S} = \{\mathfrak{L}, \mathrm{C}\}$.

6.1. *For every formula a in the predicate calculus $\mathfrak{S} = \{\mathfrak{L}, \mathrm{C}\}$ the following conditions are equivalent:*

(i) *a is derivable in $\mathfrak{S}$;*

(ii) *a is valid in every realization of $\mathfrak{L}$;*

(iii) *a is valid in every realization of $\mathfrak{L}$ in any complete Boolean algebra;*

(iv) *α is valid in every realization of $\mathfrak{L}$ in the field $\mathfrak{B}(X)$ of all subsets of any set $X \neq 0$;*

(v) *α is valid in every semantic realization of $\mathfrak{L}$ in any enumerable set;*

(vi) *there exists a non-degenerate Boolean algebra $\boldsymbol{A}$ and an infinite set J such that α is valid in any realization of $\mathfrak{L}$ in J and $\boldsymbol{A}$;*

(vii) $\alpha_{R^0}(\mathfrak{i}) = V$ *for the canonical realization R^0 of $\mathfrak{L}$ in $\mathfrak{A}(\mathfrak{S})$ and the identity valuation* $\mathfrak{i}$;

(viii) *α is a predicate tautology.*

Since the equivalence of (i)-(vii) is a particular case of 5.5, in order to prove 6.1 it suffices to observe that (iii) implies (viii) and (viii) implies (v).

The equivalence of (i) and (viii) is called the *completeness theorem* [1] for predicate calculi and is one of the most important theorems in metamathematics. The equivalence shows that the classical consequence operation C satisfies the condition (C') from V, § 10. In § 9 we shall prove that C also satisfies the stronger condition (C) from V, § 10.

By the equivalence of (i) and (viii), the meaning of the phrases "predicate tautology" and "formula derivable in $\mathfrak{S}$" is the same. In practice we shall use the first more frequently.

We shall now give some examples of tautologies. By V, 7.1 (or by 2.1), if δ is a propositional tautology in a language $\mathfrak{L}_0$ of zero order, and s is a substitution from $\mathfrak{L}_0$ into $\mathfrak{L}$, then the formula $s\delta$ (i.e. the result of the substitution s in δ) is a predicate tautology. Hence it follows that all formulas (T_0), (T_{13})-(T_{30}) mentioned on p. 260, where α, β, γ are any formulas in $\mathfrak{L}$, are predicate tautologies and, consequently, derivable in $\mathfrak{S}$.

We shall complete the last remark by the following two theorems.

6.2. *If a substitution s is a one-to-one mapping from the set V_0 of all propositional variables of a language $\mathfrak{L}_0$ of zero order into the set of all elementary formulas of a language $\mathfrak{L}$ of the first order, then for every formula δ in $\mathfrak{L}_0$, δ is a propositional tautology if and only if $s\delta$ is a predicate tautology.*

It suffices to prove that if δ is not a propositional tautology, then $s\delta$ is not a predicate tautology.

By VII, § 1, (10), there exists a valuation $v^0 = \{v_a^0\}_{a \in V_0} \in A_0^{V_0}$ (where A_0 is the two-element Boolean algebra) such that

$$\delta_{A_0}(v^0) = \Lambda .$$

We define a realization R of $\mathfrak{L}$ in the set T of all terms in $\mathfrak{L}$ as follows. R is a canonical realization of terms, i.e.

$$\varphi_R(\tau_1, \ldots, \tau_m) = \varphi(\tau_1 \ldots \tau_m)$$

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[1] The theorem stating the equivalence of (i) and (v) is another version of the Gödel completeness theorem [1].

for every m-argument functor φ ($m = 0, 1, 2, \ldots$). For every m-argument predicate ϱ ($m = 1, 2, \ldots$),

$$\varrho_R(\tau_1, \ldots, \tau_m) = \begin{cases} v_a^0 & \text{if } sa \text{ is the elementary formula } \varrho(\tau_1 \ldots \tau_m), \\ \mathbb{V} & \text{otherwise.} \end{cases}$$

Let $\mathfrak{i}: \boldsymbol{V} \to T$ be the identity valuation in $\mathfrak{k}$. We have

$$sa_R(\mathfrak{i}) = v_a^0$$

and consequently

$$(s\delta)_R(\mathfrak{i}) = \delta_{A_0}(s\mathfrak{i}) = \delta_{A_0}(v^0) = \Lambda$$

on account of VI, 6.9, since

$$s\mathfrak{i} = \{sa_R(\mathfrak{i})\}_{a \in V_0} = \{v_a^0\}_{a \in V_0} = v^0.$$

Since $s\delta_R$ is not identically equal to $\mathbb{V}$, $s\delta$ is not a predicate tautology, by 4.1.

Observe that in the above theorem it suffices to assume that s is one-to-one only on the set of all propositional variables which appear in δ.

Every open formula α in a formalized language $\mathfrak{k}$ of the first order is of the form $s\delta$, where s is a one-to-one mapping from a language $\mathfrak{k}_0$ of zero order (with an enumerable set V_0 of propositional variables) into the set of all elementary formulas in $\mathfrak{k}$. Hence, by 6.2, follows

6.3. *An open formula in a formalized language of the first order is a predicate tautology if and only if it is a substitution of a propositional tautology.*

We shall now quote predicate tautologies which are not results of substitutions performed on propositional tautologies.

6.4. *All formulas of any of the following forms are predicate tautologies.*

(T_{31}) $(\bigcap_\xi \alpha(\xi) \Rightarrow \alpha(\tau))$ *where* τ *is an arbitrary term,*

(T_{32}) $(\alpha(\tau) \Rightarrow \bigcup_\xi \alpha(\xi))$ *where* τ *is an arbitrary term,*

(T_{23}) $(\bigcap_\xi \alpha(\xi) \Rightarrow \bigcup_\xi \alpha(\xi))$,

(T_{34}) $(-\bigcap_\xi \alpha(\xi) \Rightarrow \bigcup_\xi -\alpha(\xi))$,

(T_{35}) $(\bigcup_\xi -\alpha(\xi) \Rightarrow -\bigcap_\xi \alpha(\xi))$,

(T_{36}) $(-\bigcup_\xi \alpha(\xi) \Rightarrow \bigcap_\xi -\alpha(\xi))$,

(T_{37}) $(\bigcap_\xi -\alpha(\xi) \Rightarrow -\bigcup_\xi \alpha(\xi))$,

(T_{38}) $(\bigcap_\xi (\alpha(\xi) \Rightarrow \beta) \Rightarrow (\bigcup_\xi \alpha(\xi) \Rightarrow \beta))$,

(T_{39}) $((\bigcup_\xi \alpha(\xi) \Rightarrow \beta) \Rightarrow \bigcap_\xi (\alpha(\xi) \Rightarrow \beta))$,

(T_{40}) $(\bigcap_\xi (\alpha \Rightarrow \beta(\xi)) \Rightarrow (\alpha \Rightarrow \bigcap_\xi \beta(\xi)))$,

(T_{41}) $((\alpha \Rightarrow \bigcap_\xi \beta(\xi)) \Rightarrow \bigcap_\xi (\alpha \Rightarrow \beta(\xi)))$,

(T_{42}) $(\bigcup_\xi (\alpha(\xi) \Rightarrow \beta) \Rightarrow (\bigcap_\xi \alpha(\xi) \Rightarrow \beta))$,

(T_{43}) $((\bigcap_\xi \alpha(\xi) \Rightarrow \beta) \Rightarrow \bigcup_\xi (\alpha(\xi) \Rightarrow \beta))$,

(T_{44}) $(\bigcup_\xi (\alpha \Rightarrow \beta(\xi)) \Rightarrow (\alpha \Rightarrow \bigcup_\xi \beta(\xi)))$,

(T_{45}) $((\alpha \Rightarrow \bigcup_\xi \beta(\xi)) \Rightarrow \bigcup_\xi (\alpha \Rightarrow \beta(\xi)))$,

(T_{46}) $(\bigcap_\xi (\alpha \cap \beta(\xi)) \Rightarrow (\alpha \cap \bigcap_\xi \beta(\xi)))$,

(T_{47}) $((\alpha \cap \bigcap_\xi \beta(\xi)) \Rightarrow \bigcap_\xi (\alpha \cap \beta(\xi)))$,

(T_{48}) $(\bigcap_\xi (\alpha \cup \beta(\xi)) \Rightarrow (\alpha \cup \bigcap_\xi \beta(\xi)))$,

(T_{49}) $((\alpha \cup \bigcap_\xi \beta(\xi)) \Rightarrow \bigcap_\xi (\alpha \cup \beta(\xi)))$,

(T_{50}) $(\bigcup_\xi (\alpha \cap \beta(\xi)) \Rightarrow (\alpha \cap \bigcup_\xi \beta(\xi)))$,

(T_{51}) $((\alpha \cap \bigcup_\xi \beta(\xi)) \Rightarrow \bigcup_\xi (\alpha \cap \beta(\xi)))$,

(T_{52}) $(\bigcup_\xi (\alpha \cup \beta(\xi)) \Rightarrow (\alpha \cup \bigcup_\xi \beta(\xi)))$,

(T_{53}) $((\alpha \cup \bigcup_\xi \beta(\xi)) \Rightarrow \bigcup_\xi (\alpha \cup \beta(\xi)))$,

(T_{54}) $(\bigcap_\xi (\alpha(\xi) \Rightarrow \beta(\xi)) \Rightarrow (\bigcap_\xi \alpha(\xi) \Rightarrow \bigcap_\xi \beta(\xi)))$,

(T_{55}) $(\bigcap_\xi (\alpha(\xi) \Rightarrow \beta(\xi)) \Rightarrow (\bigcup_\xi \alpha(\xi) \Rightarrow \bigcup_\xi \beta(\xi)))$,

(T_{56}) $(\bigcap_\xi (\alpha(\xi) \cap \beta(\xi)) \Rightarrow (\bigcap_\xi \alpha(\xi) \cap \bigcap_\xi \beta(\xi)))$,

(T_{57}) $((\bigcap_\xi \alpha(\xi) \cap \bigcap_\xi \beta(\xi)) \Rightarrow \bigcap_\xi (\alpha(\xi) \cap \beta(\xi)))$,

(T_{58}) $(\bigcup_\xi (\alpha(\xi) \cup \beta(\xi)) \Rightarrow (\bigcup_\xi \alpha(\xi) \cup \bigcup_\xi \beta(\xi)))$,

(T_{59}) $((\bigcup_\xi \alpha(\xi) \cup \bigcup_\xi \beta(\xi)) \Rightarrow \bigcup_\xi (\alpha(\xi) \cup \beta(\xi)))$,

(T_{60}) $((\bigcap_\xi \alpha(\xi) \cup \bigcap_\xi \beta(\xi)) \Rightarrow \bigcap_\xi (\alpha(\xi) \cup \beta(\xi)))$,

(T_{61}) $(\bigcup_\xi (\alpha(\xi) \cap \beta(\xi)) \Rightarrow (\bigcup_\xi \alpha(\xi) \cap \bigcup_\xi \beta(\xi)))$,

(T_{62}) $(\bigcup_\xi \bigcap_\eta \alpha(\xi, \eta) \Rightarrow \bigcap_\eta \bigcup_\xi \alpha(\xi, \eta))$,

(T_{63}) $(\bigcap_\xi \bigcap_\eta \alpha(\xi, \eta) \Rightarrow \bigcap_\eta \bigcap_\xi \alpha(\xi, \eta))$,

(T_{64}) $(\bigcup_\xi \bigcup_\eta \alpha(\xi, \eta) \Rightarrow \bigcup_\eta \bigcup_\xi \alpha(\xi, \eta))$.

The tautology (T_{31}) is called the *dictum de omni* law. The formulas (T_{34})-(T_{37}) are *De Morgan's laws*. The tautologies (T_{38})-(T_{45}) are *laws on a transfer of quantifiers across* $\Rightarrow$. The tautologies (T_{46})-(T_{53}) are *laws on a transfer of quantifiers* across $\cap$ and $\cup$. The formulas (T_{54})-(T_{61}) are *distributivity laws*. The formulas (T_{62})-(T_{64}) are *laws on alterations of quantifiers*.

The validity of each of the formulas (T_{31})-(T_{64}) in any realization in any set $J \neq 0$ and in any Boolean algebra A in which such a realization can be defined can easily be proved by using corresponding laws from the theory of Boolean algebras. We indicate below the number of statements in Chapters I and II from which the validity of the formulas follows:

(T_{31}) I, 7.1, (3);	(T_{32}) I, 7.1, (2);	(T_{33}) I, 7.1, (6);
(T_{34}) II, 1.4, (20);	(T_{35}) II, 1.4, (20);	(T_{36}) II, 1.4, (19);
(T_{37}) II, 1.4, (19);	(T_{38}) II, 1.4, (25);	(T_{39}) II, 1.4, (25);
(T_{40}) II, 1.4, (28);	(T_{41}) II, 1.4, (28);	(T_{42}) II, 1.4, (26);
(T_{43}) II, 1.4, (26);	(T_{44}) II, 1.4, (27);	(T_{45}) II, 1.4, (27);
(T_{46}) I, 7.1, (10);	(T_{47}) I, 7.1, (10);	(T_{48}) II, 1.4, (24);

(T_{49}) II, 1.4, (24); (T_{50}) II, 1.4, (23); (T_{51}) II, 1.4, (23);
(T_{52}) I, 7.1, (9); (T_{53}) I, 7.1, (9); (T_{54}) II, 1.4, (29);
(T_{55}) II, 4.1, (30); (T_{56}) I, 7.1, (15); (T_{57}) I, 7.1, (15);
(T_{58}) I, 7.1, (14); (T_{59}) I, 7.1, (14); (T_{60}) I, 7.1, (16);
(T_{61}) I, 7.1, (17); (T_{62}) I. 7.1, (13); (T_{63}) I, 7.1, (12);
(T_{64}) I, 7.1, (11).

By 6.1 all the formulas (T_{31})-(T_{64}) have formal proofs in S, i.e. they can be derived from (T_1)-(T_{12}) by means of the rules of inference (r_1)-(r_6) (see V, § 8). We shall not quote here any formal proofs for them. In § 7 we shall explain a general method of finding a formal proof for any tautology.

6.5. *If formulas α and β are congruent, then the implications $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are predicate tautologies.*

Indeed, for any realization R in any Boolean algebra, we have $\alpha_R(v) = \beta_R(v)$ by VI, 6.2. Consequently

$$(\alpha \Rightarrow \beta)_R(v) = \alpha_R(v) \Rightarrow \beta_R(v) = V .$$

Thus $(\alpha \Rightarrow \beta)$ is valid in every realization, i.e. it is a predicate tautology. Similarly we verify that $(\beta \Rightarrow \alpha)$ is a predicate tautology. Another proof of 6.5 follows from VI, 11.3, (5).

Theorem 6.1 gives a simple method of how to verify whether a given formula α is derivable in S or not [1]. To prove that α is not derivable in S, it suffices to find a realization in which α is not valid. To prove that α is derivable it suffices to verify that α is a predicate tautology or that α satisfies one of the conditions (ii)-(vii) in 6.1. However the last method enables us to state that α is derivable but it does not give any formal proof for α in S. At the end of the next section we shall quote a general method which enables us to find a formal proof in S for every predicate tautology α.

§ 7. Diagrams of formulas. In this section we shall prove theorems analogous to theorems VII, 6.1, 6.2. The first theorem can be considered, similarly as theorem VII, 6.1, as an (infinite) method of verification of whether a given formula α_0 is a predicate tautology.

We shall consider a fixed language $\mathfrak{L}$ of the first order such that the set of terms in $\mathfrak{L}$ is enumerable. This hypothesis is not an essential restriction since while examining a single formula α_0 we can always replace the whole language by a sublanguage having an enumerable set of terms and such that α_0 belongs to this sublanguage.

[1] However, there is no decision procedure for determining by means of a finite number of steps whether a given formula is derivable in S or not (Church [1]).

Let

(1) $$\tau_1, \tau_2, \ldots$$

be a fixed infinite sequence containing every term in $\mathfrak{L}$ exactly once. In particular every free individual variable appears in (1) exactly once.

The following notation and definitions are analogous to those in VII, § 6.

The letter Γ (with indices, if necessary) will always denote finite sequences

(2) $$\alpha_1, \ldots, \alpha_m$$

of formulas in $\mathfrak{L}$. The empty sequence (i.e. the case of $m = 0$) is admitted.

If Γ is the sequence (2), then δ_Γ will denote the disjunction $\bigcup_{i=1}^{m} \alpha_i$.

If Γ is a sequence (2), Γ' is a sequence

$$\beta_1, \ldots, \beta_n$$

and α, β are any formulas, then Γ, α, Γ' and $\Gamma, \alpha, \beta, \Gamma'$ and $\Gamma, \alpha, \Gamma', \beta$ will denote the sequences

$$\alpha_1, \ldots, \alpha_m, \alpha, \beta_1, \ldots, \beta_n$$

and

$$\alpha_1, \ldots, \alpha_m, \alpha, \beta, \beta_1, \ldots, \beta_n$$

and

$$\alpha_1, \ldots, \alpha_m, \alpha, \beta_1, \ldots, \beta_n, \beta$$

respectively.

A formula α is said to be *indecomposable* if either it is elementary or it is the negation of an elementary formula. Otherwise it is called *decomposable*.

A sequence Γ of formulas is said to be *indecomposable* if it is formed exclusively of indecomposable formulas. Otherwise it is said to be *decomposable*.

In particular, the empty sequence is *indecomposable*.

A sequence Γ is said to be *fundamental* if it contains simultaneously a formula α and its negation $-\alpha$.

By a *schema* we shall understand a pair $\{\Gamma, \Gamma^0\}$ or a triple $\{\Gamma, \Gamma^0, \Gamma^1\}$ (of non-empty sequences of formulas) usually written in the form

(3) $$\frac{\Gamma}{\Gamma^0}$$

or

(4) $$\frac{\Gamma}{\Gamma^0 ; \Gamma^1}$$

respectively. Γ is called the *conclusion* of the schema, and Γ^0, Γ^1 are called its *premises*. If the schema is of the form (4), then Γ^0 and Γ^1 are called the *left* and the *right premise*, respectively.

In the sequel we shall consider only the following schemas where Γ' always denotes an indecomposable sequence (which may be empty):

(D) $$\frac{\Gamma', (\alpha \cup \beta), \Gamma''}{\Gamma', \alpha, \beta, \Gamma''},$$

(−D) $$\frac{\Gamma', -(\alpha \cup \beta), \Gamma''}{\Gamma', -\alpha, \Gamma'' ; \Gamma', -\beta, \Gamma''},$$

(C) $$\frac{\Gamma', (\alpha \cap \beta), \Gamma''}{\Gamma', \alpha, \Gamma'' ; \Gamma', \beta, \Gamma''},$$

(−C) $$\frac{\Gamma', -(\alpha \cap \beta), \Gamma''}{\Gamma', -\alpha, -\beta, \Gamma''},$$

(I) $$\frac{\Gamma', (\alpha \Rightarrow \beta), \Gamma''}{\Gamma', -\alpha, \beta, \Gamma''},$$

(−I) $$\frac{\Gamma', -(\alpha \Rightarrow \beta), \Gamma''}{\Gamma', \alpha, \Gamma'' ; \Gamma', -\beta, \Gamma''},$$

(−N) $$\frac{\Gamma', --\alpha, \Gamma''}{\Gamma', \alpha, \Gamma''},$$

(E) $$\frac{\Gamma', \bigcup_\xi \alpha(\xi), \Gamma''}{\Gamma', \alpha(\tau), \Gamma'', \bigcup_\xi \alpha(\xi)}$$ where τ is a term,

(−E) $$\frac{\Gamma', -\bigcup_\xi \alpha(\xi), \Gamma''}{\Gamma', \bigcap_\xi -\alpha(\xi), \Gamma''},$$

(U) $$\frac{\Gamma', \bigcap_\xi \alpha(\xi), \Gamma''}{\Gamma', \alpha(x), \Gamma''}$$ where x is a free individual variable which does not appear in any formula in the conclusion

(−U) $$\frac{\Gamma', -\bigcap_\xi \alpha(\xi), \Gamma''}{\Gamma', \bigcup_\xi -\alpha(\xi), \Gamma''}.$$

Note that the schemas

(5) (D), (−C), (I), (−N), (E), (−E), (U), (−U)

are of type (3), i.e. they have only one premise. The schemas

(6) (−D), (C), (−I)

are of type (4), i.e. they have two premises. If a sequence Γ of formulas is decomposable, then Γ is the conclusion of one of the schemas (5) or (6). The hypothesis that Γ' denotes exclusively an indecomposable sequence implies that Γ is then the conclusion of exactly one of the schemas (5), (6).

The letters i, j will always denote finite sequences

(7) $$i_1, \ldots, i_n$$

or infinite sequences

(8) $$i_1, i_2, \ldots$$

composed of the numbers 0 and 1.

We shall write $i \leqslant j$ if i is an initial (proper or improper) segment of j, i.e. if j is an extension of i. If i is the sequence (7), then $i, 0$ and $i, 1$ will denote the sequences

$$i_1, \ldots, i_n, 0 \quad \text{and} \quad i_1, \ldots, i_n, 1$$

respectively. The empty sequence (the case of $n = 0$ in (7)) is admitted and will be denoted by O. By definition, $O \leqslant j$ for every j.

By the *diagram* of a formula α_0 in $\mathfrak{L}$ we shall mean any mapping which, with some finite sequences i, associates certain non-empty finite sequences Γ_i of formulas, and which is defined by induction as follows:

1) Γ_O is the sequence formed only of the formula α_0.

2) If Γ_i is defined but is either fundamental or indecomposable, then $\Gamma_{i,0}$ and $\Gamma_{i,1}$ are not defined.

3) If Γ_i is defined and is decomposable but not fundamental, then Γ_i is the conclusion of exactly one of the schemas (5), (6). If Γ_i is the conclusion of a schema (5), then $\Gamma_{i,0}$ and $\Gamma_{i,1}$ are the left and the right premises of this schema. If Γ_i is the conclusion of a schema (5), then $\Gamma_{i,0}$ is the only premise of this schema and $\Gamma_{i,1}$ is not defined. Moreover, if this schema is (E), we assume additionally that the term τ mentioned in (E) is the first term in the sequence (1) such that $\alpha(\tau)$ does not appear in any sequence Γ_j with $j \leqslant i$. If this schema is (U), we assume additionally that the variable x mentioned in (U) is the first individual variable in the sequence (1) such that x does not appear in any formula in Γ_i.

4) If Γ_i is not defined, then $\Gamma_{i,0}$ and $\Gamma_{i,1}$ are not defined.

The diagram $\{\Gamma_i\}$ of α_0 is uniquely determined by α_0. The diagram $\{\Gamma_i\}$ is said to be *finite* (*infinite*) if the set of all sequences i for which Γ_i is defined is finite (infinite).

Γ_i is said to be an *end sequence* of the diagram of α_0 if Γ_i is either fundamental or indecomposable, i.e. if $\Gamma_{i,0}$ and $\Gamma_{i,1}$ are not defined.

7.1. *A formula α_0 is a predicate tautology if and only if the diagram of α_0 is finite and all the end sequences are fundamental.*

Let $\{\Gamma_i\}$ be the diagram of α_0.

It is easy to see that if Γ_i is an end sequence, then the disjunction δ_{Γ_i} (see p. 300) is a predicate tautology if and only if Γ_i is fundamental.

On the other hand, if Γ_i is not an end sequence, then Γ_i is the conclusion of exactly one of the schemas (5), (6). If Γ_i is the conclusion of one of the schemas (5), then δ_{Γ_i} is a predicate tautology if and only if $\delta_{\Gamma_{i,0}}$ is a predicate tautology. If Γ_i is the conclusion of one of the schemas (6), then δ_{Γ_i} is a predicate tautology if and only if both $\delta_{\Gamma_{i,0}}$ and $\delta_{\Gamma_{i,1}}$ are predicate tautologies. A simple proof of those statements can be given, for instance when we pass to the algebra $\mathfrak{A}(\mathfrak{S})$ of the predicate calculus $\mathfrak{S} = \{\mathfrak{L}, \mathfrak{C}\}$ (see similar arguments in VII, p. 267).

It follows from the above consideration that when the diagram of α_0 is finite, the formula α_0 is a predicate tautology if and only if all end sequences in its diagram are fundamental.

To complete the theorem it suffices to show that if the diagram of α_0 is infinite then α_0 is not a tautology.

Suppose that the diagram of α_0 is infinite. Then there exists an infinite sequence $\boldsymbol{j}$ such that, for every finite sequence $\boldsymbol{i} \leqslant \boldsymbol{j}$, Γ_i is defined. Let F' be the set of all indecomposable formulas appearing in at least one Γ_i, $\boldsymbol{i} \leqslant \boldsymbol{j}$. Note that if $\boldsymbol{i} \leqslant \boldsymbol{i}' \leqslant \boldsymbol{j}$ and an indecomposable formula appears in Γ_i, then it also appears in $\Gamma_{i'}$. Since no Γ_i ($\boldsymbol{i} \leqslant \boldsymbol{j}$) is fundamental, for every elementary formula β at most one of the formulas β and $-\beta$ is in F'.

Let R be the canonical semantic realization of $\mathfrak{L}$ in the set T of all terms, such that for every predicate ϱ

$$\varrho_R(\tau', \tau'', \ldots) = \begin{cases} \Lambda & \text{if the elementary formula } \varrho(\tau'\,\tau''\ldots) \text{ is in } F', \\ V & \text{otherwise.} \end{cases}$$

Let $\mathfrak{i}$ be the identity valuation. By definition (see VI, 7.1, (2')),

$$\beta_R(\mathfrak{i}) = \Lambda \tag{9}$$

for every indecomposable formula in F'.

To prove that α_0 is not a tautology it suffices to show that

$$\alpha_{0R}(\mathfrak{i}) = \Lambda\,. \tag{10}$$

For this purpose let us assume the following inductive definition of the *order* $\operatorname{ord}\alpha$ of a formula α. If α is an elementary formula, then $\operatorname{ord}\alpha = 1$. If $\operatorname{ord}\alpha = n$, then $\operatorname{ord}-\alpha = n+1$. If $\operatorname{ord}\alpha \leqslant n$ and $\operatorname{ord}\beta \leqslant n$ and the equality holds either for α or for β, then $\operatorname{ord}(\alpha \cup \beta) = \operatorname{ord}(\alpha \cap \beta) = \operatorname{ord}(\alpha \Rightarrow \beta) = n+1$. If $\operatorname{ord}\alpha(x) = n$, then $\operatorname{ord}\bigcup_\xi \alpha(\xi) = \operatorname{ord}\bigcap_\xi \alpha(\xi) = n+1$.

Suppose that (10) does not hold, i.e. that $\alpha_{0R}(\mathfrak{i}) = V$. Thus the set F'' of all formulas α such that α appears in one of the sequences $\Gamma_{\mathfrak{i}}$ with $\mathfrak{i} \leqslant \mathfrak{j}$ and

$$\alpha_R(\mathfrak{i}) = V \tag{11}$$

is not empty because α_0 belongs to F''. Let α' be a formula in F'' such that $\operatorname{ord}\alpha' \leqslant \operatorname{ord}\alpha$ for every formula α in F''. There exists an $\mathfrak{i} \leqslant \mathfrak{j}$ such that $\Gamma_{\mathfrak{i}}$ is of the form $\Gamma', \alpha', \Gamma''$ with an indecomposable Γ'.

We may suppose that α' is not of the form

$$-\bigcup_\xi \alpha(\xi) \quad \text{or} \quad -\bigcap_\xi \alpha(\xi)\,, \tag{12}$$

for if (12) is in F'', then also the formula $\bigcap_\xi -\alpha(\xi)$ or $\bigcup_\xi -\alpha(\xi)$ is in F'' and the orders of the two formulas are equal.

The formula α' cannot be of one of the forms

$$\begin{gathered}(\alpha \cup \beta),\ -(\alpha \cup \beta)\,, \quad (\alpha \cap \beta),\ -(\alpha \cap \beta)\,,\\ (\alpha \Rightarrow \beta)\,, \quad -(\alpha \Rightarrow \beta),\ --\alpha,\ \bigcap_\xi \alpha(\xi)\,.\end{gathered} \tag{13}$$

For if α' is one of the forms (13) and $\mathfrak{i}'$ is the successor of $\mathfrak{i}$ in $\mathfrak{j}$ (i.e. $\mathfrak{i}'$ is a sequence obtained from $\mathfrak{i}$ by adding 0 or 1 at the end, such that $\mathfrak{i}' \leqslant \mathfrak{j}$), then $\Gamma_{\mathfrak{i}'}$ is of the form $\Gamma', \alpha'', \Gamma''$ with α'' in F'' and $\operatorname{ord}\alpha'' < \operatorname{ord}\alpha'$, which contradicts the definition of α'.

The formula α' cannot be of the form

$$\bigcup_\xi \alpha(\xi) \tag{14}$$

since then there exists a term τ and an $\mathfrak{i}''$ such that $\mathfrak{i} \leqslant \mathfrak{i}'' \leqslant \mathfrak{j}$, $\alpha'(\tau)$ appears in $\Gamma_{\mathfrak{i}''}$, and the formula $\alpha(\tau)$ satisfies (11). Thus $\alpha(\tau)$ belongs to F'' and $\operatorname{ord}\alpha(\tau) < \operatorname{ord}\alpha'$, which contradicts the definition of α'.

Since α' is not of the form (12), (13), (14), α' is indecomposable. Thus α' is in F' and consequently (9) holds. On the other hand, α' is one of the formulas α satisfying (11). Thus the hypothesis that (10) is not true leads to a contradiction.

This completes the proof of theorem 7.1. We shall now formulate some remarks analogous to those in the second part of VII, § 6.

If

$$\frac{\Gamma}{\overline{\overline{\Gamma^0}}} \quad \text{or} \quad \frac{\Gamma}{\overline{\overline{\Gamma^0\,;\,\Gamma^1}}} \tag{15}$$

is one of the schemas (5) or (6) respectively, then

$$\frac{\delta_{\Gamma^0}}{\delta_\Gamma} \quad \text{or} \quad \frac{\delta_{\Gamma^0}\,,\ \delta_{\Gamma^1}}{\delta_\Gamma} \tag{16}$$

is a rule of inference in the sense defined in V, § 8, p. 173. By the symbols

(17)

$$(D^*),\ (-D^*),\ (C^*),\ (-C^*),\ (I^*),\ (-I^*),\ (-N^*),\ (E^*),\ (-E^*),\ (U^*),\ (-U^*)$$

we shall denote the rules of inference corresponding to the schemas (D), (−D), (C), (−C), (I), (−I), (−N), (E), (−E), (U), (−U), respectively.

A formula in $\mathfrak{L}$ is said to be *fundamental* if it is of the form δ_Γ, where Γ is a fundamental sequence.

7.2. *The set of all predicate tautologies is the least set containing all fundamental formulas and closed under the rules of inference* (17) [1].

In fact, every fundamental formula is a predicate tautology. Since rules of inference lead from tautologies to tautologies (see V, 8.1), every formula in the least set containing all fundamental formulas and closed under (17) is a predicate tautology.

On the other hand, if α_0 is a predicate tautology, then all end sequences in the diagram $\{\Gamma_i\}$ of α_0 are fundamental. We can arrange all indexes i appearing in the diagram in a finite sequence

$$i_1, \ldots, i_k \tag{18}$$

such that if $i \leqslant j$ $(i \neq j)$, then j precedes i in the sequence (18). By definition, i_k is the empty sequence O. It follows from the arrangement of (18) that the sequence of formulas

$$\delta_{\Gamma_{i_1}}, \ldots, \delta_{\Gamma_{i_k}} \tag{19}$$

is a formal proof of α_0 from the set of fundamental formulas by means of the rules of inference (17). This proves that every predicate tautology α_0 is in the least set containing all fundamental formulas and closed under (17).

We see from the second part of the proof of 7.2 that, for every given predicate tautology α_0, we can effectively find a formal proof of α_0 from the set of all fundamental formulas by means of the rules of inference (17). More precisely, the diagram of the formula α_0 is its formal proof (from fundamental formulas by means of (17)) but written in a way different from (19). We would like to emphasize the important property of the proof represented by the diagram: the process of deduction leads here from simpler formulas to more complicated ones, and all formulas appearing in the formal proof are parts or substitutions of parts of the formula α_0.

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[1] Theorems 7.1-7.2 are a modification of a theorem of Gentzen [1]. This modification is due to Schütte [1]. Cf. also Kanger [1]. The proof quoted is due to Rasiowa and Sikorski [8]. For another modification, see Beth [2] and [4]. See also Craig [1].

In § 6 we proved that if a formula α_0 is a predicate tautology, then it is derivable in $\mathcal{S} = \{\mathfrak{L}, \mathfrak{C}\}$, i. e. there exists a formal proof of α_0 from axioms (T_1)-(T_{12}) (see p. 279) by means of the rules of inference (r_1)-(r_6) mentioned in V, § 8. However we did not quote any method of construction of such a proof. The diagram of α_0 can also be used to yield such a construction.

The construction is based on the fact that for every fundamental formula β we can effectively find a formal proof of β from (T_1)-(T_{12}) by means of (r_1)-(r_6). Moreover, if δ_Γ is obtained from $\delta_{\Gamma 0}$ or from $\delta_{\Gamma 0}$ and $\delta_{\Gamma 1}$ (see (15) and (16)) by means of one of the rules (17), then we can effectively give a formal proof of δ_Γ from (T_1)-(T_{12}) and $\delta_{\Gamma 0}$ or from (T_1)-(T_{12}) and δ_{Γ_0}, $\delta_{\Gamma 1}$ respectively, by means of (r_1)-(r_6). We shall not give these formal proofs in detail because their description is rather long.

If α_0 is a tautology, then its diagram is finite and every end sequence is fundamental. Thus for every end sequence Γ_i we can write a formal proof of δ_{Γ_i} from (T_1)-(T_{12}) by means of (r_1)-(r_6). If Γ_i is not an end sequence, we can write a formal proof of δ_{Γ_i} from (T_1)-(T_{12}) and $\delta_{\Gamma_{i,0}}$, or from (T_1)-(T_{12}) and $\delta_{\Gamma_{i,0}}$, $\delta_{\Gamma_{i,1}}$, respectively, by means of (r_1)-(r_6). By a suitable arrangement of all formulas appearing in these formal proofs we get, with the help of (18), a formal proof of α_0 from (T_1)-(T_{12}) by means of (r_1)-(r_6).

§ 8. Rich theories. We recall that a formula α (in a formalized language of the first order) is said to be existential if and only if the existential quantifier $\bigcup$ is the first sign of α, i.e. if α is of the form $\bigcup_\xi \beta(\xi)$.

A formalized theory $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A}'\}$ of the first order is said to be *rich* provided, for every existential formula $\bigcup_\xi \beta(\xi)$ in $\mathfrak{L}'$, there exists a term τ in $\mathfrak{L}'$ such that the formula

$$(\bigcup_\xi \beta(\xi) \Rightarrow \beta(\tau))$$

is a theorem in $\mathfrak{T}'$.

The aim of the first part of this section is to prove that every consistent theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ has a rich inessential extension $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A}'\}$ (see theorem 8.3 below).

First we shall construct the language $\mathfrak{L}'$. For this purpose we define by induction an enumerable sequence of formalized languages $\mathfrak{L}_n = \{A_n, T_n, F_n\}$ $(n = 0, 1, 2, \ldots)$ and an enumerable sequence of functions ψ^n $(n = 0, 1, 2, \ldots)$, such that

1) $\mathfrak{L}_0$ is identical with $\mathfrak{L}$;

2) $\mathfrak{L}_{n+1}$ is an extension of $\mathfrak{L}_n$ obtained by adding a set Ψ_n of functors $(n = 0, 1, 2, \ldots)$, the set Ψ_n being disjoint with the set S_n of all signs in $\mathfrak{L}_n$;

3) ψ^n is a one-to-one mapping from the set E_n onto Ψ_n, where E_n is the set of all existential formulas in $\mathfrak{L}_n$ which do not occur in $\mathfrak{L}_{n-1}$, such that if α is an existential formula in E_n with m free individual variables, then the image ψ^n_α of α is an m-argument functor.

Roughly speaking, $\mathfrak{L}'$ is the union of all the languages $\mathfrak{L}_n$. More exactly, the language $\mathfrak{L}' = \{A', T', F'\}$ in question is the extension of $\mathfrak{L}$ obtained by adding the union Ψ of sets

$$\Psi_0, \Psi_1, \Psi_2, \ldots$$

to the set Φ of all functors in $\mathfrak{L}$. By definition, the set E' of all existential formulas in $\mathfrak{L}'$ is the union of disjoint sets E_n, $n = 0, 1, 2, \ldots$; by 3), all the mappings ψ^n $(n = 0, 1, 2, \ldots)$ determine together a one-to-one mapping ψ from E' onto Ψ, such that

4) ψ maps E_n onto Ψ_n;

5) if α is an existential formula in $\mathfrak{L}'$ with m free individual variables, i.e. α is of the form

$$\bigcup_\xi \beta(\xi, x_1, \ldots, x_m), \tag{1}$$

then the image ψ_α of α is an m-argument functor in Ψ which does not occur in (1).

The language $\mathfrak{L}'$ just defined is called a *rich extension* of the language $\mathfrak{L}$. It is easy to see that a language $\mathfrak{L}'$ is a rich extension of a language $\mathfrak{L}$ if $\mathfrak{L}'$ is obtained from $\mathfrak{L}$ by adding a set Ψ of functors, if there exists a one-to-one mapping ψ from E' onto Ψ with property 5) and, moreover, if any sublanguage of $\mathfrak{L}'$ having the properties just mentioned (with the same mapping ψ) coincides with $\mathfrak{L}'$.

For every existential formula α in $\mathfrak{L}'$ of the form (1), let α' denote the formula

$$\beta(\psi_\alpha(x_1 \ldots x_m), x_1, \ldots, x_m), \tag{2}$$

and let α'' denote the implication

$$(\alpha \Rightarrow \alpha').$$

Let $\mathfrak{B}$ be the set of all formulas α'' where α is any existential formula in $\mathfrak{L}'$, and let $\mathfrak{A}'$ be the union of $\mathfrak{A}$ and $\mathfrak{B}$.

The theory

$$\mathfrak{C}' = \{\mathfrak{L}', \mathfrak{C}, \mathfrak{A}'\}$$

has the required properties. To prove this we have first to prove the following two auxiliary lemmas.

8.1. *Every semantic realization R of the language $\mathfrak{L}$ in a set $J \neq 0$ can be extended to a semantic realization R' of $\mathfrak{L}'$ in the set J in such a way*

that all formulas in $\mathfrak{B}$ are valid in R'. Consequently, if an existential formula α in $\mathfrak{L}'$ is valid in R', so is the formula α'.

We define, by induction, a sequence R_n $(n = 0, 1, 2, \ldots)$ of realizations such that

6) R_n is a semantic realization of $\mathfrak{L}_n$ in J $(n = 0, 1, 2, \ldots)$, R_0 being the realization R;

7) R_{n+1} is an extension of the realization R_n.

Suppose that R_n was defined. To define R_{n+1} it suffices, by 2) and 7), to define $\varphi_{R_{n+1}}$ for any functor φ in Ψ_n. By 3) and 4), we have $\varphi = \psi_\alpha$, α being an existential formula of the form (1) where $\beta(x, x_1, \ldots, x_m)$ is a formula in $\mathfrak{L}_n$. Thus, by the induction hypothesis, β_{R_n} is a well defined mapping from J^{m+1} into the two-element Boolean algebra (we shall use here the definition VI, § 6, (2) of β_{R_n}). By 3) and 5), φ is an m-argument functor. If, for given elements $j_1, \ldots, j_m \in J$, there exists $j \in J$ such that $\beta_{R_n}(j, j_1, \ldots, j_m) = V$, we assume

$$\beta_{R_{n+1}}(j_1, \ldots, j_m) = j .$$

If not, we define the element $\varphi_{R_{n+1}}(j_1, \ldots, j_m)$ in an arbitrary way.

It follows immediately from this definition that if $\alpha_{R_n}(j_1, \ldots, j_m) = V$, then $\alpha'_{R_{n+1}}(j_1, \ldots, j_m) = V$. Hence, for every α in E_n,

(3) $\qquad \alpha''$ is valid in R_{n+1}.

By 6) and 7) all the realizations R_n together determine a realization R' of $\mathfrak{L}'$, viz. the realization R' defined as follows:

$\varrho_{R'} = \varrho_R$ for every predicate ϱ in $\mathfrak{L}$,

$\varphi_{R'} = \varphi_R$ for every functor φ in $\mathfrak{L}$,

$\varphi_{R'} = \varphi_{R_{n+1}}$ for every functor φ in Ψ_n $(n = 0, 1, 2, \ldots)$.

By definition, if a formula γ is in $\mathfrak{L}_n$, then $\gamma_{R'} = \gamma_{R_n}$. Hence, by (3), α'' is valid in R' for every existential formula α in $\mathfrak{L}'$.

8.2. *If a formula γ in $\mathfrak{L}$ is not a theorem in $\mathfrak{C}$, then γ is not a theorem in $\mathfrak{C}'$.*

Suppose that γ is a theorem in $\mathfrak{C}'$. Thus, by V, 10.1, there exist finite sets $\mathcal{A}_1 \subset \mathcal{A}$ and $\mathfrak{B}_1 \subset \mathfrak{B}$ such that γ belongs to $C(\mathcal{A}_1 \cup \mathfrak{B}_1)$. Since γ is not a theorem in $\mathfrak{C}$, γ is not a theorem in the theory $\mathfrak{C}_1 = \{\mathfrak{L}, C, \mathcal{A}_1\}$. By 5.2, there exists a semantic model R for the theory $\mathfrak{C}_1$ in an enumerable set J, and a valuation v such that $\gamma_R(v) = \Lambda$. By 8.1 the realization R can be extended to a realization R' of $\mathfrak{L}'$ in the set J, such that every formula in $\mathfrak{B}$ is valid in J. Thus R' is a semantic model for the theory

$\mathfrak{C}_1' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A}_1 \cup \mathcal{B}_1\}$. Consequently, $\gamma_{R'}(v) = V$ by 2.2. On the other hand, $\gamma_{R'}(v) = \gamma_R(v) = \Lambda$ since R' is an extension of R. Contradiction!

8.3. *Every theory* $\mathfrak{C} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ *has a rich inessential extension* $\mathfrak{L}' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A} \cup \mathcal{B}\}$ *such that the sets of all terms in* $\mathfrak{L}'$, *the set of all formulas in* $\mathfrak{L}'$ *and the set of all formulas in* $\mathfrak{L}$ *have the same power. If* $\mathfrak{C}$ *is consistent, so is* $\mathfrak{C}'$.

In fact, the theory $\mathfrak{C}'$ defined at the beginning of this section satisfies all the above conditions.

8.4. *If* $\mathfrak{C} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ *is a rich theory, then every homomorphism* h *from the* Q*-algebra* $\mathfrak{A}(\mathfrak{C})$ *into any Boolean algebra* A *is a* $\mathfrak{Q}$*-homomorphism, i.e.*

$$h(\|\bigcup_\xi \beta(\xi)\|) = \bigcup_{\tau \in T}^{A} \|\beta(\tau)\|\,, \qquad h(\|\bigcap_\xi \beta(\xi)\|) = \bigcap_{\tau \in T}^{A} \|\beta(\tau)\|\,. \tag{4}$$

By 1.1, (Q),

$$\|\beta(\tau)\| \leqslant \|\bigcup_\xi \beta(\xi)\| \quad \text{for every term } \tau\,. \tag{5}$$

Since the theory $\mathfrak{C}$ is rich, there exists a term τ_0 such that the formula $(\bigcup_\xi \beta(\xi) \Rightarrow \beta(\tau_0))$ is a theorem in $\mathfrak{C}$, i.e. $\|\bigcup_\xi \beta(\xi)\| \leqslant \|\beta(\tau_0)\|$. Hence

$$\|\beta(\tau_0)\| = \|\bigcup_\xi \beta(\xi)\|\,. \tag{6}$$

It follows from (5) and (6) that

$$h(\|\beta(\tau)\|) \leqslant h(\|\bigcup_\xi \beta(\xi)\|) \quad \text{for every term } \tau$$

and

$$h(\|\beta(\tau_0)\|) = h(\|\bigcup_\xi \beta(\xi)\|)\,.$$

This proves the first of the identities (4). The proof of the second is when we pass to complements.

8.5. *Let* $\mathfrak{C} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ *be a rich theory. If a formula* a *is irrefutable in* $\mathfrak{C}$, *then there exists a semantic model* R *for* $\mathfrak{C}$ *in the set* T *of all terms, such that* a *is satisfiable in* R.

Since a is irrefutable, we have $\|a\| \neq \Lambda$. By I, 8.5, there is a prime (i.e. maximal) filter ∇ such that $\|a\| \in \nabla$. By 8.4, the natural homomorphism h of $\mathfrak{A}(\mathfrak{C})$ onto the two-element Boolean algebra $A_0 = \mathfrak{A}(\mathfrak{C})/\nabla$ is a Q-homomorphism. The canonical realization R^0 for $\mathfrak{C}$, determined by the Q-homomorphism h, is a semantic model for $\mathfrak{C}$ on account of 3.2. Moreover, by 3.1, we have for the identity valuation i:

$$a_{R^0}(\mathrm{i}) = h(\|a\|) = V_{A_0}$$

since $\|a\| \in \nabla$. Thus a is satisfiable in R^0.

8.6. *Let* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ *be a rich theory. If a formula* β *is not a theorem in* $\mathcal{T}$, *then there exists a semantic model* R *for* $\mathcal{T}$ *in the set* T *of all terms, such that* β *is not valid in* R.

Theorem 8.6 follows immediately from 8.5 where α is replaced by $-\beta$. Note that the mutual relation between 8.5 and 8.6 is the same as in the case of 5.1 and 5.2 (see the remark on p. 293).

§ 9. The existence of semantic models for arbitrary consistent theories. The following theorem is a generalization of theorem 5.1.

9.1. *If a formula* α *is irrefutable in a theory* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$, *then there exists a semantic model* R *for* $\mathcal{T}$ *such that* α *is satisfiable in* R *and the cardinal of* R *is equal to* $\max(\aleph_0, \overline{\overline{\mathcal{A}}})$.

Let $\mathfrak{m} = \max(\aleph_0, \overline{\overline{\mathcal{A}}})$.

First we shall prove 9.1 for the case in which the set of all formulas in $\mathcal{T}$ has the cardinal $\mathfrak{m}$. By 8.3, there exists a rich theory $\mathcal{T}' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A} \cup \mathcal{B}\}$ which is an inessential extension of $\mathcal{T}$ and such that the set T' of all terms in $\mathcal{T}'$ has the cardinal $\mathfrak{m}$. Since $\mathcal{T}'$ is an inessential extension of $\mathcal{T}$, the formula α is irrefutable in $\mathcal{T}'$. By 8.5, there exists a semantic model R' for $\mathcal{T}'$ in the set T' such that α is satisfiable in R'. The restriction R of the realization R' to the language $\mathfrak{L}$ is a semantic model for $\mathcal{T}$ in the set T' of the power $\mathfrak{m}$, and α is satisfiable in R.

Consider now the case of an arbitrary theory $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$. Suppose α is irrefutable in $\mathcal{T}$. There is a sublanguage $\mathfrak{L}_1$ of $\mathfrak{L}$ such that $\mathfrak{L}_1$ contains all signs occurring in α and in all the mathematical axioms in $\mathcal{A}$, and the set of all formulas in $\mathfrak{L}_1$ has the power $\mathfrak{m}$. By the part of 9.1 just proved there exists a semantic model R_1 for the theory $\mathcal{T}_1 = \{\mathfrak{L}_1, \mathfrak{C}, \mathcal{A}\}$ such that α is satisfiable in R_1, and R_1 has the cardinal $\mathfrak{m}$. Let R be a realization of $\mathfrak{L}$ such that R is an extension of R_1. By 2.7, R is a semantic model for $\mathcal{T}$, and α is satisfiable in R.

The following theorem is a generalization of 5.2.

9.2. *If a formula* β *is not a theorem in a theory* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$, *then there exists a semantic model* R *for* $\mathcal{T}$ *such that* β *is not valid in* R *and the cardinal of* R *is equal to* $\max(\aleph_0, \overline{\overline{\mathcal{A}}})$.

Theorem 9.2 follows from 9.1 where α is replaced by $-\beta$ (see § 2, (3), (4) and VI, 10.11).

From each of the above theorems follows directly:

9.3. *Every consistent theory* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ *has a semantic model the cardinal of which is* $\max(\aleph_0, \overline{\overline{\mathcal{A}}})$ [1].

---

[1] This is a generalization (Malcev [1], Henkin [1], A. Robinson [1]) of the Gödel [1] theorem. See also Beth [1] and Łoś [5]. The idea of introducing the set $\mathcal{B}$ of formulas is due to Hasenjaeger [1].

Theorems 9.1-9.3 are among the most important theorems in the metatheory of elementary formalized theories. It follows from 9.2 and 2.2 that the consequence operation C defined in V, § 11, p. 188, has the property (C) from V, § 10, i.e. it is a good formalization of the intuitive notion of the consequence. In fact, by the natural one-to-one correspondence between interpretations and semantic realizations (see § 4), the condition (C) from V, § 10, can be formulated as follows: given any set $\mathcal{A}$ of formulas, a formula α belongs to $C(\mathcal{A})$ if and only if it is true in every semantic model for $\mathcal{A}$. The last statement follows directly from 2.2 and 9.2.

The intuitive meaning of 9.3 is that a theory is consistent if and only if it can be semantically realized, i.e. if there exists a set $J \neq 0$ of elements and a system of relations in J and of functions in J with values in J which have the properties expressed by the mathematical axioms of this theory. The proof of analogous theorem 5.3 on the existence of models for enumerable theories is effective [1], i.e. the axiom of choice has not bean applied in this proof. The proof of 9.3 is not effective since it is based on the well-ordering principle (used in the proof of 8.1 and 8.4). This non-effectiveness is essential since 9.3 is equivalent effectively to several known statements which cannot be proved without the axiom of choice. This problem will be discussed in § 21.

Axiomatic mathematical theories are usually formalized by means of an at most enumerable set of axioms. Therefore the case of enumerable theories seems to be the most natural. However, in some applications of the theorem on the existence of models to mathematics it is convenient to consider formalized theories based on a non-enumerable set of axioms. Theorem 9.3 therefore constitutes a powerful and important apparatus in mathematical proofs. For instance a considerable number of interesting theorems in modern algebra can be proved by means of theorem 9.3 [2].

Theorems 5.3 and 9.3 also lead to some results which seem rather paradoxical. Examples will be discussed in § 20.

The next theorem explains in greater detail the connections which exist between the consistency of a theory and the existence of various kinds of models. It is a generalization of theorem 5.4.

9.4. *The following conditions are equivalent for any theory* $\mathcal{T} = \{\mathfrak{L}, C, \mathcal{A}\}$:

(i) $\mathcal{T}$ *is consistent,*

---

[1] The proof quoted in this book is based on the Baire theorem I, 2.2. For the case of enumerable theories, the Baire theorem is applied to compact spaces with an enumerable basis. In this case the proof of the Baire theorem is effective.

[2] See e.g. A. Robinson [1].

(ii) *there exists a model for* $\mathfrak{T}$,

(iii) *there exists a model for* $\mathfrak{T}$ *in a set* $J \neq 0$ *and in the field* $\mathfrak{B}(X)$ *of all subsets of a set* $X \neq 0$,

(iv) *there exists an adequate model for* $\mathfrak{T}$,

(v) *there exists a semantic model for* $\mathfrak{T}$ *in a set* J, *the cardinal of which is equal to* $\max(\aleph_0, \overline{\overline{A}})$.

In fact, (i) implies (v) by 9.3; (v) implies (iii) since the two-element Boolean algebra is isomorphic to the field of all subsets of a one-element set X; (iii) implies (ii) in an obvious way and (ii) implies (i) by 2.4. Thus conditions (i), (ii), (iii), (v) are equivalent. On the other hand (iv) implies (i) by 2.4 and (i) implies (iv) by 3.2.

The following theorem is a generalization of 5.5.

9.5. *For every formula* α *in a consistent theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ *the following conditions are equivalent*:

(i) α *is a theorem in* $\mathfrak{T}$;

(ii) α *is valid in every model for* $\mathfrak{T}$;

(iii) α *is valid in every model for* $\mathfrak{T}$ *in any complete Boolean algebra*;

(iv) α *is valid in every model for* $\mathfrak{T}$ *in the field* $\mathfrak{B}(X)$ *of all subsets of any set* $X \neq 0$;

(v) α *is valid in every semantic model for* $\mathfrak{T}$ *in any set of the power* $\max(\aleph_0, \overline{\overline{\mathfrak{A}}})$;

(vi) *there exists a non-degenerate Boolean algebra* $\boldsymbol{A}$ *and a set* J *of a power* $\geqslant \max(\aleph_0, \overline{\overline{\mathfrak{A}}})$ *such that* α *is valid in every model for* $\mathfrak{T}$ *in* J *and* $\boldsymbol{A}$;

(vii) $a_{R^0}(\mathfrak{i}) = V$ *for the canonical model* R^0 *for* $\mathfrak{T}$ *in* $\mathfrak{A}(\mathfrak{T})$ *and the identity valuation* $\mathfrak{i}$.

The proof of 9.5 is the same as the proof of 5.5 (instead of 5.2 we apply 9.2).

We round off this section with the following remarks.

9.6. *For any theories* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ *and* $\mathfrak{T}' = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}'\}$ *the following conditions are equivalent*:

(i) $\mathfrak{T}$ *and* $\mathfrak{T}'$ *are equivalent*;

(ii) $\mathfrak{T}$ *and* $\mathfrak{T}'$ *have the same models*;

(iii) $\mathfrak{T}$ *and* $\mathfrak{T}'$ *have the same semantic models.*

(i) implies (ii) by 2.2. Obviously (ii) implies (iii). (iii) implies (i) because of the equivalence of 9.5 (i) and 9.5 (v).

We shall now prove a theorem stating that sometimes the cardinal $\max(\aleph_0, \overline{\overline{\mathfrak{A}}})$ can be replaced in 9.1, 9.2, 9.3 and 9.5 by another cardinal less than $\max(\aleph_0, \overline{\overline{\mathfrak{A}}})$. First we have to prove an auxiliary lemma.

9.7. *If a sublanguage $\mathfrak{L}_0$ of $\mathfrak{L}$ contains all predicates and functors which occur in the set $\mathcal{A}$ of axioms of a theory $\mathcal{T} = \{\mathfrak{L}, C, \mathcal{A}\}$, then there exists a set $\mathcal{A}'$ of formulas in $\mathfrak{L}_0$ such that the theory $\mathcal{T}' = \{\mathfrak{L}_0, C, \mathcal{A}'\}$ is equivalent to $\mathcal{T}$.*

The construction of the set $\mathcal{A}'$ is as follows. Let V_0 and Ξ_0 be the sets of all free individual variables and of all bound individual variables in $\mathfrak{L}_0$. For every formula α in $\mathcal{A}$, let α^* be a formula congruent to α and such that all bound individual variables in α^* belong to Ξ_0. Let $\mathcal{A}^*$ be the set of all formulas α^* (α in $\mathcal{A}$). By 6.5, both formulas $(\alpha \Rightarrow \alpha^*)$ and $(\alpha^* \Rightarrow \alpha)$ are predicate tautologies. Thus, by *modus ponens*, $\alpha \in C(\alpha^*) \subset C(\mathcal{A}^*)$ and $\alpha^* \in C(\alpha) \subset C(\mathcal{A})$. Hence $\mathcal{A} \subset C(\mathcal{A}^*)$ and $\mathcal{A}^* \subset C(\mathcal{A})$ and consequently $C(\mathcal{A}) \subset CC(\mathcal{A}^*) = C(\mathcal{A}^*) \subset CC(\mathcal{A}) = C(\mathcal{A})$ (see V § 10 (c_2) and (c_3'), p. 181), which proves that

$$C(\mathcal{A}) = C(\mathcal{A}^*) .$$

Now, let α' be a formula obtained from α^* by a one-to-one substitution of some free individual variables in V_0 for free individual variables in α^*. Since the substitution is one-to-one, α' is also the result of a substitution (viz. the inverse substitution) in α^*. Let $\mathcal{A}'$ be the set of all formulas α' (α in $\mathcal{A}$). By the rule of substitution (r_1), $\alpha' \in C(\alpha^*) \subset C(\mathcal{A}^*)$ and $\alpha^* \subset C(\alpha') \subset C(\mathcal{A}')$. Hence, by the same argument as above,

$$C(\mathcal{A}^*) = C(\mathcal{A}') .$$

Consequently $C(\mathcal{A}) = C(\mathcal{A}')$, i.e. theories $\mathcal{T}$ and $\mathcal{T}' = \{\mathfrak{L}_0, C, \mathcal{A}'\}$ are equivalent.

9.8. *In theorems* 9.1, 9.2, 9.3, 9.5 *the cardinal* $\max(\aleph_0, \overline{\overline{\mathcal{A}}})$ *may be replaced by the cardinal* $\max(\aleph_0, \overline{\overline{P}}_{\mathcal{A}}, \overline{\overline{\Phi}}_{\mathcal{A}})$ *where $P_{\mathcal{A}}$ and $\Phi_{\mathcal{A}}$ are respectively the sets of all predicates and all functors occurring in formulas in $\mathcal{A}$.*

Let $\mathfrak{L}_0$ be the sublanguage of $\mathfrak{L}$ such that $\mathfrak{L}_0$ contains only an enumerable set V_0 of free individual variables, an enumerable set Ξ_0 of bound individual variables, functors from the set $\Phi_{\mathcal{A}}$ and predicates from the set $P_{\mathcal{A}}$. By 9.7, there is a set $\mathcal{A}'$ of formulas in $\mathfrak{L}_0$ such that the theory $\mathcal{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ is equivalent to $\mathcal{T}' = \{\mathfrak{L}_0, C, \mathcal{A}'\}$. It is easy to verify that $\max(\aleph_0, \overline{\overline{\mathcal{A}}}') = \max(\aleph_0, \overline{\overline{P}}_{\mathcal{A}}, \overline{\overline{\Phi}}_{\mathcal{A}})$. Applying the theorems mentioned in 9.8 to the theory $\mathcal{T}'$ we obtain the required statements for the theory $\mathcal{T}$ by 9.6 (i), (iii).

§ 10. The deduction theorems. This name is given to the following two theorems [1].

10.1. *Let α be a closed formula in $\mathfrak{L}$. A formula β is a theorem in a theory $\mathcal{T}' = \{\mathfrak{L}, C, \mathcal{A} \cup [\alpha]\}$ if and only if the formula $(\alpha \Rightarrow \beta)$ is a theorem in the theory $\mathcal{T} = \{\mathfrak{L}, C, \mathcal{A}\}$.*

[1] Herbrand [1], [2] and Tarski [1].

If $(\alpha \Rightarrow \beta)$ is a theorem in $\mathfrak{C}$, then it is also a theorem in $\mathfrak{C}'$. Since α is an axiom in $\mathfrak{C}'$, by *modus ponens* the formula β is a theorem in $\mathfrak{C}'$.

If $(\alpha \Rightarrow \beta)$ is not a theorem in $\mathfrak{C}$, then by 9.2 there is a semantic model R for $\mathfrak{C}$ and a valuation v such that $(\alpha \Rightarrow \beta)_R(v) = \Lambda$, i.e. $\alpha_R(v) \Rightarrow \beta_R(v) = \Lambda$. Hence

$$\alpha_R(v) = V \quad \text{and} \quad \beta_R(v) = \Lambda .$$

Since α is a closed formula, the function α_R is constant. Thus α_R is identically equal to V, i.e. R is a model for α. Consequently R is a model for $\mathfrak{C}'$ but β is not valid in R. This proves, by 2.2, that β is not a theorem in $\mathfrak{C}'$.

10.2. *A formula β is a theorem in a theory $\mathfrak{C} = \{\mathfrak{L}, C, \mathcal{A}\}$ with a non-empty set $\mathcal{A}$ of axioms if and only if there exists a conjunction α of closures of a finite number of axioms in $\mathcal{A}$ such that the implication $(\alpha \Rightarrow \beta)$ is a predicate tautology.*

Suppose that β is a theorem in $\mathfrak{C}$, i.e. β belongs to $C(\mathcal{A})$. Then there exists a finite non-empty set $\mathcal{A}_1 \subset \mathcal{A}$ such that β is in $C(\mathcal{A}_1)$ (see V, 10.1). Let α be a conjunction of closures of all formulas in $\mathcal{A}_1$. By *modus ponens*, the tautology (T_6) and VI, 11.5, $\mathcal{A}_1 \subset C([\alpha])$. Consequently β belongs to $C([\alpha])$, i.e. β is a theorem in the theory $\{\mathfrak{L}, C, [\alpha]\}$. By 10.1 (where $\mathcal{A}$ is the empty set), the implication $(\alpha \Rightarrow \beta)$ is a theorem in $\{\mathfrak{L}, C, 0\}$, i.e. it is a predicate tautology.

Conversely, if there exists a conjunction α of closures of axioms in $\mathcal{A}$ such that $(\alpha \Rightarrow \beta)$ is a predicate tautology, then α is a theorem in $\mathfrak{C}$ (see VI, 11.5 and VI, 10.8) and, by *modus ponens*, β is also a theorem in $\mathfrak{C}$.

§ 11. The connection between theories and filters [1]. In this section (which is an analogue to VII, § 9) the symbol $\|\alpha\|$ will always denote the element of the Boolean algebra $\mathfrak{A}(\mathfrak{S})$ determined by a formula α. We recall that $\mathfrak{S} = \{\mathfrak{L}, C\}$ is the predicate calculus based on $\mathfrak{L}$.

A filter $\mathcal{F}$ in $\mathfrak{A}(\mathfrak{S})$ is said to be an $\bigcap$*-filter* if it has the following property:

if $\|\alpha(x)\|$ is in $\mathcal{F}$, then the element $\|\bigcap_\xi \alpha(\xi)\|$ (i.e. the meet $\bigcap_{\tau \in T} \|\alpha(\tau)\|$) is also in $\mathcal{F}$.

In other words, a filter $\mathcal{F}$ in $\mathfrak{A}(\mathfrak{S})$ is said to be a $\bigcap$-filter if it has the following property:

if $\|\alpha\| \in \mathcal{F}$, and $\bar{\alpha}$ is a closure of the formula α, then also $\|\bar{\alpha}\| \in \mathcal{F}$.

The symbol $\overline{\mathfrak{A}}(\mathfrak{S})$ will denote the set of all elements $\|\alpha\| \in \mathfrak{A}(\mathfrak{S})$ where α is a closed formula. It is easy to see that $\overline{\mathfrak{A}}(\mathfrak{S})$ is closed under

[1] For the results in this section, see Tarski [5], [7].

the Boolean operations $\cup, \cap, \Rightarrow, -$ in $\mathfrak{A}(\mathcal{S})$, i.e. $\overline{\mathfrak{A}}(\mathcal{S})$ is a subalgebra of the Boolean algebra $\mathfrak{A}(\mathcal{S})$.

For any set $\mathcal{A}$ of formulas in $\mathfrak{L}$, the symbol $\nabla_{0\mathcal{A}}$ will denote the set of all $\|a\| \in \mathfrak{A}(\mathcal{S})$ where a is in $\mathcal{A}$, and the symbol $\overline{\nabla}_{0\mathcal{A}}$ will denote the set of all $\|\bar{a}\| \in \overline{\mathfrak{A}}(\mathcal{S})$ where $\bar{a}$ is a closure of a formula a in $\mathcal{A}$ (all closures of a formula a determine the same element of $\mathfrak{A}(\mathcal{S})$ by 6.5). The symbol $\nabla_{\mathcal{A}}$ will denote the set of all $\|a\| \in \mathfrak{A}(\mathcal{S})$ where a is a theorem in the theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$, and $\overline{\nabla}_{\mathcal{A}}$ will denote the set of all $\|a\|$ where a is a closed formula and a theorem in $\mathfrak{T}$. The definition of $\nabla_{\mathcal{A}}$ can be formulated as follows:

(1) $\|a\| \in \nabla_{\mathcal{A}}$ if and only if a is a theorem in $\mathfrak{T}$.

The set $\overline{\nabla}_{\mathcal{A}}$ is the intersection

$$\overline{\nabla}_{\mathcal{A}} = \nabla_{\mathcal{A}} \cap \overline{\mathfrak{A}}(\mathcal{S}) . \tag{2}$$

A set S of elements in $\mathfrak{A}(\mathcal{S})$ is said to $\bigcap$-*generate* a $\bigcap$-filter ∇ in $\mathfrak{A}(\mathcal{S})$ (or: to be a set of $\bigcap$-*generators* for ∇) if ∇ is the least $\bigcap$-filter containing S.

11.1. *The set $\nabla_{\mathcal{A}}$ is a $\bigcap$-filter in $\mathfrak{A}(\mathcal{S})$. The set $\nabla_{0\mathcal{A}}$ $\bigcap$-generates the $\bigcap$-filter $\nabla_{\mathcal{A}}$. The set $\overline{\nabla}_{0\mathcal{A}}$ generates the filter $\nabla_{\mathcal{A}}$ in $\mathfrak{A}(\mathcal{S})$.*

The set $\overline{\nabla}_{\mathcal{A}}$ is a filter in $\overline{\mathfrak{A}}(\mathcal{S})$. The set $\overline{\nabla}_{0\mathcal{A}}$ generates the filter $\overline{\nabla}_{\mathcal{A}}$ in $\overline{\mathfrak{A}}(\mathcal{S})$.

By the deduction theorem 10.2, a is a theorem in $\mathfrak{T}$ if and only if there exists a conjunction $\bigcap_{i=1}^{m} \bar{a}_i$ of closures $\bar{a}_i$ of some formulas a_i in $\mathcal{A}$, such that the implication $(\bigcap_{i=1}^{m} \bar{a}_i \Rightarrow a)$ is a predicate tautology, i.e. if

$$\|(\bigcap_{i=1}^{m} \bar{a}_i \Rightarrow a)\| = V \in \mathfrak{A}(\mathcal{S}) .$$

In other words, a is a theorem in $\mathfrak{T}$ if and only if there exist formulas $a_1, \ldots, a_m$ in $\mathcal{A}$ such that

$$\bigcap_{i=1}^{m} \|\bar{a}_i\| \leqslant \|a\| ,$$

$\bar{a}_i$ being a closure of a_i. This proves the first part of 11.1, by (1) and I, 8.1.

The second part follows from the first part and (2).

We have deduced the first part of 11.1 from the deduction theorem 10.2. It is easy to see that, conversely, 10.2 follows directly from the first part of 11.1. Thus the first part of 11.1 is only another algebraic formulation of the deduction theorem 10.2. It also explains the algebraic sense of the set of axioms and the set of all theorems in a theory.

11.2. *A theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ is consistent if and only if the filter $\nabla_{\mathcal{A}}$ is proper, i.e. if the filter $\overline{\nabla}_{\mathcal{A}}$ is proper.*

A closed formula a is irrefutable in the theory $\mathfrak{T}$ if and only if the theory $\{\mathfrak{L}, \mathfrak{C}, \mathfrak{A} \cup [a]\}$ is consistent.

A closed formula β is not a theorem in $\mathfrak{T}$ if and only if the theory $\{\mathfrak{L}, \mathfrak{C}, \mathfrak{A} \cup [-\beta]\}$ is consistent.

The first part of 11.2 follows immediately from the definition of $\nabla_{\mathfrak{A}}$ or $\overline{\nabla}_{\mathfrak{A}}$ (see V, 14.1).

The remaining part of 11.2 follows from the part just proved. For the filter generated by $\overline{\nabla}_{0\mathfrak{A}}$ and $\|a\|$ (i.e. generated by $\overline{\nabla}_{\mathfrak{A}}$ and $\|a\|$) is proper if and only if $-\|a\| \notin \nabla_{\mathfrak{A}}$ (see I, 13.9). Similarly, the filter generated by $\overline{\nabla}_{0\mathfrak{A}}$ and $-\|\beta\|$ (i.e. generated by $\overline{\nabla}_{\mathfrak{A}}$ and $-\|\beta\|$) is proper if and only if $\|\beta\| \notin \nabla_{\mathfrak{A}}$ (see II, 5.3).

Without the hypothesis that the formula β in question is closed, the third part of 9.2 can be formulated as follows, on account of VI, 11.5. A formula β is not a theorem in $\mathfrak{T}$ if and only if the theory $[\mathfrak{L}, \mathfrak{C}, \mathfrak{A} \cup [-\bar{\beta}]\}$ where $\bar{\beta}$ is a closure of β is consistent.

Replacing β by $-a$ in the last statement, we obtain the following generalization of the second part of 11.2: A formula $a(x_1, \ldots, x_n)$ (where $x_1, \ldots, x_n$ are all free individual variables occurring in this formula) is irrefutable in $\mathfrak{T}$ if and only if the theory $\{\mathfrak{L}, \mathfrak{C}, \mathfrak{A} \cup [\bigcup_{\xi_1} \ldots \bigcup_{\xi_n} a(\xi_1, \ldots, \xi_n)]\}$ is consistent.

11.3. *A theory $\mathfrak{T}' = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}'\}$ is a linguistically invariant extension of a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ if and only if $\nabla_{\mathfrak{A}} \subset \nabla_{\mathfrak{A}'}$, i.e. if $\overline{\nabla}_{\mathfrak{A}} \subset \overline{\nabla}_{\mathfrak{A}'}$.*

By definition, $\mathfrak{T}'$ is a linguistically invariant extension of $\mathfrak{T}$ if and only if $\mathfrak{C}(\mathfrak{A}) \subset \mathfrak{C}(\mathfrak{A}')$. By (1) this is equivalent to the inclusion $\nabla_{\mathfrak{A}} \subset \nabla_{\mathfrak{A}'}$. By (2) and 11.1 the last inclusion is equivalent to $\overline{\nabla}_{\mathfrak{A}} \subset \overline{\nabla}_{\mathfrak{A}'}$.

11.4. *Theories $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ and $\mathfrak{T}' = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}'\}$ are equivalent if and only if $\nabla_{\mathfrak{A}} = \nabla_{\mathfrak{A}'}$, i.e. if $\overline{\nabla}_{\mathfrak{A}} = \overline{\nabla}_{\mathfrak{A}'}$.*

This follows immediately from 11.3.

We shall complete theorem 11.1 by the following remark.

11.5. *For every filter $\overline{\nabla}$ in $\overline{\mathfrak{A}}(\mathfrak{S})$ there exists a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ such that $\overline{\nabla} = \overline{\nabla}_{\mathfrak{A}}$. Moreover, for every set ∇_0 of generators of $\overline{\nabla}$ we can choose the set $\mathfrak{A}$ of axioms for $\mathfrak{T}$ in such a way that $\nabla_0 = \overline{\nabla}_{0\mathfrak{A}}$.*

For every $\bigcap$-filter ∇ in $\mathfrak{A}(\mathfrak{S})$ there exists a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ such that $\nabla = \nabla_{\mathfrak{A}}$. Moreover, for every set ∇_0 of $\bigcap$-generators of ∇ we can choose the set $\mathfrak{A}$ of axioms for $\mathfrak{T}$ in such a way that $\nabla_0 = \nabla_{0\mathfrak{A}}$.

For every element $e \in \nabla_0$, let a_e be a formula such that $\|a_e\| = e$, and let $\mathfrak{A}$ be the set of all $a_e, \frown e \in \nabla_0$. The theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ has the required property.

Similarly we prove the second part of 11.5.

Theorems 11.1, 11.4 and 11.5 state that there is a natural one-to-one correspondence between $\bigcap$-filters in $\mathfrak{A}(\mathcal{S})$, filters in $\overline{\mathfrak{A}}(\mathcal{S})$ and formalized theories based on $\mathcal{S}$, two theories being identified if and only if they are equivalent.

To prove the next theorem we shall use the more precise notation introduced in VI, § 10, p. 245: for every formula α, the symbol $\|\alpha\|_{\mathcal{S}}$ will denote the corresponding element in the algebra $\mathfrak{A}(\mathcal{S})$ of the predicate calculus $\mathcal{S} = \{\mathfrak{L}, \mathfrak{C}\}$ and the symbol $\|\alpha\|_{\mathcal{T}}$ will denote the corresponding element of the algebra $\mathfrak{A}(\mathcal{T})$ of a theory $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$.

If formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are derivable in $\mathcal{S}$, then they are also theorems in $\mathcal{T}$. Hence it follows that the equation

$$h(\|\alpha\|_{\mathcal{S}}) = \|\alpha\|_{\mathcal{T}}$$

defines a mapping from $\mathfrak{A}(\mathcal{S})$ onto $\mathfrak{A}(\mathcal{T})$. It is easy to verify that h is a Boolean homomorphism and even a Q-homomorphism (this follows from 1.1, (2) and (Q)). By the definition of h and $\nabla_{\mathfrak{A}}$, the element $h(\|\alpha\|_{\mathcal{S}})$ is the unit element of $\mathfrak{A}(\mathcal{T})$ if and only if α is a theorem in $\mathcal{T}$, i.e. if $\|\alpha\|_{\mathcal{S}} \in \nabla_{\mathfrak{A}}$. Hence by I, 13.4, follows

11.6. *The algebra $\mathfrak{A}(\mathcal{T})$ of a theory $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ is isomorphic to the algebra $\mathfrak{A}(\mathcal{S})/\nabla_{\mathfrak{A}}$.*

§ 12. Maximal and prime theories. According to V, § 14, a consistent theory $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ is said to be maximal if every linguistically invariant consistent extension of $\mathcal{T}$ is equivalent to $\mathcal{T}$.

In the first characterization of maximal theories we shall use the notation $\nabla_{\mathfrak{A}}$, $\overline{\nabla}_{\mathfrak{A}}$ from § 11. We recall that $\mathfrak{A}(\mathcal{S})$ is the Boolean algebra of the predicate calculus $\mathcal{S} = \{\mathfrak{L}, \mathfrak{C}, 0\}$. A $\bigcap$-filter ∇ in $\mathfrak{A}(\mathcal{S})$ is said to be a *maximal $\bigcap$-filter* provided it is proper (i. e. it is not equal to the whole $\mathfrak{A}(\mathcal{S})$) and it is not a proper subset of a proper $\bigcap$-filter.

12.1. *The following conditions are equivalent for every theory $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ of the first order:*

(i) *$\mathcal{T}$ is maximal;*

(ii) *$\nabla_{\mathfrak{A}}$ is a maximal $\bigcap$-filter in $\mathfrak{A}(\mathcal{S})$;*

(iii) *$\overline{\nabla}_{\mathfrak{A}}$ is a maximal (i.e. prime) filter in $\overline{\mathfrak{A}}(\mathcal{S})$.*

Theorem 12.1 follows easily from the one-to-one natural correspondence between $\bigcap$-filters in $\mathfrak{A}(\mathcal{S})$, filters in $\overline{\mathfrak{A}}(\mathcal{S})$ and classes of equivalent theories, and from theorem 11.3.

12.2. *A theory $\mathcal{T}$ is maximal if and only if, for every closed formula α, exactly one of the formulas α and $-\alpha$ is a theorem of $\mathcal{T}$. In other words, $\mathcal{T}$ is maximal if and only if, in the Q-algebra $\mathfrak{A}(\mathcal{T})$, the class of all elements $\|\alpha\| \in \mathfrak{A}(\mathcal{T})$ where α is a closed formula is a two-element subalgebra of $\mathfrak{A}(\mathcal{T})$, i.e. the subalgebra consisting only of $\Lambda, V \in \mathfrak{A}(\mathcal{T})$, $\Lambda \neq V$.*

The first part of 12.2 follows immediately from 12.1 (i), (iii) (see II, 5.2). The second follows from the first.

12.3. *For every theory* $\mathfrak{T}$ *the following conditions are equivalent*:

(i) $\mathfrak{T}$ *is maximal*;

(ii) $\mathfrak{T}$ *is consistent and every semantic model for* $\mathfrak{T}$ *is adequate*;

(iii) $\mathfrak{T}$ *has an adequate semantic model.*

Suppose that R is a semantic model for a maximal theory $\mathfrak{T}$. Let $\mathcal{A}'$ be the set of all formulas which are valid in R. By definition, the theory $\mathfrak{T}' = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}'\}$ is an extension of $\mathfrak{T}$. $\mathfrak{T}'$ is consistent since R is a model for $\mathfrak{T}'$. Since $\mathfrak{T}$ is maximal, $\mathfrak{T}'$ is equivalent to $\mathfrak{T}$. By definition of $\mathfrak{T}'$, R is an adequate model for $\mathfrak{T}'$. Thus R is also an adequate model for the equivalent theory $\mathfrak{T}$.

Thus (i) implies (ii). Since (ii) trivially implies (iii), it remains to prove that (iii) implies (i). Suppose that R is an adequate semantic model for $\mathfrak{T}$. For every closed formula α, either α_R is the constant $\vee$, or α_R is the constant $\wedge$, i. e. $(-\alpha)_R$ is the constant $\vee$. Since R is adequate, we infer that exactly one of the formulas α and $-\alpha$ is a theorem in $\mathfrak{T}$. Thus $\mathfrak{T}$ is maximal.

12.4. *If* R *is a semantic realization of a formalized language of the first order and* $\mathcal{A}_0$ *is the set of all formulas which are valid in* R, *then every theory* $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ *equivalent to* $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}_0\}$ *is maximal.*

In fact, R is then an adequate semantic model for $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$.

12.5. *If a theory* $\mathfrak{T}$ *with a sign of equality* $=$ *has ordinary semantic models of powers* n *and* $\mathfrak{n}$ *where* $n \neq \mathfrak{n}$ *and* $n < \aleph_0$, *then* $\mathfrak{T}$ *is not maximal.*

Let γ be the conjunction of the following two formulas:

$$\bigcup_{\xi_1} \dots \bigcup_{\xi_n} ((\xi_1 \neq \xi_2) \cap ((\xi_1 \neq \xi_3) \cap \dots \cap ((\xi_1 \neq \xi_n) \cap ((\xi_2 \neq \xi_3) \cap \dots \cap (\xi_{n-1} \neq \xi_n)) \dots),$$

$$\bigcup_{\xi_1} \dots \bigcup_{\xi_n} \bigcap_{\xi} ((\xi = \xi_1) \cup ((\xi = \xi_2) \cup \dots \cup (\xi = \xi_n)) \dots),$$

where $(\xi_i \neq \xi_j)$ is an abbreviation for $-(\xi_i = \xi_j)$. The first formula states that there exist at least n elements. The second formula states that there exist at most n elements. Thus the closed formula γ states that there exist exactly n elements. Neither γ nor $-\gamma$ is a theorem. Suppose, for example that γ is a theorem. Then $\gamma_R = \vee$ for every ordinary semantic model. This is impossible since $\gamma_R = \wedge$ in the ordinary semantic model of power $\mathfrak{n}$. Similarly we prove that $-\gamma$ is not a theorem since $(-\gamma)_R = \wedge$ in the ordinary semantic model of power n.

12.6. *Every consistent theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ *can be extended to a maximal theory* $\mathfrak{T}_0 = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}_0\}$ [1].

[1] This result is due to A. Lindenbaum (see A. Tarski [1]).

Since $\mathfrak{T}$ is consistent, the filter $\overline{\nabla}_{\mathcal{A}}$ in $\overline{\mathfrak{A}}(\mathfrak{S})$ is proper (see 11.2). By I, 8.4, the filter $\overline{\nabla}_{\mathcal{A}}$ can be extended to a maximal filter $\overline{\nabla}$ in $\overline{\mathfrak{A}}(\mathfrak{S})$. Let $\mathcal{A}_0$ be any set of formulas such that $\overline{\nabla} = \overline{\nabla}_{\mathcal{A}_0}$ (see 11.5). By 12.1, the theory $\mathfrak{T}_0 = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}_0\}$ is maximal. It is an extension of $\mathfrak{T}$ since $\overline{\nabla}_{\mathcal{A}} \subset \overline{\nabla}_{\mathcal{A}_0}$.

Another proof of 12.6 is as follows: Let R be a semantic realization of $\mathfrak{T}$ and let $\mathcal{A}_0$ be the set of all formulas which are valid in R. The theory $\mathfrak{T}_0 = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}_0\}$ is an extension of $\mathfrak{T}$ and is maximal, by 12.4.

Theorem 12.6 has only a theoretical meaning. It is never used in mathematical practice although formalized mathematical theories are maximal only exceptionally. For instance, none of the formalized theories quoted in V, § 12, A)-E) is maximal.

This is not surprising in the cases of the theory of ordering, lattice theory, the theory of Boolean algebras and theory of groups (the proof follows easily from 12.5). Roughly speaking, all formalized mathematical theories can be included in one of the following two categories:

1) theories whose aim is to formalize and axiomatize some general mathematical notions like ordered sets, lattices, Boolean algebras, groups, etc. and which have many essentially different models in intuitive mathematics;

2) theories whose aim is to formalize and axiomatize some specific mathematical notions (like the set of positive integers, etc.) and which have only one model in intuitive mathematics (up to isomorphism).

In the case of a theory of the first category, we can expect that certain closed formulas have true interpretations in some models, and false interpretations in others. Thus neither those formulas nor their negations are theorems, and consequently the theory is not maximal. It is also aimless to turn such a theory into a maximal one by adding additional axioms by means of 12.6. For instance, if we axiomatize the theory of arbitrary Boolean algebras, it is aimless to add axioms which hold not for all Boolean algebras but only for some of them.

From the above point of view it is surprising that there exist formalized theories of the first category which are maximal. For instance, add to the axioms (A_1)-(A_4) of the formalized theory of linear ordering (see V, § 13, A) the following three axioms:

$$\bigcap_\xi \bigcup_\eta ((\xi \leqslant \eta) \cap -(\xi = \eta)),$$

$$\bigcap_\xi \bigcup_\eta ((\eta \leqslant \xi) \cap -(\xi = \eta)),$$

$$\bigcap_{\xi_1} \bigcap_{\xi_2} ((-(\xi_1 = \xi_2) \cap (\xi_1 \leqslant \xi_2)) \Rightarrow \bigcup_\eta ((\xi_1 \leqslant \eta) \cap ((\eta \leqslant \xi_2) \cap (-(\xi_1 = \eta) \cap -(\xi_2 = \eta))))).$$

The formalized theory so defined, i.e. the formalized theory of dense linear ordering without the first and the last elements, is maximal (1). We also know other examples of mathematically interesting maximal theories of the first category.

We expect theories of the second category to be maximal. The formalized arithmetic (V, § 13, D)) is a typical example of a theory of the second category. The discovery that the formalized arithmetic is not maximal is surprizing (2). It is a very significant result of metaarithmetic. Its proof must be omitted here because it requires completely different methods than those used in this book.

On the one hand the axiomatization of arithmetic quoted in V, § 12, D) seems to be completely satisfactory because all the theorems which are customarily included in intuitive arithmetic are theorems in our formalized arithmetic. In intuitive arithmetic we do not make use of any other properties of positive integers than those described in the set of axioms quoted, and their consequences. The axioms quoted in V, § 12, D) are, in fact, the well-known axioms of Peano for intuitive axiomatic arithmetic. On the other hand, there exists a closed formula α such that its natural interpretation in the set of all positive integers is a true sentence but neither α nor $-\alpha$ is a theorem of formalized arithmetic. Among all these formulas α we do not know any one whose contents are immediately interesting from the point of view of pure arithmetic. They do not have the character of forgotten or omitted axioms of the theory of positive integers. Therefore, from the mathematical point of view, there is no necessity to add them to the set of axioms of formalized arithmetic. There is no need, to make the arithmetic a maximal theory, using theorem 12.6. Moreover, the structure of known formulas α of this kind is very complicated; they are very long, so that it would be very difficult to join any of them to the set of axioms in practice. But it should also be mentioned that among such formulas α there exist some which are very important statements from metaarithmetic, when translated suitably into the language of metamathematics. The reader can find details in special works on this subject (3). We mention here also that some formalized fragments of arithmetic, e.g. the additive arithmetic, are maximal theories (4).

(1) See Langford [1], Tarski [7].

(2) This theorem is due to Gödel [2]. For another form of the Gödel incompleteness theorem see Rosser [1]. For various proofs of the Gödel theorem see e.g. Tarski, Mostowski, Robinson [1], Kleene [3], Kleene [5], Post [2], Mostowski [2], Grzegorczyk [1]. An exposition of the theory of K. Gödel can be found in Mostowski [5]. See also Smullyan [1].

(3) See e.g. papers cited in the footnote (2).

(4) Presburger [1].

It is easy to imagine that formalized set theory is not a maximal theory. For this purpose let us recall that a cardinal $\mathfrak{n} > \aleph_0$ is said to be *inaccessible* if $\mathfrak{n}$ is a limit cardinal such that

$$\text{if } \mathfrak{m}_\varkappa < \mathfrak{n} \text{ for all } \varkappa \in K, \text{ and } \overline{\overline{K}} < \mathfrak{n}, \text{ then } \sum_{\varkappa \in K} \mathfrak{m}_\varkappa < \mathfrak{n};$$

$$\text{if } \mathfrak{m} < \mathfrak{n}, \text{ then } 2^{\mathfrak{m}} < \mathfrak{n}.$$

In formalized set theory (V, § 12, E)) there exists a formula $\beta(x)$ whose intuitive meaning is:

the cardinal of the set x is inaccessible.

Neither the formula

(1) $$\bigcup_\xi \beta(\xi)$$

nor its negation are theorems of formalized set theory. In fact, adding (1) to the set of axioms of set theory we obtain a consistent theory. Thus the negation of (1) cannot be a theorem in the system described in V, § 12, E). On the other hand, if in the natural model for set theory in intuitive mathematics we restrict ourselves to sets whose cardinals are not greater than an inaccessible cardinal, than we obtain a model for formalized set theory in which the natural interpretation of (1) is a false statement. Thus (1) cannot be a theorem in formalized set theory.

We come back to an examination of an arbitrary theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$.

We recall that $\mathfrak{T}$ is said to be prime (see V, § 14) provided, for any closed formulas β, γ,

(2) if $(\beta \cup \gamma)$ is a theorem in $\mathfrak{T}$, then either β or γ is a theorem in $\mathfrak{T}$.

$\mathfrak{T}$ is said to be a strongly prime theory provided (2) holds for all (closed or not) formulas β, γ.

Note that the statement converse to (2) is always true: if either α or β is a theorem in $\mathfrak{T}$, then $(\beta \cup \gamma)$ is a theorem in $\mathfrak{T}$ by VI, 10.8.

12.7. *The following conditions are equivalent for every theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$:

(i) *the theory* $\mathfrak{T}$ *is prime*;

(ii) *the filter* $\overline{V}_{\mathcal{A}}$ *is prime*;

(iii) *the theory* $\mathfrak{T}$ *is maximal.*

The equivalence of (i) and (ii) follows immediately from the definition. By II, 5.2, the notions of prime filters and maximal filters coincide in Boolean algebras. Thus, by 12.1, (ii) is equivalent to (iii).

12.8. *For every theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ *the following conditions are equivalent*:

(i) *the theory* $\mathfrak{T}$ *is strongly prime*;

(ii) *the filter* $\nabla_{\mathfrak{A}}$ *is prime*;

(iii) *for every* m*-argument predicate* ϱ $(m = 1, 2, \ldots)$, *either the formula*

$$\varrho(x_1 \ldots x_m) \tag{3}$$

or the formula

$$-\varrho(x_1 \ldots x_m) \tag{4}$$

is a theorem in $\mathfrak{T}$, $x_1, \ldots, x_m$ *being different free individual variables.*

The equivalence of (i) and (ii) follows immediately from the definition. Clearly (i) implies (iii) since $(\varrho(x_1 \ldots x_m) \cup -\varrho(x_1 \ldots x_m))$ is a tautology (T_{12}) and therefore a theorem in $\mathfrak{T}$.

Suppose that (iii) holds. Let R be any model for $\mathfrak{T}$. If (3) is a theorem in $\mathfrak{T}$, then ϱ_R is a constant function equal to $\curlyvee$. If (4) is a theorem in $\mathfrak{T}$, then ϱ_R is a constant function equal to $\curlywedge$. Thus, for every formula α, α_R is a constant function equal to $\curlyvee$ or $\curlywedge$, and the value of α_R does not depend on the model R. Hence it follows by 9.5 that α is a theorem in $\mathfrak{T}$ if and only if $\alpha_R \equiv \curlyvee$. If $(\beta \cup \gamma)$ is a theorem in $\mathfrak{T}$, then $(\beta \cup \gamma)_R \equiv \curlyvee$ and consequently either $\beta_R \equiv \curlyvee$ or $\gamma_R \equiv \curlyvee$, i.e. either β or γ is a theorem in $\mathfrak{T}$.

Theorem 12.8 shows that strongly prime theories based on the classical logic are not interesting from the mathematical point of view (we shall see in X, §§ 12-13, that this is not so if the theory is based on a non-classical logic).

Of course, every strongly prime theory is prime, i.e. maximal. The converse statement is not true. For instance, suppose that a theory $\mathfrak{T}_0 = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}_0\}$ contains only two axioms:

$$\bigcup_\xi \varrho(\xi) \quad \text{and} \quad \bigcup_\xi -\varrho(\xi), \tag{5}$$

where ϱ is a one-argument predicate. $\mathfrak{T}_0$ is consistent and, by 12.6, can be extended to a maximal theory $\mathfrak{T}$. Since formulas (5) are theorems in $\mathfrak{T}$, none of the formulas $\varrho(x)$, $-\varrho(x)$ is a theorem in the consistent theory $\mathfrak{T}$. By 12.8, the prime theory $\mathfrak{T}$ is not strongly prime.

This remark explains, by 12.1, 12.7 and 12.8, that the sentences

$$\nabla \text{ is a maximal } \bigcap\text{-filter in } \mathfrak{A}(\mathfrak{S})$$

and

$$\nabla \text{ is a maximal filter and a } \bigcap\text{-filter in } \mathfrak{A}(\mathfrak{S})$$

are not, in general, equivalent.

An existential formula $\bigcup_\xi \beta(\xi)$ can be conceived of from the intuitive point of view, as an infinite disjunction of all sentences $\beta(\tau)$

where τ is any term. Thus the following notions can be considered as an infinite analogue of the notion of strongly prime theories: a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ is said to be a $\bigcup$-*theory* provided that, for every existential formula $\bigcup_\xi \beta(\xi)$,

(6) if $\bigcup_\xi \beta(\xi)$ is a theorem in $\mathfrak{T}$, then there exists a term τ such that $\beta(\tau)$ is a theorem in $\mathfrak{T}$.

A filter ∇ in the algebra $\mathfrak{A}(\mathfrak{S})$ of the predicate calculus $\mathfrak{S} = \{\mathfrak{L}, \mathfrak{C}\}$ is said to be a $\bigcup$-*filter* provided that, for every existential formula $\bigcup_\xi \beta(\xi)$,

(7) if $\|\bigcup_\xi \beta(\xi)\| \in \nabla$, then there exists a term τ such that $\|\beta(\tau)\| \in \nabla$.

The following theorem follows immediately from the definition.

12.9. *A theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ *is a* $\bigcup$-*theory if and only if* $\nabla_{\mathfrak{A}}$ *is a* $\bigcup$-*filter in* $\mathfrak{A}(\mathfrak{S})$.

Every strongly prime theory $\mathfrak{T}$ is a $\bigcup$-theory. In fact, let $\mathfrak{T}$ be a strongly prime theory and let R be a model for $\mathfrak{T}$. We have shown in the proof of 12.8 that, for every formula a, a_R is a constant function equal to $\vee$ or $\wedge$, and a is a theorem in $\mathfrak{T}$ if and only if $a_R \equiv \vee$. If $\bigcup_\xi \beta(\xi)$ is a theorem in $\mathfrak{T}$, then $\left(\bigcup_\xi \beta(\xi)\right)_R = \vee$. By the inductive definition of a_R (see VI, § 6, (4)), the element $\left(\bigcup_\xi \beta(\xi)\right)_R$ is the join of elements $\left(\beta(\tau)\right)_R$ equal to $\vee$ or $\wedge$, τ being any term. Thus there exists a term τ such that $\beta(\tau)_R \equiv \vee$, i.e. $\beta(\tau)$ is a theorem in $\mathfrak{T}$.

The converse statement is not true, in general. There is a $\bigcup$-theory which is not a strongly prime theory, and even is not a prime (i.e. maximal) theory. For instance, any rich inessential extension $\mathfrak{T}$ of a non-maximal theory (see 8.3) is not a maximal theory. It is a $\bigcup$-theory by the first part of the following theorem.

12.10. *If a theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ *is rich, then it is a* $\bigcup$-*theory, i.e.* $\nabla_{\mathfrak{A}}$ *is a* $\bigcup$-*filter.*

More generally, *every filter* ∇ *in* $\mathfrak{A}(\mathfrak{S})$ *such that* $\nabla_{\mathfrak{A}} \subset \nabla$ *is a* $\bigcup$-*filter.*

It suffices to prove only the second part of 12.10.

Suppose that $\nabla_{\mathfrak{A}} \subset \nabla$ and $\|\bigcup_\xi a(\xi)\|$ is in ∇. Since $\mathfrak{T}$ is rich, there exists a term τ such that $\|(\bigcup_\xi a(\xi) \Rightarrow a(\tau))\|$ is in $\nabla_{\mathfrak{A}}$. Consequently $\|\bigcup_\xi a(\xi)\| \Rightarrow \|a(\tau)\|$ is in ∇. Hence it follows that $\|a(\tau)\|$ is in ∇ (see I, 13.1), i.e. ∇ is a $\bigcup$-filter.

12.11. *A maximal filter in* $\mathfrak{A}(\mathfrak{S})$ *is a* $\bigcup$-*filter if and only if it is a* Q-*filter.*

This follows directly from the definitions of Q-filters in $\mathfrak{A}(\mathfrak{S})$ (see § 1, (a), p. 281).

§ 13. Extensions of theories to theories with equality. Suppose that $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ is an arbitrary theory. Let $\mathfrak{L}'$ be the formalized language obtained from $\mathfrak{L}$ by the addition of a new sign $\mathfrak{e}$ of a binary predicate which corresponds to the relation of equality. Let $\mathfrak{A}'$ be the set of formulas in $\mathfrak{L}'$ consisting of all formulas in $\mathfrak{A}$ and of the axioms V, § 12, (e_1)-(e_5) for equality. The theory $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}, \mathfrak{A}'\}$ is said to be an *extension of* $\mathfrak{T}$ *to a theory with equality.*

13.1. *For any theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ *the extension* $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}, \mathfrak{A}'\}$ *of* $\mathfrak{T}$ *to a theory with equality is an inessential extension of* $\mathfrak{T}$. *Consequently* $\mathfrak{T}$ *is consistent if and only if* $\mathfrak{T}'$ *is consistent.*

For every semantic realization R of $\mathfrak{L}$ in a set $J \neq 0$ let R' be a semantic realization of $\mathfrak{L}'$ in J such that R' is an extension of R and, for every j_1, j_2 in J,

$$\mathfrak{e}_{R'}(j_1, j_2) = \begin{cases} V & \text{if} \quad j_1 = j_2\,, \\ \Lambda & \text{if} \quad j_1 \neq j_2\,. \end{cases}$$

It is easy to see that R' is a model for all axioms V, § 12, (e_1)-(e_5). Moreover, by VI, 6.4, for every formula α in $\mathfrak{L}$ and for every valuation v in J we have

$$(1) \qquad \alpha_R(v) = \alpha_{R'}(v)\,.$$

It follows from these remarks that R is a model for $\mathfrak{T}$ if and only if R' is a model for $\mathfrak{T}'$. If a formula α in $\mathfrak{L}$ is not a theorem in $\mathfrak{T}$ then there exists a semantic model R for $\mathfrak{T}$ in which α is not valid (see 9.2). Then R' is a model for $\mathfrak{T}'$ and, by (1), α is not valid in R'. Thus, by 2.2, α is not a theorem in $\mathfrak{T}'$. Conversely every formula in $\mathfrak{L}$ which is a theorem in $\mathfrak{T}$ is also a theorem in $\mathfrak{T}'$, since $\mathfrak{T}'$ is an extension of $\mathfrak{T}$. Thus $\mathfrak{T}'$ is an inessential extension of $\mathfrak{T}$.

§ 14. The inessentiality of definitions. Suppose that $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ is a formalized theory of the first order.

Let S_1 be a set of formulas

$$(1) \qquad \alpha(x_1, \ldots, x_m) \qquad (m > 0)$$

where the sequence $x_1, \ldots, x_m$ contains all free individual variables occurring in (1). Let S_2 be a set of closed formulas of the form

$$(2) \qquad \bigcap_{\xi_1} \ldots \bigcap_{\xi_m} \bigcup_{\eta} \beta(\xi_1, \ldots, \xi_m, \eta) \qquad (m \geqslant 0)$$

which are theorems in $\mathfrak{T}$. Let S_3 be a set of terms

$$(3) \qquad \tau(x_1, \ldots, x_m) \qquad (m \geqslant 0)\,,$$

where the sequence $x_1, \ldots, x_m$ contains all free individual variables occurring in (3). Some of the sets S_1, S_2, S_3 may be empty. If $\mathfrak{T}$ does not contain the sign of equality $\mathfrak{e}$, then S_3 is presumed to be empty.

We form an extension $\mathfrak{L}'$ of $\mathfrak{L}$ by adding to $\mathfrak{L}$ a set of predicates ϱ_α and of functors φ_β, ψ_τ which do not appear in $\mathfrak{L}$. Namely, for every formula (1) in S_1, ϱ_α is an m-argument predicate. For every formula (2) in S_2, φ_β is an m-argument functor. For every functor (3) in S_3, ψ_τ is an m-argument functor. All the predicates ϱ_α are different. All the functors φ_β, ψ_τ are different.

Let $\mathcal{A}'$ be the set composed of all formulas in $\mathcal{A}$ and of all formulas

$$(\varrho_\alpha(x_1 \dots x_m) \Rightarrow \alpha(x_1, \dots, x_m)) , \quad (\alpha(x_1, \dots, x_m) \Rightarrow \varrho_\alpha(x_1 \dots x_m)) , \tag{4}$$

$$\bigcap_{\xi_1} \dots \bigcap_{\xi_m} \beta(\xi_1, \dots, \xi_m, \varphi_\beta(\xi_1 \dots \xi_m)) , \tag{5}$$

$$\mathfrak{e}(\psi_\tau(x_1 \dots x_m)\, \tau(x_1, \dots, x_m)) . \tag{6}$$

Of course, formulas (6) are included only if $\mathfrak{C}$ contains the sign of equality.

By V, § 15, the theory $\mathfrak{C}' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A}'\}$ is said to be *obtained from* $\mathfrak{C}$ *by simultaneously assuming definitions* (4), (5), (6). Under the above hypotheses we shall prove

14.1. *Every semantic model for* $\mathfrak{C}$ *in a set* J *can be extended to a model for* $\mathfrak{C}'$ *in* J.

Let a semantic realization R be a model for $\mathfrak{C}$ in J. Let

$$\varrho_{R'} = \varrho_R \quad \text{and} \quad \varphi_{R'} = \varphi_R \tag{7}$$

for all predicates ϱ and for all functors φ in $\mathfrak{L}$. The definition of the realization of the remaining predicates and functors in $\mathfrak{L}'$ is as follows (we adopt notation VI, § 6, (2) and VI, § 3, (7)). The mapping $\varrho_{\alpha R'}$ is defined by the equation

$$\varrho_{\alpha R'}(j_1, \dots, j_m) = \alpha_R(j_1, \dots, j_m) \tag{8}$$

for every formula (1). For every formula (2), if $j_1, \dots, j_m \in J$, then there exists an element $j \in J$ such that

$$\beta_R(j_1, \dots, j_m, j) = V . \tag{9}$$

(This follows from the fact that (2) is a theorem in $\mathfrak{C}$ and consequently (2) is valid in R). We define

$$\varphi_{\beta R'}(j_1, \dots, j_m) = j . \tag{10}$$

For every term (3), the mapping $\psi_{\tau R'}$ is defined by the equation

$$\psi_{\tau R'}(j_1, \dots, j_m) = \tau_R(j_1, \dots, j_m) . \tag{11}$$

The realization R' is an extension of R, by (7). Therefore it is a model for $\mathcal{A}$. It follows from (8), (9) and (10), (11) respectively that R' is also a model for formulas (4), (5), (6). Thus R' is a semantic model for $\mathfrak{C}'$.

It is easy to verify that if R' is an extension of R and a model for $\mathfrak{T}'$, then R' has to be defined by (7), (8), (10) and (11). Hence it follows that $\varrho_{aR'}$ and $\psi_{\tau R'}$ are uniquely determined by R. However there is an arbitrariness in the choice of $\varphi_{\beta R'}$ since, for given $j_1, \ldots, j_m$, there exist, in general, many elements j satisfying (9). As we have asserted in V, § 15, we introduce definitions (4) in practice only when, besides formula (2), the formula

$$\bigcap_{\xi_1} \ldots \bigcap_{\xi_m} \bigcap_{\eta_1} \bigcap_{\eta_2} ((\beta(\xi_1, \ldots, \xi_m, \eta_1) \frown \beta(\xi_1, \ldots, \xi_m, \eta_2)) \Rightarrow \mathfrak{e}(\eta_1 \eta_2))$$

is also a theorem in $\mathfrak{T}$. Then, if R is an ordinary semantic model for $\mathfrak{T}$, there exists exactly one j satisfying (9) and, consequently, $\varphi_{\beta R'}$ is also uniquely determined by R.

In V, § 15, we have asserted that, from the intuitive point of view, the assumption of some definitions does not introduce any new mathematical contents to the theory. The next theorem is a precise formulation of this statement. All the above hypotheses and notation are assumed below.

15.2. *The theory $\mathfrak{T}'$ obtained from $\mathfrak{T}$ by assuming definitions* (4), (5), (6) *is an inessential extension of $\mathfrak{T}$.*

Since $\mathscr{A} \subset \mathscr{A}'$, the theory $\mathfrak{T}'$ is an extension of $\mathfrak{T}$. To complete the proof it suffices to show that if a formula γ in $\mathfrak{L}$ is not a theorem in $\mathfrak{T}$ then it is not a theorem in $\mathfrak{T}'$.

By 9.2 there exists a semantic model R for $\mathfrak{T}$ in a set $J \neq 0$ and a valuation v such that

$$\gamma_R(v) \neq V .$$

By 15.1, we can extend R to a model R' for $\mathfrak{T}'$ in the set J. Since R' is an extension of R, we have

$$\gamma_{R'}(v) = \gamma_R(v) \neq V$$

which proves, by 2.2, that γ is not a theorem in $\mathfrak{T}'$.

§ 15. Open theories. We recall that a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathscr{A}\}$ is said to be open if all formulas in $\mathscr{A}$ are open, i.e. if the mathematical axioms of the theory do not contain quantifiers.

If $\mathfrak{L} = \{A, T, F\}$ is a formalized language of the first order, then $\mathfrak{L}^0 = \{A^0, T, F^0\}$ will denote the language of open formulas in $\mathfrak{L}$, according to V, § 14, and VI, § 9 (p. 205 and p. 243). We recall that A^0 is the alphabet obtained from the alphabet A of $\mathfrak{L}$ by ommitting quantifiers and bound individual variables, and the set F^0 of all formulas in $\mathfrak{L}^0$ is the set of all open formulas in $\mathfrak{L}$.

The symbol C^0 will always denote the consequence operation in $\mathfrak{L}^0$, determined (see V, § 10, (S), p. 182) by *modus ponens* (r_1) and the rule of substitution for free individual variables (r_2) and by the set $\mathscr{A}_1^0$ of logical axioms composed of all formulas of the form V, § 11, (T_1)-(T_{12}) where α, β, γ are any formulas in F^0. According to the definition, for any set S of formulas in $\mathfrak{L}^0$, the set $C^0(S)$ consists of all formulas in $\mathfrak{L}^0$ which are derivable from S by means of (r_1), (r_2) and the logical axioms in $\mathscr{A}_1^0$ (see V, § 9). By V, 10.2, $C^0(S)$ is the least set (of formulas in $\mathfrak{L}^0$) containing S and $\mathscr{A}_1^0$ and closed under the rules of inference (r_1) and (r_2) in $\mathfrak{L}^0$. By definition, $C^0(S)$ consists of open formulas in $\mathfrak{L}$ only. Moreover,

$$C^0(S) \subset C(S) . \tag{1}$$

Any set $\mathscr{A}$ of open formulas in $\mathfrak{L}$ uniquely determines two theories, viz. the open theory $\mathfrak{T} = \{\mathfrak{L}, C, \mathscr{A}\}$ of the first order and the theory $\mathfrak{T}^0 = \{\mathfrak{L}^0, C^0, \mathscr{A}\}$ which has an auxiliary character. Formulas in $C^0(\mathscr{A})$ are called *theorems* in $\mathfrak{T}^0$.

According to VI, § 10, the algebra $\mathfrak{A}(\mathfrak{T}^0)$ of the theory $\mathfrak{T}^0$ is the quotient algebra $F^0/\sim$, where $\sim$ is the congruence relation in the algebra $\{F^0, \cup, \cap, \Rightarrow, -\}$ of open formulas, defined as follows:

(2) $\alpha \sim \beta$ if and only if both formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are theorems in $\mathfrak{T}^0$, i.e. belong to $C^0(\mathscr{A})$.

The following theorem results immediately from VI, 10.4, 10.6 and 10.7.

15.1. *The algebra* $\mathfrak{A}(\mathfrak{T}^0)$ *of* $\mathfrak{T}^0 = \{\mathfrak{L}^0, C^0, \mathscr{A}\}$ *is a Boolean algebra. Moreover, for any formulas* α, β *in* $\mathfrak{T}^0$,

$$\begin{aligned} |\alpha| \cup |\beta| &= |(\alpha \cup \beta)| , \quad |\alpha| \cap |\beta| = |(\alpha \cap \beta)| , \\ |\alpha| \Rightarrow |\beta| &= |(\alpha \Rightarrow \beta)| , \quad -|\alpha| = |-\alpha| . \end{aligned} \tag{3}$$

The relation

$$|\alpha| \leqslant |\beta| \tag{4}$$

holds if and only if $(\alpha \Rightarrow \beta)$ *is a theorem in* $\mathfrak{T}^0$. *The element* $|\alpha|$ *is the unit element of* $\mathfrak{A}(\mathfrak{T}^0)$ *if and only if* α *is a theorem in* $\mathfrak{T}^0$. *The Boolean algebra* $\mathfrak{A}(\mathfrak{T}^0)$ *is non-degenerate if and only if the theory* $\mathfrak{T}^0$ *is consistent*, i.e. $C^0(\mathscr{A}) \neq F^0$.

Suppose that the theory $\mathfrak{T}^0$ is consistent. Let h be a Boolean homomorphism of $\mathfrak{A}(\mathfrak{T}^0)$ into a complete Boolean algebra $\boldsymbol{A}$. Let R^0 be the realization of the language $\mathfrak{L}$ in the set T of all terms and the Boolean algebra $\boldsymbol{A}$, defined as follows:

(a) R^0, restricted to all functors in $\mathfrak{L}$, is the canonical realization of terms, i.e. (see VI, § 7, (1))

(5) $\varphi_{R^0}(\tau_1, \ldots, \tau_m) = \varphi(\tau_1 \ldots \tau_m)$ for every m-argument functor φ in $\mathfrak{L}$ $(m = 0, 1, 2, \ldots)$;

(b) for every m-argument predicate ϱ in $\mathfrak{L}$ $(m = 1, 2, \ldots)$,

(6) $$\varrho_{R^0}(\tau_1, \ldots, \tau_m) = h(|\varrho(\tau_1 \ldots \tau_m)|)\,,$$

$\tau_1, \ldots, \tau_m$ being arbitrary terms.

We recall that any valuation $v\colon V \to T$ is also a substitution in $\mathfrak{L}$. For every term τ and for every formula α, the symbols $v\tau$ and $v'\alpha$ denote the result of the substitution v in τ and α, respectively. In particular, for the identity valuation $\mathfrak{i}$, we have $\mathfrak{i}\tau = \tau$ and $\mathfrak{i}'\alpha = \alpha$. Under the above hypotheses and notation the following theorem holds.

15.2. *For every term τ and for every open formula α in $\mathfrak{L}$,*

(7) $$\tau_{R^0}(v) = v\tau\,,$$

(8) $$\alpha_{R^0}(v) = h(|v'\alpha|)\,,$$

for any valuation $v\colon V \to T$. In particular, for the identity valuation $\mathfrak{i}$,

(9) $$\alpha_{R^0}(\mathfrak{i}) = h(|\alpha|)\,.$$

Identity (7) was proved in VI, 7.1.

By VI, 9.3, for every fixed v, both sides of (8), considered as functions of α, are homomorphisms from the algebra F^0 of open formulas into $\boldsymbol{A}$. It follows from (6) and (7) that both homomorphisms coincide on the set of all elementary formulas. Since elementary formulas generate F^0, the two homomorphisms are equal, i.e. (8) holds.

(9) follows immediately from (8).

From 15.2, (8), results immediately

15.3. *R^0 is a model for the open theory $\mathfrak{T}$ in the set $\mathfrak{T}$ of all terms and the Boolean algebra $\boldsymbol{A}$.*

Copying the terminology adopted in VI, § 7, for canonical realizations, we shall call R^0 the *canonical model determined by the homomorphism h.*

As an application of 15.3 we shall prove the following theorem.

15.4. *Every consistent open theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ has a semantic model of a power $\leqslant \overline{\overline{T}}$ where T is the set of all terms in $\mathfrak{T}$. More precisely, $\mathfrak{T}$ has a semantic model of the power $\mathfrak{m} = \max(\aleph_0, \overline{\overline{\Phi}}_{\mathcal{A}})$ where $\Phi_{\mathcal{A}}$ is the set of all functors occurring in formulas in $\mathcal{A}$.*

Consider first the case in which $\overline{\overline{T}} = \mathfrak{m}$. Since the theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ is consistent, the algebra $\mathfrak{A}(\mathfrak{T}^0)$ of the corresponding auxiliary

theory $\mathfrak{T}^0 = \{\mathfrak{L}^0, \mathfrak{C}^0, \mathfrak{A}\}$ is not degenerate and, consequently, it has a maximal filter V. Let h be the natural homomorphism from $\mathfrak{A}(\mathfrak{T}^0)$ onto the two-element Boolean algebra $\boldsymbol{A}_0 = \mathfrak{A}(\mathfrak{T}^0)/V$. The canonical model R^0 determined by the homomorphism h has the required properties.

If $\overline{\overline{T}} > \mathfrak{m}$, let $\mathfrak{L}_1$ be a sublanguage of $\mathfrak{L}$ such that $\mathfrak{L}_1$ contains all functors and predicates occurring in formulas in $\mathfrak{A}$ and $\mathfrak{m} = \overline{\overline{T}}_1$, where T_1 is the set of all terms in $\mathfrak{L}_1$. By 9.7 there exists a set $\mathfrak{A}_1$ of formulas in $\mathfrak{L}_1$ such that $\mathfrak{C}_{\mathfrak{L}}(\mathfrak{A}) = \mathfrak{C}_{\mathfrak{L}}(\mathfrak{A}_1)$. Since $\mathfrak{T}$ is consistent, the open theory $\mathfrak{T}_1 = \{\mathfrak{L}_1, \mathfrak{C}, \mathfrak{A}_1\}$ is also consistent. By the part of 15.4 just proved, there exists a semantic model for $\mathfrak{T}_1$ of power $\mathfrak{m}$. Any extension of this model to a realization of $\mathfrak{L}$ is a semantic model for $\mathfrak{T}$ of the power $\mathfrak{m}$.

Theorem 15.4 is a supplement to theorems 9.3 and 9.8. Theorem 15.4 does not follow from these theorems because it gives another estimation of the power of the semantic model in question. In general, theorem 15.4 asserts the existence of a semantic model of a lesser power than 9.3 and 9.8, but only for open theories.

Modifications of 15.4 which are analogues of theorems 9.1 and 9.2 will be proved later (17.5 and 17.6).

15.5. *An open formula β is a theorem in an open theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ if and only if it is a theorem in the theory $\mathfrak{T}^0 = \{\mathfrak{L}^0, \mathfrak{C}^0, \mathfrak{A}\}$* [1].

In symbols,

$$\mathfrak{C}^0(\mathfrak{A}) = \mathfrak{C}(\mathfrak{A}) \cap F^0 \tag{10}$$

where F^0 is the set of all open formulas in $\mathfrak{T}$.

By (1), it suffices to prove that if an open formula β is not a theorem in $\mathfrak{T}^0$, it is not a theorem in $\mathfrak{T}$. If β is not in $\mathfrak{C}^0(\mathfrak{A})$, then $|\beta|$ is not the unit element of the Boolean algebra $\mathfrak{A}(\mathfrak{T}^0)$ by 15.1. By I, 8.5, there exists a maximal ideal Δ in $\mathfrak{A}(\mathfrak{T}^0)$ such that $|\beta| \in \Delta$. Let h be the natural homomorphism from $\mathfrak{A}(\mathfrak{T}^0)$ onto the two-element Boolean algebra $\boldsymbol{A}_0 = \mathfrak{A}(\mathfrak{T}^0)/\Delta$ and let R^0 be the canonical model for $\mathfrak{T}$ determined by the homomorphism h. Since $|\beta| \in \Delta$, we have

$$\beta_{R^0}(\mathrm{i}) = h(|\beta|) = \Lambda \in \boldsymbol{A}_0$$

by (9). This proves that β is not a theorem in $\mathfrak{T}$ (see 2.2).

Theorem 15.5 is called the *theorem on elimination of quantifiers in formal proofs*. To explain the name, let us recall that a formula α is a theorem in $\mathfrak{T}$ if there exists a sequence $\alpha_1, \ldots, \alpha_n$ which is a formal proof of α in $\mathfrak{T}$. Theorem 15.5 asserts that if $\mathfrak{T}$ is open, α is open and if α has a formal proof in $\mathfrak{T}$ then α has a formal proof $\alpha_1, \ldots, \alpha_n$ formed only of formulas without quantifiers.

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[1] Hilbert and Bernays [1], vol. II. The quoted proof is due to Rasiowa [6], [7].

Let $\mathfrak{A}(\mathfrak{C})$ be the algebra of $\mathfrak{C} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$. We recall that elements of $\mathfrak{A}(\mathfrak{C})$ are equivalence classes $\|\alpha\|$ of the relation $\approx$ defined as follows:

(11) $\alpha \approx \beta$ if and only if both the formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are theorems in $\mathfrak{C}$.

It follows immediately from definitions (2) and (11) and from theorem 15.5 that, for any open formulas α, β,

(12) $$\alpha \sim \beta \quad \text{if and only if} \quad \alpha \approx \beta .$$

Let $\mathfrak{A}^0(\mathfrak{C})$ denote the subalgebra of $\mathfrak{A}(\mathfrak{C})$ consisting of all elements $\|\alpha\|$ where α is any open formula.

15.6. *The mapping*

(13) $$h_0(|\alpha|) = \|\alpha\| \quad (\alpha \text{ open})$$

is an isomorphism from the Boolean algebra $\mathfrak{A}(\mathfrak{C}^0)$ *onto the Boolean algebra* $\mathfrak{A}^0(\mathfrak{C})$.

In fact, it follows immediately from (12) that h_0 is a one-to-one mapping from $\mathfrak{A}(\mathfrak{C}^0)$ onto $\mathfrak{A}^0(\mathfrak{C})$. It follows immediately from the definitions of Boolean operations in $\mathfrak{A}(\mathfrak{C}^0)$ and $\mathfrak{A}(\mathfrak{C})$ (see (3) and § 1, (2)) that h_0 preserves all those operations, i.e. h_0 is an isomorphism.

§ 16. The prenex form. To simplify the notation in the next sections it is convenient to introduce the following abbreviations.

Letters $\boldsymbol{x}, \boldsymbol{y}, \ldots, \boldsymbol{\xi}, \boldsymbol{\eta}, \ldots$, with indices if necessary, will denote some finite sequences of distinct (free or bound, respectively) individual variables:

$$\boldsymbol{x} = \{x_1, \ldots, x_m\}, \qquad \boldsymbol{y} = \{y_1, \ldots, y_n\},$$

$$\boldsymbol{x}_i = \{x_{i1}, \ldots, x_{im_i}\}, \qquad \boldsymbol{y}_i = \{y_{i1}, \ldots, y_{in_i}\},$$

$$\boldsymbol{\xi} = \{\xi_1, \ldots, \xi_m\}, \qquad \boldsymbol{\eta} = \{y_1, \ldots, y_n\},$$

$$\boldsymbol{\xi}_i = \{\xi_{i1}, \ldots, \xi_{im_i}\}, \qquad \boldsymbol{\eta}_i = \{\eta_{i1}, \ldots, \eta_{in_i}\}.$$

Consequently expressions of the form

$$\beta(x_1, \ldots, x_m, \xi_{11}, \ldots, \xi_{1m_1}, \ldots, \xi_{r1}, \ldots, \xi_{rm_r})$$

will be denoted in brief by

$$\beta(\boldsymbol{x}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r).$$

Of course,

$$\bigcup_{\boldsymbol{\xi}_1} \bigcap_{\boldsymbol{\xi}_2} \ldots \bigcup_{\boldsymbol{\xi}_r} \beta(\boldsymbol{x}, \boldsymbol{\xi}_1, \ldots, \boldsymbol{\xi}_r)$$

will be an abbreviation for

$$\bigcup \xi_{11} \dots \bigcup \xi_{1m_1} \bigcap \xi_{21} \dots \bigcap \xi_{2m_2} \dots \bigcup \xi_{r1} \dots \bigcup \xi_{rm_r} \beta(x_1, \dots, x_m, \xi_{11}, \dots, \xi_{1m_1}, \dots, \xi_{r1}, \dots, \xi_{rm_r})$$

etc.

$\boldsymbol{x}, \boldsymbol{y}, \dots$ are called *complex free variables,* and $\boldsymbol{\xi}, \boldsymbol{\eta}, \dots$ are called *complex bound variables.*

We shall say that the complex free variables $\boldsymbol{x}$ and $\boldsymbol{y}$ are *disjoint* if no free individual variable occurs simultaneously in the sequences $\boldsymbol{x}$ and $\boldsymbol{y}$. The same terminology is assumed for complex bound variables $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$. Hereafter we shall always assume that all complex variables denoted by different symbols are disjoint.

Sometimes the case in which some of the numbers m, m_i, n, n_i are equal to zero is admitted. This should be so interpreted that the complex variable does not occur at all on the place where it is written.

An analogous notation will be applied for terms and functors. The letters τ and φ, with indices if necessary, will denote finite sequences of terms or functors respectively:

$$\tau = \{\tau_1, \dots, \tau_q\}, \qquad \varphi = \{\varphi_1, \dots, \varphi_p\},$$
$$\tau_i = \{\tau_{i1}, \dots, \tau_{iq_i}\}, \qquad \varphi_i = \{\varphi_{i1}, \dots, \varphi_{ip_i}\}.$$

The sense of symbols like

$$\varphi(\tau),$$
$$\varphi(x_1 \dots x_k),$$
$$\beta\big(x, \varphi_1(x_1), \varphi_2(x_1\, x_2)\big),$$
$$\beta(x_1, x_2, \tau_1, \tau_2)$$

is obvious: they are abbreviations for

$$\{\varphi_1(\tau_1 \dots \tau_q), \dots, \varphi_p(\tau_1 \dots \tau_q)\},$$
$$\{\varphi_1(x_{11} \dots x_{1m_1} \dots x_{k1} \dots x_{km_k}), \dots, \varphi_p(x_{11} \dots x_{1m_1} \dots x_{k1} \dots x_{km_k})\},$$
$$\beta\big(x_1 \dots x_{m_1} \varphi_{11}(x_{11} \dots x_{1m_1}), \dots, \varphi_{1p_1}(x_{11} \dots x_{1m_1}), \varphi_{21}(x_{11} \dots x_{1m_1} x_{21} \dots x_{2m_2}), \dots, \varphi_{2p_2}(x_{11} \dots x_{1m_1} x_{21} \dots x_{2m_2})\big),$$
$$\beta(x_{11}, \dots, x_{1m_1}, x_{21}, \dots, x_{2m_2}, \tau_{11}, \dots, \tau_{1q_1}, \tau_{21}, \dots, \tau_{2q_2}),$$

respectively.

τ and φ will be called *complex terms* and *complex functors*, respectively.

The symbol Ω_{ξ} will always denote any finite sequence of signs

(1) $$\Xi^1_{\xi_1} \Xi^2_{\xi_2} \dots \Xi^m_{\xi_m}$$

where, for $i = 1, \ldots, m$, Ξ^i is either the sign $\bigcup$ or the sign $\bigcap$ (the case of $m = 0$ is admitted). Thus, if $\beta(\boldsymbol{x})$ is any formula, then $\Omega_{\boldsymbol{\xi}}\beta(\boldsymbol{\xi})$ denotes the formula

$$\Xi^1_{\xi_1}\Xi^2_{\xi_2} \ldots \Xi^m_{\xi_m}\beta(\xi_1, \ldots, \xi_m) .$$

If $\Omega_{\boldsymbol{\xi}}$ denotes the sequence (1), then $\bar{\Omega}_{\boldsymbol{\xi}}$ denote the sequence

$$\bar{\Xi}^1_{\xi_1}\bar{\Xi}^2_{\xi_2} \ldots \bar{\Xi}^m_{\xi_m}$$

where, for $i = 1, \ldots, m$, $\bar{\Xi}^i$ is the sign $\bigcap$ (the sign $\bigcup$) if Ξ^i is the sign $\bigcup$ (the sign $\bigcap$).

The symbols $\Omega'_{\boldsymbol{\xi}_1}$ and $\Omega''_{\boldsymbol{\xi}_2}$ will also denote sequences of quantifiers of form (1).

A formula α is said to be *prenex* provided it is of the form

$$\Omega_{\boldsymbol{\xi}}\beta(\boldsymbol{\xi}) \tag{2}$$

where $\beta(\boldsymbol{x})$ is an open formula (which may also contain free individual variables other than those which occur in $\boldsymbol{x}$). Roughly speaking, α is prenex if all quantifiers occurring in α are collected (together with the associated bound variables) at the beginning of the formula, the remaining part of α being an expression obtained from an open formula by a substitution of the bound variables associated with quantifiers for some free individual variables. In particular, every open formula is prenex (Ω is then the empty sequence!).

A prenex formula α is said to be a *prenex form of a formula* γ provided both implications

$$(\alpha \Rightarrow \gamma), \quad (\gamma \Rightarrow \alpha) \tag{3}$$

are predicate tautologies. In particular, every prenex formula is a prenex form for itself.

16.1. *Every formula γ in a formalized language of the first order has a prenex form α.*

The proof of 16.1 is by induction on the length of γ.

This statement is true when γ is an elementary formula since γ is then open and consequently γ is a prenex form for γ.

If a formula α of the form (2) is a prenex form of γ, then the formula α':

$$\bar{\Omega}_{\boldsymbol{\xi}} - \beta(\boldsymbol{\xi})$$

is a prenex form for $-\gamma$. In fact, (3) means that, for every realization R of $\mathfrak{L}$ is any complete Boolean algebra,

$$\alpha_R = \gamma_R .$$

It follows easily from the De Morgan laws for Boolean algebra (see II, 1.4, (19), (20)) that

$$a'_R = -a_R$$

for every realization R. Hence

$$a'_R = (-\gamma)_R$$

for every realization R, i.e. both the implications

$$(\alpha' \Rightarrow -\gamma), \quad (-\gamma \Rightarrow \alpha')$$

are predicate tautologies (see 6.1).

Suppose now that $\Omega'_{\xi_1}\beta_1(\xi_1)$ and $\Omega''_{\xi_2}\beta_2(\xi_2)$ are prenex normal forms of formulas γ_1 and γ_2 respectively. We can always suppose that ξ_1 and ξ_2 are disjoint (for if not, we can replace $\Omega''_{\xi_2}\beta_2(\xi_2)$ by a congruent formula with a set of bound variables disjoint from that in ξ_1; this congruent formula will also be a prenex form of γ_2). Then

$$\Omega'_{\xi_1}\Omega''_{\xi_2}(\beta_1(\xi_1) \cup \beta_2(\xi_2)) \quad \text{is a prenex form of} \quad (\gamma_1 \cup \gamma_2),$$

$$\Omega'_{\xi_1}\Omega''_{\xi_2}(\beta_1(\xi_1) \cap \beta_2(\xi_2)) \quad \text{is a prenex form of} \quad (\gamma_1 \cap \gamma_2),$$

$$\bar{\Omega}'_{\xi_1}\Omega''_{\xi_2}(\beta_1(\xi_1) \Rightarrow \beta_2(\xi_2)) \quad \text{is a prenex form of} \quad (\gamma_1 \Rightarrow \gamma_2).$$

The proof is similar to that in the case of negation. It is based on the Boolean laws II, 1.4, (23)-(28) and I, 7.1, (9), (10).

If $\alpha(x)$ is a prenex form of $\beta(x)$, then $\bigcap_\xi \alpha(\xi)$ and $\bigcup_\xi \alpha(\xi)$ are prenex forms of $\bigcap_\xi \beta(\xi)$ and $\bigcup_\xi \beta(\xi)$ respectively.

Another proof of 16.1 can be obtained by applications of tautologies (T_{34})-(T_{53}) instead of the corresponding Boolean identities quoted above.

We shall usually write the prenex form α of γ either in the form

$$\bigcup_{\xi_1}\bigcap_{\eta_1} \dots \bigcup_{\xi_r}\bigcap_{\eta_r} \beta(x, \xi_1, \eta_1, \dots, \xi_r, \eta_r) \tag{4}$$

where $r \geqslant 0$ and (using the notation from p. 330)

$$m \geqslant 0, \quad m_1 \geqslant 0, \quad m_2, \dots, m_r > 0$$

$$n_1, \dots, n_{r-1} > 0, \quad n_r \geqslant 0,$$

or in the form

$$\bigcap_{\xi_1}\bigcup_{\eta_1} \dots \bigcap_{\xi_r}\bigcup_{\eta_r} \beta(x, \xi_1, \eta_1, \dots, \xi_r, \eta_r) \tag{5}$$

where r, m, m_i, n_i satisfy the same conditions. Of course, $m = 0$, or $m_1 = 0$, or $n_r = 0$ means that the corresponding variables and quantifiers do not occur at all. In (4) and (5),

$$\beta(x, x_1, y_1, \dots, x_r, y_r)$$

is an open formula.

§ 17. The elimination of quantifiers from the axioms of a theory. Open theories have many special properties which do not hold for arbitrary theories. In a certain sense, they are the simplest and the most regular theories of the first order. For instance, we have seen in § 15 that the fundamental theorem 15.4 on the existence of semantic models for open theories can be proved much more simply than the general theorem 9.3 on the existence of semantic models for arbitrary theories. In § 23 and § 26 we shall also quote some properties which characterize open theories only.

The main purpose of this section is to show that, in a sense, the difference between open theories and arbitrary theories is not essential. Namely, every theory of the first order can be turned into an open theory by adding a set of new functors and by replacing axioms of the theory by open formulas which, roughly speaking, express the same mathematical contents. A little more precisely, we shall prove that for every theory $\mathfrak{T}$ there exists an inessential extension $\mathfrak{T}^*$ which is an open theory, and the mathematical axioms of $\mathfrak{T}^*$ are obtained from the mathematical axioms of $\mathfrak{T}$ by a suitable elimination of quantifiers (see 17.3 below).

Let $\mathfrak{L} = \{A, T, F\}$ be a formalized language of the first order and let $\mathfrak{L}' = \{A', T', F'\}$ be a rich extension of $\mathfrak{L}$ (see § 8, p. 307). We recall that $\mathfrak{L}'$ was obtained from $\mathfrak{L}$ by adding a set Ψ of functors. Moreover, there exists a mapping ψ from the set E' of all existential formulas in $\mathfrak{L}'$ onto Ψ, such that if α is an existential formula with m free individual variables, i.e. if α is of the form

$$\bigcup_\xi \beta(\xi, x_1, \ldots, x_m), \tag{1}$$

then ψ_α is an m-argument functor which does not occur in α. Then α' will denote the formula

$$\beta(\psi_\alpha(x_1 \ldots x_m), x_1, \ldots, x_m) \tag{2}$$

and α'' will denote the implication $(\alpha \Rightarrow \alpha')$. The set of all formulas α'' where α is in E' is denoted by $\mathfrak{B}$. The fundamental property of the rich extension of $\mathfrak{L}$ is expressed by the theorem 8.1 stating that every semantic realization R of $\mathfrak{L}$ can be extended to a semantic realization R' of $\mathfrak{L}'$ such that all formulas in $\mathfrak{B}$ are valid in R'.

Let α be a formula in $\mathfrak{L}'$ such that the first sign in α is a quantifier. If this quantifier is existential, i.e. if α is of the form (1), then α' (i.e. formula (2)) will be called the *formula obtained from α by elimination of the first quantifier*. If the first sign of α is the universal quantifier, i.e. if α is of the form

$$\bigcap_\xi \beta(\xi, x_1, \ldots, x_m),$$

then each formula of the form

$$\beta(x, x_1, \ldots, x_m),$$

where x is a free individual variable which does not occur in α, is called the *formula obtained from α by elimination of the first quantifier.*

17.1. *If γ is a formula obtained from a formula α by elimination of the first quantifier,* then

(i) *α belongs to* $\mathfrak{C}_{\mathfrak{L}'}(\gamma)$;

(ii) *if a realization R' of $\mathfrak{L}'$ is a model for $\mathfrak{B}$, and α is valid in R', then γ is also valid in R'.*

When the first quantifier in α is universal, (i) follows from the rule of generalization (see VI, 11.4), and (ii) follows from the inductive definition of α_R (see VI, § 6, (4)).

When α is an existential formula, the formula $(\gamma \Rightarrow \alpha)$ is a predicate tautology (see § 6, (T_{32})). By *modus ponens* we obtain (i). Since $(\alpha \Rightarrow \gamma)$ is in $\mathfrak{B}$, we have

$$\alpha_{R'}(v) \Rightarrow \gamma_{R'}(v) = (\alpha \Rightarrow \gamma)_{R'}(v) = \mathrm{V}$$

for every valuation v. Hence if $\alpha_{R'}(v) = \mathrm{V}$ for every v, then also $\gamma_{R'}(v) = \mathrm{V}$ for every v, i.e. (ii) holds.

Let α be a formula in $\mathfrak{L}'$. An open formula α^* is said to be *obtained from α by elimination of quantifiers* provided there exists a sequence

$$\alpha_1, \ldots, \alpha_n \tag{3}$$

of formulas, such that

1^0 α_1 is a prenex form of α;

2^0 α_{k+1} is obtained from α_k by elimination of the first quantifier ($k = 1, \ldots, n-1$);

3^0 α_n is the formula α^*.

By 16.1 we may assume that α_1 is of the form

$$\bigcap_{\xi_1} \bigcup_{\eta_1} \cdots \bigcap_{\xi_r} \bigcup_{\eta_r} \beta(x, \xi_1, \eta_1, \ldots, \xi_r, \eta_r).$$

Then the formula α^* is of the form

$$\beta\big(x, x_1, \varphi_1(x\, x_1), \ldots, x_r, \varphi_r(x\, x_1 \ldots x_r)\big)$$

where all complex individual variables $x, x_1, \ldots, x_r$ are disjoint and all complex functors $\varphi_i = \{\varphi_{i1}, \ldots, \varphi_{im_i}\}$ are formed of different functors in Ψ.

For every formula α in $\mathfrak{L}'$ there exists a sequence (3) such that 1^0 and 2^0 hold and α_n is an open formula. Thus for every formula α there exists an open formula α^* which is obtained from α by elimination of quantifiers. If α is open, then we may assume α to be the formula α^*.

17.2. *If α^* is obtained from a formula α by elimination of quantifiers, then*

(i) *α belongs to $C_{\mathfrak{L}'}(\alpha^*)$;*

(ii) *if a realization R' of $\mathfrak{L}'$ is a model for $\mathfrak{B}$, and α is valid in R', then α^* is also valid in R'.*

Theorem 17.2 follows immediately from 17.1.

Now let $\mathcal{A}$ be a set of formulas in the language $\mathfrak{L}$. Assign, to every formula α in $\mathcal{A}$, an open formula α^* (in $\mathfrak{L}'$) obtained from α by elimination of quantifiers. Let $\mathcal{A}^*$ be the set of all the formulas α^* (α in $\mathcal{A}$). Let $\mathfrak{L}^*$ be the least sublanguage of $\mathfrak{L}'$ such that $\mathfrak{L}^*$ is an extension of $\mathfrak{L}$ and $\mathfrak{L}^*$ contains all the formulas in $\mathcal{A}^*$. In other words, $\mathfrak{L}^*$ is the language obtained from $\mathfrak{L}$ by adding the set Ψ^* of all the functors from Ψ which occur in formulas in $\mathcal{A}^*$. Note that

$$\max(\aleph_0, \overline{\overline{\Psi^*}}) \leqslant \max(\aleph_0, \overline{\overline{\mathcal{A}}}) . \tag{4}$$

The theory $\{\mathfrak{L}^*, C, \mathcal{A}^*\}$ is said to be *obtained from the theory* $\{\mathfrak{L}, C, \mathcal{A}\}$ *by the elimination of quantifiers from axioms of* $\mathfrak{C}$.

17.3. *The open theory $\mathfrak{C}^* = \{\mathfrak{L}^*, C, \mathcal{A}^*\}$ is an inessential extension of the theory $\mathfrak{C} = \{\mathfrak{L}, C, \mathcal{A}\}$* [1].

Every semantic model R for $\mathfrak{C}$ can be extended to a semantic model R^* for $\mathfrak{C}^*$. Conversely, for every model R^* for $\mathfrak{C}^*$, the restriction R of R^* to the language $\mathfrak{L}$ is a model for $\mathfrak{C}$.

It follows from 17.2, (i), that $\mathcal{A} \subset C_{\mathfrak{L}^*}(\mathcal{A}^*)$. Consequently $C_{\mathfrak{L}}(\mathcal{A}) \subset C_{\mathfrak{L}^*}(\mathcal{A}) \subset C_{\mathfrak{L}^*}(\mathcal{A}^*)$. Thus the theory $\mathfrak{C}^*$ is an extension of the theory $\mathfrak{C}$. Moreover, if R^* is a model for $\mathfrak{C}^*$, the restriction R of R^* to $\mathfrak{L}$ is a model for $\mathfrak{C}$ by 2.2.

Suppose that R is a semantic model for $\mathfrak{C}$. Let R' be an extension of R such that R' is a realization of $\mathfrak{L}'$ and all formulas in $\mathfrak{B}$ are valid in R'. The restriction R^* of R' to the language $\mathfrak{L}^*$ is an extension of R and a model for $\mathfrak{C}^*$. This follows from 17.2, (ii), since $\beta_{R'}(v) = \beta_{R^*}(v)$ for every formula β in $\mathfrak{L}^*$ (see 2.7).

If a formula β in $\mathfrak{L}$ is not a theorem in $\mathfrak{C}$, then, by 9.2, there exists a semantic model R for $\mathfrak{C}$ and a valuation v such that $\beta_R(v) = \Lambda$. By the part of 17.3 just proved, R can be extended to a model R^* for $\mathfrak{C}^*$. Since $\beta_{R^*}(v) = \beta_R(v) = \Lambda$ (see 2.7), β is not a theorem in $\mathfrak{C}^*$ on account of 2.2. This proves that $\mathfrak{C}^*$ is an inessential extension of $\mathfrak{C}$.

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[1] Hilbert and Bernays [1], vol II. The above method of elimination of quantifiers is due to Skolem. The quoted proof is a modification of an algebraic proof given by Rasiowa [7].

In the above proof of 17.3 we have applied theorem 9.2 on the existence of semantic models for arbitrary (in general non-enumerable) consistent theories. However, by a slight modification of the above proof, we can prove 17.3 using only a weaker form of 9.2, viz. theorem 5.2 on the existence of semantic models for enumerable theories. Namely, we can prove the inessentiality of $\mathfrak{T}^*$ as follows: Suppose that a formula β in $\mathfrak{T}$ is a theorem in $\mathfrak{T}^*$ but it is not a theorem in $\mathfrak{T}$. Then there exists a finite subset $\mathcal{A}_0^* \subset \mathcal{A}^*$ such that β belongs to $C_{\mathfrak{L}^*}(\mathcal{A}_0^*)$, i.e. that β is a theorem of the theory $\{\mathfrak{L}^*, \mathfrak{C}, \mathcal{A}_0^*\}$. Denote by $\mathcal{A}_0$ the corresponding finite subset of $\mathcal{A}$. Since β is not a theorem in $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}_0\}$, by 5.2 there is a semantic model R for $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}_0\}$ and a valuation v such that $\beta_R(v) = \Lambda$. The model R can be extended to a model R^* for the theory $\{\mathfrak{L}^*, \mathfrak{C}, \mathcal{A}_0^*\}$. Moreover, $\beta_{R^*}(v) = \beta_R(v) = \Lambda$. This contradicts the hypothesis that β is a theorem in $\{\mathfrak{L}^*, \mathfrak{C}, \mathcal{A}_0^*\}$.

It follows from the above remark that theorems 15.4 and 17.3 can be used (together with theorem 5.2) as a basis for another proof [1] of the fundamental theorem 9.3 on the existence of semantic models for consistent (in general, non-enumerable) theories. This new proof is as follows: Suppose that $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ is consistent. By the first part of 17.3, the open theory $\mathfrak{T}^* = \{\mathfrak{L}^*, \mathfrak{C}, \mathcal{A}^*\}$ is also consistent (see V, 14.2). By 15.4 there exists a semantic model R^* for $\mathfrak{T}^*$ in a set $J \neq 0$. By 17.3, the restriction R of R^* to $\mathfrak{L}$ is a semantic model for $\mathfrak{T}$ in the set J.

The above proof of theorem 9.3 yields, in general, a lower estimation of the power of the set J since, by (4), $\max(\aleph_0, \overline{\overline{\mathcal{A}}}) \geqslant \max(\aleph_0, \overline{\overline{\Phi}}_{\mathcal{A}^*})$ where $\Phi_{\mathcal{A}^*}$ is the set of all functors occurring in formulas in $\mathcal{A}^*$.

More precisely, we have

17.4. *Suppose that a set $\mathcal{A}$ of formulas in $\mathfrak{L}$ consists of a set $\mathcal{A}_0$ of open formulas and a set $\mathcal{A}_1$ of non-open formulas. If the theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ is consistent, it has a semantic model of power $\max(\aleph_0, \overline{\overline{\mathcal{A}_1}}, \overline{\overline{\Phi}}_{\mathcal{A}_0})$ where $\Phi_{\mathcal{A}_0}$ is the set of all functors in formulas in $\mathcal{A}_0$.*

We repeat the argumentation just used. For every α in $\mathcal{A}_1$, let α^* be an open formula obtained from α by elimination of quantifiers. Let $\mathcal{A}_1^*$ be the set of all formulas α^* (α in $\mathcal{A}_1$) and let $\mathcal{A}^* = \mathcal{A}_0 \cup \mathcal{A}_1^*$. By 17.3, the theory $\mathfrak{T}^* = \{\mathfrak{L}^*, \mathfrak{C}, \mathcal{A}^*\}$ is consistent. By 15.4, the open theory $\mathfrak{T}^*$ has a semantic model of power $\max(\aleph_0, \overline{\overline{\Phi}}_{\mathcal{A}^*})$ where $\Phi_{\mathcal{A}^*}$ is the set of all functors in formulas in $\mathcal{A}^*$. This cardinal is equal to $\max(\aleph_0, \overline{\overline{\mathcal{A}_1}}, \overline{\overline{\Phi}}_{\mathcal{A}_0})$. Thus the restriction R of R^* to $\mathfrak{L}$ is the required semantic model for $\mathfrak{T}$, by 17.3.

Now we are able to prove the following supplements to theorem 15.4.

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[1] Łoś [5].

17.5. *If a is an irrefutable formula in an open theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$, then there exists a semantic model R for $\mathfrak{T}$ of the power $\mathfrak{m} = \max(\aleph_0, \overline{\overline{\Phi}}_{\mathfrak{A}})$ (where $\Phi_{\mathfrak{A}}$ is the set of all functors occurring in formulas in $\mathfrak{A}$) such that a is satisfiable in R.*

Suppose that $x_1, \ldots, x_n$ are all free individual variables in a and denote a by $a(x_1, \ldots, x_n)$. By VI, 11.6, the closed formula

$$\bigcup_{\xi_1} \ldots \bigcup_{\xi_n} a(\xi_1, \ldots, \xi_n),$$

denoted for brevity by γ, is also irrefutable. By 11.2 the theory $\{\mathfrak{L}, \mathfrak{C}, \mathfrak{A} \cup [\gamma]\}$ is consistent. By 17.4 (where $\mathfrak{A}_0$ is replaced by $\mathfrak{A}$ and $\mathfrak{A}_1$ consists only of the formula γ), this theory has a semantic model R of the power $\mathfrak{m}$. This implies that R is a semantic model for $\mathfrak{T}$ and γ is satisfiable in R. By 4.2, the formula a is also satisfiable in R.

17.6. *If a formula β is not a theorem in an open theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$, then there exists a semantic model R for $\mathfrak{T}$ of the power $\mathfrak{m} = \max(\aleph_0, \overline{\overline{\Phi}}_{\mathfrak{A}})$ (where $\Phi_{\mathfrak{A}}$ is the set of all functors occurring in formulas in $\mathfrak{A}$) such that β is not valid in R.*

17.6 follows from the dual statement 17.5 by replacing a by $-\beta$ (see § 2, (3), (4) and VI, 10.11).

§ 18. Products of semantic realizations modulo a prime filter. Let $\mathfrak{L}$ be a formalized language of the first order. Let $\mathfrak{F}$ be a prime filter in the Boolean algebra of all subsets of a non-empty set N and, for every $n \in N$, let R_n be a semantic realization of $\mathfrak{L}$ in a set $J_n \neq 0$. By the *product of the realizations R_n modulo the prime filter* (1) $\mathfrak{F}$ we shall mean the semantic realization R of $\mathfrak{L}$ in a two-element Boolean algebra $\boldsymbol{A}_0$ and in the set

$$J = P_{n \in N} J_n$$

defined as follows: for any points $j_1 = \{j_{1,n}\}_{n \in N}, \ldots, j_m = \{j_{m,n}\}_{n \in N}$ in J,

1) if φ is an m-argument functor ($m = 0, 1, 2, \ldots$), then

$$\varphi_R(j_1, \ldots, j_m) = \{\varphi_{R_n}(j_{1,n}, \ldots, j_{m,n})\}_{n \in N} \in J ;$$

2) if ϱ is an m-argument predicate ($m = 1, 2, \ldots$), then

$$\varrho_R(j_1, \ldots, j_m) = \begin{cases} V & \text{if} \quad Nj_1, \ldots, j_m \in \mathfrak{F}, \\ \Lambda & \text{if} \quad Nj_1, \ldots, j_m \notin \mathfrak{F}, \end{cases}$$

where $Nj_1, \ldots, j_m$ is the set of all $n \in N$ such that

$$\varrho_{R_n}(j_{1,n}, \ldots, j_{m,n}) = V .$$

(1) This notion and theorem 18.2 is due to Łoś [7]. See also Frayne, Scott, Tarski [1], Frayne, Morel, Scott [1], [2], Morel and Tarski [1], Frayne and Scott [1].

In other words, for every $n \in N$, ϱ_{R_n} is the characteristic of an m-argument relation, say r_n. Then ϱ_R is the characteristic of the compound relation r which holds between some points $j_1, \ldots, j_m \in J$ if and only if there exists a set $N' \in V$ such that, for every $n \in N'$, the relation r_n holds between the nth coordinates of $j_1, \ldots, j_m$.

The definition of the product R of all R_n modulo V can also be formulated in another way, with the help of the notion of the product of realizations introduced in VI, § 8. By definition, R_n is a realization of $\mathfrak{L}$ in J_n and in a two-element Boolean algebra A_n, $n \in N$. Let R_0 be the product of all the realizations R_n (see VI, § 8, 1° and 2°). We recall that R_0 is a realization of $\mathfrak{L}$ in the set J and in the product $A = P_{n \in N} A_n$ of the two-element complete Boolean algebras

$$\{A_n, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}.$$

According to the remark after theorem II, 7.3, we can conceive of A as the field $\mathfrak{B}(N)$ of all subsets of N. Consequently we can conceive of V as a prime filter in A. Let h be the natural homomorphism of the Boolean algebra A onto the two-element Boolean algebra $A_0 = A/V$. It follows from 1) and 2) and the definition of R_0 from VI, § 8, 1°, 2° (see also VI, § 8, (9)) that

$$R = hR_0 .$$

This is the second equivalent definition of the product R of the semantic realizations R_n modulo the prime filter V.

Let V denote, as usual, the set of all free individual variables in $\mathfrak{L}$. We recall (see VI, § 8, p. 241) that the valuations $v = \{v_x\}_{x \in V} \in J^V$ in J are mappings which, to every free individual variable x, assign a point $v_x = \{v_{x,n}\}_{n \in N} \in J$. By definition, $v_{x,n} \in J_n$. Thus, for every fixed $n \in N$, $v^n = \{v_{x,n}\}_{x \in V} \in J_n^V$ is a valuation in J_n. Conversely if, for every $n \in N$, $v^n = \{v_{x,n}\}_{x \in V}$ is a valuation in J_n, then putting $v_x = \{v_{x,n}\}_{n \in N}$ we define a valuation $v = \{v_x\}_{x \in V}$ in J.

Using the above notation we have the following theorem:

18.1. *For every formula a in $\mathfrak{L}$,*

$$a_R(v) = V \quad \textit{if and only if} \quad N(a, v) \in V$$

where $N(a, v)$ is the set of all $n \in N$ such that

$$a_{R_n}(v^n) = V .$$

By VI, 8.3, and II, 7.5,

$$a_R(v) = a_{hR_0}(v) = h\big(a_{R_0}(v)\big) .$$

By VI, 8.2, $a_{R_0}(v) = \{a_{R_n}(v^n)\}_{n \in N}$. Hence

$$a_R(v) = h\left(\{a_{R_n}(v^n)\}_{n \in N}\right).$$

The last identity is another equivalent formulation of theorem 18.1 (1).

Theorem 18.1 can also be proved directly, by induction on the length of a.

18.2. *For any formula a, if the set $N(a)$ of all n such that R_n is a model for a belongs to $\mathcal{V}$, then R is a model for a.*

In fact, $N(a) \subset N(a, v)$ for every valuation v. Since $N(a) \in \mathcal{V}$, we also have $N(a, v) \in \mathcal{V}$. By 18.1, $a_R(v) = \mathcal{V}$ for every valuation v, i.e. R is a model for a.

As another application of theorem 18.1 we shall quote the third proof of the fundamental theorem 9.3 on the existence of semantic models for consistent theories $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ with arbitrary (finite, enumerable or non-enumerable) sets $\mathcal{A}$ of axioms.

The case in which $\mathcal{A}$ is finite or enumerable is very easy to prove. It was obtained by us (see 5.2) as a corollary from a general theorem II, 9.3, on the existence of Q-filters for any enumerable set (Q) of finite joins and meets in a Boolean algebra. The infinite joins and meets in the proof were the joins and meets corresponding to logical quantifiers in the algebra of a consistent theory. The general theorem II, 9.3, on the existence of Q-filters cannot be generalized to the case of an non-enumerable set Q of infinite joins and meets and, therefore, the above method cannot be applied to prove the existence of semantic models for consistent theories with an non-enumerable set of axioms. Consequently the non-enumerable case was proved by us in § 9 by another method which, roughly speaking, is based on a trivialization or elimination of quantifiers. Another proof of the existence of semantic models for consistent theories with arbitrary sets of axioms was given in § 17. It was based on the idea to pass to open theories, i.e. also on the elimination of quantifiers. However, both proofs of the general theorem on the existence of semantic models for $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ were based on the theorem on the existence of semantic models for consistent theories $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}'\}$ with finite sets $\mathcal{A}'$ of axioms. This is closely connected with the fact that the theory $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ is consistent if and only if all theories $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}'\}$, where $\mathcal{A}'$ is any finite subset of $\mathcal{A}$, are consistent. We recall that the last remark is an immediate result of the finiteness of the consequence operation—see V, § 10, (c_4), p. 181.

The proof (2) which we now present is not based on elimination of quantifiers. Similarly as in the first and second proofs we shall use

(1) This proof of 18.1 was given by Sikorski [15].

(2) Due to Scott.

the fact that every consistent theory $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}'\}$ with a finite set $\mathcal{A}'$ of axioms has a semantic model. More precisely, we shall construct a semantic model for $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ from the semantic models for the theories $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}'\}$, where $\mathcal{A}'$ ranges over all finite subsets of $\mathcal{A}$. This is the subject of theorem 18.3 below. We precede this theorem by a few definitions.

Suppose that $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ is a consistent theory. Let N be the set of all finite subsets of $\mathcal{A}$. In order to preserve the notation introduced at the beginning of this section, finite subsets of $\mathcal{A}$ (i.e. elements of N) will be denoted by the letter n. For every formula α in $\mathcal{A}$, let N_α be the set of all $n \in N$ such that α is in n. By definition,

$$n \in N_\alpha \quad \text{if and only if} \quad \alpha \in n . \tag{1}$$

We have

$$N_{\alpha_1} \cap \ldots \cap N_{\alpha_k} \neq 0 \tag{2}$$

for any $\alpha_1, \ldots, \alpha_k$ in $\mathcal{A}$. In fact, let n be the set formed of all the formulas $\alpha_1, \ldots, \alpha_k$. Since $\alpha_i \in n$, we have $n \in N_{\alpha_i}$ for $i = 1, \ldots, k$ by (1), which proves (2).

It follows from (2) that the filter generated by all the sets N_α (α in $\mathcal{A}$) in the Boolean algebra of all subsets of N is proper. By I, 8.5, the filter can be extended to a maximal (i.e. prime) filter $\mathfrak{V}$.

By hypothesis, each of the theories $\mathfrak{T}_n = \{\mathfrak{L}, \mathfrak{C}, n\}$ is consistent ($n \in N$). Thus, by 5.3, there exists a semantic model R_n for $\mathfrak{T}_n$ in a set $J_n \neq 0$. With the above notation we shall prove

18.3. *The product R of the realizations R_n modulo the prime filter $\mathfrak{V}$ is a semantic realization for the consistent theory $\mathfrak{T}$ in the set $J = \boldsymbol{P}_{n \in N} J_n$* [1].

For every formula α in $\mathcal{A}$, it follows from the definition of N and R_n that

$$N_\alpha \subset N(\alpha) ,$$

the set $N(\alpha)$ being defined as in 18.2. Thus $N(\alpha) \in \mathfrak{V}$. By 18.2, R is a model for α. Thus R is a semantic model for $\mathfrak{T}$ in J.

§ 19. Cardinals of models. We recall that if R is a model for a consistent theory $\mathfrak{T}$ in a set J of a power $\mathfrak{m}$ and a Boolean algebra $\boldsymbol{A}$, then the cardinal $\mathfrak{m}$ is called the cardinal of the model R (see § 2, p. 283), or R is said to be a model of power $\mathfrak{m}$.

19.1. *If a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ has a model of a power $\mathfrak{m}$ in a complete Boolean algebra $\boldsymbol{A}$, and $\mathfrak{n} \geqslant \mathfrak{m}$, then $\mathfrak{T}$ also has a model of power $\mathfrak{n}$ in the algebra $\boldsymbol{A}$.*

In particular, if $\mathfrak{T}$ has a semantic model of a power $\mathfrak{m}$ and $\mathfrak{n} \geqslant \mathfrak{m}$, then $\mathfrak{T}$ also has a semantic model of power $\mathfrak{n}$.

[1] See Morel and Tarski [1].

Suppose that R_1 is a model for $\mathfrak{T}$ in a set J_1, $\overline{\overline{J}}_1 = \mathfrak{m}$. Let J be any set of power $\mathfrak{n}$, and let f be any mapping from J onto J_1. We can always define a realization R in J and A such that R satisfies the hypotheses of theorem VI, 6.8. In fact, for every given m-argument functor φ ($m = 0, 1, 2, \ldots$) and any elements $j_1, \ldots, j_m \in J$, let $\varphi_R(j_1, \ldots, j_m)$ be an element in the set $f^{-1}(\varphi_{R_1}(f(j_1), \ldots, f(j_m)))$ (this set is not empty because f maps J onto J_1). By definition,

$$f(\varphi_R(j_1, \ldots, j_m)) = \varphi_{R_1}(f(j_1), \ldots, f(j_m)) . \tag{1}$$

For every m-argument predicate ϱ ($m = 1, 2, \ldots$), let ϱ_R be the function

$$\varrho_R(j_1, \ldots, j_m) = \varrho_{R_1}(f(j_1), \ldots, f(j_m)) . \tag{2}$$

Since (1) and (2) are hypotheses of theorem VI, 6.8, we have

$$a_R(v) = a_{R_1}(fv)$$

for every valuation v: $V \to J$ and for every formula a. In particular, $a_R(v) = V$ for every valuation v: $V \to J$ and for every formula a in $\mathfrak{A}$ since R_1 is a model for $\mathfrak{T}$. This proves that R is a model for $\mathfrak{T}$.

Denote by $\operatorname{card}\mathfrak{T}$ the least cardinal of a model for $\mathfrak{T}$, by $\operatorname{card}_A\mathfrak{T}$ the least cardinal of a model for $\mathfrak{T}$ in a complete Boolean algebra A, and by $\operatorname{card}_2\mathfrak{T}$ the least cardinal of a semantic model for $\mathfrak{T}$. By definition

(3) $\operatorname{card}\mathfrak{T} \leqslant \operatorname{card}_A\mathfrak{T}$ for every complete Boolean algebra A;

in particular

$$\operatorname{card}\mathfrak{T} \leqslant \operatorname{card}_2\mathfrak{T} . \tag{4}$$

Theorem 19.1 can now be formulated in the following way.

19.2. *A consistent theory $\mathfrak{T}$ has a model of a cardinal $\mathfrak{n}$ in a complete Boolean algebra A if and only if $\mathfrak{n} \geqslant \operatorname{card}_A\mathfrak{T}$. In particular, a consistent theory $\mathfrak{T}$ has a semantic model of a cardinal $\mathfrak{n}$ if and only if $\mathfrak{n} \geqslant \operatorname{card}_2\mathfrak{T}$.*

A consistent theory $\mathfrak{T}$ has a model of a cardinal $\mathfrak{n}$ if and only if $\mathfrak{n} \geqslant \operatorname{card}\mathfrak{T}$.

Theorems 3.3, 9.3, 15.3 yield the following estimations where T is the set of all terms in $\mathfrak{T}$.

19.3. *For every consistent theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$,*

$$\operatorname{card}\mathfrak{T} \leqslant \overline{\overline{T}} , \tag{5}$$

$$\operatorname{card}_2\mathfrak{T} \leqslant \max(\aleph_0, \overline{\overline{A}}) . \tag{6}$$

For every open theory $\mathfrak{T}$,

$$\operatorname{card}_2\mathfrak{T} \leqslant \max(\aleph_0, \overline{\overline{T}}) . \tag{6'}$$

All the above estimations are exact, i.e. they cannot be replaced by any better ones.

For instance, let $\mathfrak{L}$ be a formalized language of the first order, containing an enumerable set of free and bound individual variables, such that the set of all functors in $\mathfrak{L}$ contains $\mathfrak{n}$ individual constants ($\mathfrak{n} \geqslant \aleph_0$), and the set of all predicates in $\mathfrak{L}$ contains only the equality sign $\mathfrak{e}$. Let $\mathcal{A}$ be the set consisting of the axioms for equality (V, § 12, $(\mathfrak{e}_1)$, $(\mathfrak{e}_2)$, $(\mathfrak{e}_3)$) and of all the following axioms

$$-\mathfrak{e}(c_1\, c_2)\,, \tag{7}$$

where c_1, c_2 are any distinct individual constants. It is easy to see that the theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ is consistent because $\mathfrak{T}$ has a natural ordinary semantic model in the set of all individual constants. If R is a model for $\mathfrak{T}$ in a set $J \neq 0$ and a complete Boolean algebra $\boldsymbol{A}$, then

$$\mathfrak{e}_R(j, j) = V \quad \text{for every} \quad j \in J$$

by the axiom for equality V, § 12, $(\mathfrak{e}_1)$, and

$$\mathfrak{e}_R(c_{1R}, c_{2R}) = \Lambda$$

for every pair of distinct individual constants c_1, c_2, because of (7). This implies that $c_{1R} \neq c_{2R}$ for $c_1 \neq c_2$. Thus the set J has at least the cardinal $\mathfrak{n}$. The above consideration shows that

$$\operatorname{card}\mathfrak{T} = \mathfrak{n} = \overline{\overline{T}} \quad \text{and} \quad \operatorname{card}_2\mathfrak{T} = \mathfrak{n} = \max(\aleph_0, \overline{\overline{\mathcal{A}}}) = \max(\aleph_0, \overline{\overline{T}})\,,$$

i.e. the equality holds in all the estimations (5), (6), (6′).

Inequality (6) cannot be replaced by the inequality $\operatorname{card}_2\mathfrak{T} \leqslant \overline{\overline{\mathcal{A}}}$. In fact, let $\mathfrak{L}$ be a formalized language of the first order with an enumerable set of free and bound individual variables, the empty set of functors, and the set of predicates consisting only of the equality sign $\mathfrak{e}$ and another binary predicate ϱ. Let $\mathcal{A}$ be the set consisting of the four axioms for equality V, § 12, $(\mathfrak{e}_1)$, $(\mathfrak{e}_2)$, $(\mathfrak{e}_3)$, $(\mathfrak{e}_5)$, and of the three following axioms:

$$\bigcap_\xi \bigcap_\eta \bigcap_\zeta (\varrho(\xi\,\eta) \Rightarrow (\varrho(\eta\,\zeta) \Rightarrow \varrho(\xi\,\zeta)))\,, \tag{8}$$

$$\bigcap_\xi \bigcap_\eta (\varrho(\xi\,\eta) \Rightarrow -\mathfrak{e}(\xi\,\eta))\,, \tag{9}$$

$$\bigcap_\xi \bigcup_\eta \varrho(\xi\,\eta)\,. \tag{10}$$

The theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ has an enumerable ordinary semantic model (e.g. in the set of all integers, ϱ being interpreted as the characteristic of the arithmetical relation $<$) and consequently is consistent. The set $\mathcal{A}$ of axioms for $\mathfrak{T}$ is finite. However, the theory $\mathfrak{T}$ has no semantic model in a finite set. In fact, if R is a semantic model for $\mathfrak{T}$ in a set J,

then writing $j_1 < j_2$ if $\varrho_R(j_1, j_2) = V$, we have by (8), (9), (10) for all $j_1, j_2, j_3 \in J$:

if $j_1 < j_2$ and $j_2 < j_3$, then $j_1 < j_3$;
if $j_1 < j_2$, then $j_1 \neq j_2$;
for every j_1 there exists a j_2 such that $j_1 < j_2$.

Thus J must be infinite since it contains a sequence $j_1 < j_2 < j_3 < \ldots$ of different elements.

Note also that the inequality (6′) is not true, in general, for non-open theories. For instance, let $\mathfrak{L}$ be a formalized language of the first order, such that the set of all free and bound individual variables is enumerable, the set of functors is empty, the set of predicates consisting of the equality sign $\mathfrak{e}$ and of a set of unary predicates of a power $\mathfrak{n} > \aleph_0$. Let $\mathcal{A}$ consists of the axioms for equality and of all formulas

$$\bigcap_\xi \bigcap_\eta ((\varrho_1(\xi) \cap \varrho_2(\eta)) \Rightarrow -\mathfrak{e}(\xi\, \eta)),$$

where ϱ_1 and ϱ_2 are any distinct unary predicates, and of all formulas

$$\bigcup_\xi \varrho(\xi),$$

where ϱ is any unary predicate. The theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ has semantic models only in sets of powers $\geqslant \mathfrak{n}$ and $\overline{\overline{T}} = \aleph_0 < \mathfrak{n}$. Note that, on the other hand, this theory has an adequate but non-semantic model in an enumerable set on account of 3.3.

Inequality (4) cannot be replaced by an equation. In fact, in the example just quoted we have $\operatorname{card}\mathfrak{T} = \aleph_0$ (see (5)) and $\operatorname{card}_2\mathfrak{T} = \mathfrak{n} > \aleph_0$.

Theorem 19.2 explains exactly the structure of the class of all cardinals $\mathfrak{n}$ such that a given consistent theory $\mathfrak{T}$ has a model (or a semantic model) of the power $\mathfrak{n}$: this class consists of all cardinals equal to or greater than a fixed cardinal. The structure of the class of all cardinals $\mathfrak{n}$ such that a given theory $\mathfrak{T}$ has an ordinary semantic model of power $\mathfrak{n}$ is a little more complicated. It follows from 4.4 that $\operatorname{card}_2\mathfrak{T}$ is the least cardinal in this class. However it may happen that this class contains only the cardinal $\operatorname{card}_2\mathfrak{T}$.

For instance, let $\mathfrak{L}$ be a formalized language of the first order, containing the equality sign $\mathfrak{e}$. Let α_n be the formula

$$(11)\quad \bigcup_{\xi_1}\bigcup_{\xi_2} \ldots \bigcup_{\xi_n} (-\mathfrak{e}(\xi_1\, \xi_2) \cap (-\mathfrak{e}(\xi_1\, \xi_3) \cap \ldots \cap C-\mathfrak{e}(\xi_1\, \xi_n) \cap \cap \ldots \cap -\mathfrak{e}(\xi_{n-1}\, \xi_n))\ldots)$$

stating that there exist at least n elements; let β_n be the formula

$$(12)\quad \bigcup_{\xi_1}\bigcup_{\xi_2} \ldots \bigcup_{\xi_n} \bigcap_\xi ((\mathfrak{e}(\xi\, \xi_1) \cup ((\mathfrak{e}(\xi\, \xi_2) \cup \ldots \cup \mathfrak{e}(\xi\, \xi_n)\ldots)$$

stating that there exist at most n elements. Their conjunction γ_n:

$$(a_n \frown \beta_n) \tag{13}$$

states that there exist exactly n elements.

Let $\mathcal{A}$ be the set consisting of four formulas: the axioms for equality V, § 12, (e_1), (e_2), (e_3) and of the formula γ_n where n is a fixed positive integer. The theory $\mathcal{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ has ordinary semantic models only in n-element sets.

The above example shows that we have no analogue of 19.1 for ordinary semantic models. Instead of 19.1 we have only the three following theorems.

19.4. *If a theory* $\mathcal{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ *has an infinite ordinary semantic model, then it has an ordinary semantic model of any power* $\mathfrak{n} \geqslant \max(\aleph_0, \overline{\overline{A}})$ [1].

If an open theory $\mathcal{T}$ *has an infinite ordinary semantic model, then it has an ordinary semantic model of any power* $\mathfrak{n} \geqslant \max(\aleph_0, \overline{\overline{T}})$, T *being the set of terms in* $\mathcal{T}$.

Let $\mathfrak{L}_1$ be the formalized language obtained from $\mathfrak{L}$ by adding a set C of individual constants, $\overline{\overline{C}} = \mathfrak{n}$. Let $\mathcal{A}_0$ be the set of all formulas (7):

$$-\mathfrak{e}(c_1\, c_2)\,,$$

where c_1 and c_2 are any distinct individual constants in C. Every theory $\{\mathfrak{L}_1, C, \mathcal{A} \cup \mathcal{A}_1\}$, where $\mathcal{A}_1$ is a finite subset of $\mathcal{A}_0$, is consistent since it has a model. In fact, $\mathcal{T}$ has an ordinary semantic model R in an infinite set J. Since J is infinite, we can extend the realization R of $\mathfrak{L}$ to a realization (denoted by the same letter R) of $\mathfrak{L}_1$ in such a way that all the elements $c_R \in J$, where $c \in C$ occurs in a formula in $\mathcal{A}_0$ are different. Thus the extended realization R is also a model for all formulas in $\mathcal{A}_0$, i.e. it is a model for $\{\mathfrak{L}_1, C, \mathcal{A} \cup \mathcal{A}_1\}$.

It follows from the second part of V, 14.1, that the theory $\mathcal{T}_1 = \{\mathfrak{L}_1, C, \mathcal{A} \cup \mathcal{A}_0\}$ is consistent.

If $\mathfrak{n} \geqslant \max(\aleph_0, \mathcal{A})$, then $\mathcal{T}_1$ has $\mathfrak{n}$ axioms, and $\mathcal{T}_1$ has an ordinary semantic model R_1 in a set J_1 of a power $\leqslant \mathfrak{n}$ (see p. 292).

If $\mathcal{T}$ is open and $\mathfrak{n} \geqslant \max(\aleph_0, \overline{\overline{T}})$, then $\mathcal{T}_1$ is open and the set of all terms in $\mathcal{T}_1$ has the power $\mathfrak{n}$. Consequently $\mathcal{T}_1$ has an ordinary semantic model R_1 in a set J_1 of a power $\leqslant \mathfrak{n}$ (see 15.4 and 4.4).

On the other hand, by the same argument as that on p. 343 (see (7) and below) we verify that every semantic model of $\mathcal{T}_1$ has a power $\geqslant \mathfrak{n}$. Hence it follows that $\overline{\overline{J}}_1 = \mathfrak{n}$.

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[1] This remark is due to Tarski; see Skolem [3], *Bemerkungen der Redaktion*. See also Henkin [1].

19.5. *If a theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ *has an ordinary semantic model of a power* $\mathfrak{m} \geqslant \aleph_0$, *then it has an ordinary semantic model of any power* $\mathfrak{n} \geqslant 2^{\mathfrak{m}}$.

It suffices to prove 19.5 in the case for which $\mathfrak{L}$ has an enumerable set of free and bound individual variables (for if this set is non-enumerable, we can replace $\mathfrak{T}$, in the consideration below, by an auxiliary theory $\mathfrak{T}_1 = \{\mathfrak{L}_1, \mathfrak{C}, \mathfrak{A}_1\}$, where $\mathfrak{L}_1$ is obtained from $\mathfrak{L}$ by omitting a set of free and bound individual variables, the set of all free and bound individual variables in $\mathfrak{L}_1$ is enumerable, and axioms in $\mathfrak{A}_1$ are obtained from axioms in $\mathfrak{A}$ by a suitable renaming of some individual variables—see 9.7; $\mathfrak{T}$ and $\mathfrak{T}_1$ have exactly the same models).

Let R be an ordinary semantic model for $\mathfrak{T}$ in a set J of a power $\mathfrak{m} \geqslant \aleph_0$. Let $\mathfrak{T}^* = \{\mathfrak{L}^*, \mathfrak{C}, \mathfrak{A}^*\}$ be an open theory obtained from $\mathfrak{T}$ by elimination of quantifiers from the axioms of $\mathfrak{T}$ (see § 17), $\mathfrak{L}^*$ being obtained from $\mathfrak{L}$ by adding some functors. By 17.3, the model R can be extended to an ordinary semantic model R^* for $\mathfrak{T}^*$ in the same set J.

Two m-argument functors φ and φ_1 in $\mathfrak{L}^*$ ($m = 0, 1, 2, \ldots$) are said to be *equivalent*, in symbols

$$\varphi \sim \varphi_1 \tag{14}$$

if $\varphi_R = \varphi_{1R}$, i.e. if

$$\varphi_R(j_1, \ldots, j_m) = \varphi_{1R}(j_1, \ldots, j_m) \quad \text{for all} \quad j_1, \ldots, j_m \in J\,. \tag{15}$$

Choose one functor φ from every equivalence class of relation (14). Let $\mathfrak{L}^{**}$ be the formalized language obtained from $\mathfrak{L}^*$ by omitting all those functors which were not chosen. The set of all terms in $\mathfrak{L}^{**}$ has a power $\leqslant 2^{\mathfrak{m}}$ since the set of all operations in J has the cardinal $2^{\mathfrak{m}}$.

For every formula α^* in $\mathfrak{L}^*$, let α^{**} be the formula obtained from α^* by replacing every functor by the corresponding equivalent chosen functor in $\mathfrak{L}^{**}$. By definition, α^{**} is in $\mathfrak{L}^{**}$. Denoting by R^{**} the restriction of R^* to functors and predicates in $\mathfrak{L}^*$, we have

$$\alpha^*_{R^*}(v) = \alpha^{**}_{R^{**}}(v) \quad \text{for every valuation } v \tag{16}$$

by (15). Let $\mathfrak{A}^{**}$ be the set of all the formulas α^{**} where α^* is a formula in $\mathfrak{A}^*$. By (16), R^{**} is an ordinary semantic model for the theory $\mathfrak{T}^{**} = \{\mathfrak{L}^{**}, \mathfrak{C}, \mathfrak{A}^{**}\}$ in the set J.

Suppose $\mathfrak{n} \geqslant 2^{\mathfrak{m}}$. By the second part of 19.4, the open theory $\mathfrak{T}^{**}$ has an ordinary semantic model R_0^{**} in a set J_0, $\overline{\overline{J}}_0 = \mathfrak{n}$. Extend the realization R_0^{**} of $\mathfrak{L}^{**}$ to a realization R_0^* of $\mathfrak{L}^*$ as follows: for any m-argument functor φ_1 in $\mathfrak{L}^*$ ($m = 0, 1, 2, \ldots$), let φ be the only chosen functor in $\mathfrak{L}^{**}$ such that (15) (i.e. (16)) holds, and let

$$\varphi_{1R_0^*} = \varphi_{R_0^{**}}\,.$$

It follows easily from this definition that

$$a^*_{R_0^*}(v) = a^{**}_{R_0^{**}}(v)$$

for every formula a^* in $\mathfrak{L}^*$ and for every valuation v. Hence it follows that R_0^* is an ordinary semantic model for $\mathfrak{C}^*$. By 2.2 and 17.3, the restriction R of R_0^* to functors and predicates in $\mathfrak{L}$ is an ordinary semantic model for $\mathfrak{C}$ in the set J_0 of power $\mathfrak{n}$.

For every consistent theory $\mathfrak{C}$, let $N_{\mathfrak{C}}$ denote the set of all positive integers m such that $\mathfrak{C}$ has an ordinary semantic model of power m.

19.6. *If the set $N_{\mathfrak{C}}$ is infinite, then the theory $\mathfrak{C}$ has an ordinary semantic model in an infinite set of a power $\leqslant 2^{\aleph_0}$* (1).

Suppose that $m_1 < m_2 < \ldots$ is an infinite sequence of positive integers in $N_{\mathfrak{C}}$. By definition, $\mathfrak{C}$ has an ordinary semantic model R_n in a set

$$J_n = (j_{n,1}, \ldots, j_{n,m_n})$$

containing exactly m_n elements, $n = 1, 2, \ldots$

Let N be the set of all positive integers, and let $\mathfrak{V}$ by any prime filter in the Boolean algebra of all subsets of N, such that no finite subset of N belongs to $\mathfrak{V}$ (the existence of $\mathfrak{V}$ follows, e.g., from I, 8.4, where Δ_0 contains all sets $N - N_0$ where N_0 is finite).

Let R be the product of all R_n modulo $\mathfrak{V}$. By 18.2, R is a semantic model for $\mathfrak{C}$ in the set $J = P_{n \in N} J_n$ of the power $2^{\aleph_0}$. By § 4 (2), 4.3 $\mathfrak{e}_R$ is the characteristic of an equivalence relation $\sim$ in J. Identifying points j, j' in J if and only if $j \sim j'$ (i.e. if $\mathfrak{e}_R(j, j') = \mathfrak{V}$), we get by 4.4 an ordinary semantic model R' for $\mathfrak{C}$ in the set $J' = J/\sim$. By definition, $\overline{\overline{J'}} \leqslant 2^{\aleph_0}$. It remains to prove that J' is infinite. For this purpose, let us note that for points $j = \{j_n\}_{n \in N} \in J$ and $j' = \{j'_n\}_{n \in N} \in J$ we have $j \sim j'$ if and only if the set of all $n \in N$ such that $j_n = j'_n$ belongs to $\mathfrak{V}$ (see § 18, 2). Thus denoting

$$j^n = \{j_{1,1}, \ldots, j_{n-1,1}, j_{n,n}, j_{n+1,n}, j_{n+2,n}, \ldots\}$$

for $n = 1, 2, \ldots$, we do not have $j^n \sim j^m$ for $n \neq m$. This proves that J is infinite.

The following theorem which is a substitute for 19.2 for theories with the equality sign follows immediately from 19.4, 19.5, 19.6 and the generalized Cantor hypothesis $2^{\aleph_\nu} = \aleph_{\nu+1}$.

19.7. *For every consistent theory $\mathfrak{C}$ one of the following statements holds:*

(i) *the set $N_{\mathfrak{C}}$ is non-empty and finite, and $\mathfrak{C}$ has no infinite ordinary semantic model;*

(1) This theorem was communicated to us by A. Ehrenfeucht.

(ii) *the set* $N_{\mathcal{C}}$ *is finite and there exists an infinite cardinal* $\mathfrak{m}$ *such that* $\mathcal{C}$ *has an ordinary semantic model of an infinite cardinal* $\mathfrak{n}$ *if and only if* $\mathfrak{n} \geqslant \mathfrak{m}$;

(iii) *the set* $N_{\mathcal{C}}$ *is infinite and there exists an infinite cardinal* $\mathfrak{m} \leqslant 2^{\aleph_0}$ *such that* $\mathcal{C}$ *has an ordinary semantic model of an infinite cardinal* $\mathfrak{n}$ *if and only if* $\mathfrak{n} \geqslant \mathfrak{m}$.

We round off this section with three examples showing that theorem 19.7 is exact.

For every non-empty finite set N_0 of positive integers there exists a theory $\mathcal{C}$ such that $N_{\mathcal{C}} = N_0$ and $\mathcal{C}$ has no infinite ordinary semantic model. For example, let $\mathfrak{L}$ contain the sign of equality $\mathfrak{e}$ and let $\mathcal{A}$ be the set consisting of the three axioms for equality V, § 12, (e_1), (e_2), (e_3) and of the fourth formula which is the disjunction of the formulas γ_n (see (13)) where $n \in N_0$. We recall that γ_n states that there exist exactly n elements. Thus $\mathcal{C} = \{\mathfrak{L}, \mathrm{C}, \mathcal{A}\}$ has only ordinary semantic models of powers $\mathfrak{n} \in N_0$.

If N_0 is any finite set of positive integers, and $\mathfrak{m}$ is a given infinite cardinal, then there exists a theory $\mathcal{C}$ such that $\mathcal{C}$ has ordinary semantic models of powers $\mathfrak{n} \in N_0$ and $\mathfrak{n} \geqslant \mathfrak{m}$ only. In fact, let $\mathfrak{L}$ be a formalized language (with an enumerable set of individual variables) containing a set of $\mathfrak{m}$ individual constants and three predicates: the equality sign $\mathfrak{e}$, and two binary predicates ϱ and π. Let γ_0 be a conjunction of the formulas (8), (9), (10) and let δ be the formula

$$(\gamma_0 \Rightarrow \bigcap_{\xi} \bigcap_{\eta} (\pi(\xi\,\eta) \Rightarrow -\,\mathfrak{e}(\xi\,\eta))) .$$

Let $\mathcal{A}$ be the set consisting of the four axioms for equality V, § 12, (e_1), (e_2), (e_3), (e_5), of all the formulas

$$\pi(c_1\,c_2) ,$$

where c_1 and c_2 are distinct individual constants of the formula δ and of the disjunction of all the formulas γ_n where $n = 0$ or $n \in N_0$ (see (13)). The theory $\mathcal{C} = \{\mathfrak{L}, \mathrm{C}, \mathcal{A}\}$ has the required properties.

For every given infinite set N_0 of positive integers there exists a theory $\mathcal{C}$ such that $N_{\mathcal{C}} = N_0$ and $\mathcal{C}$ has no infinite semantic model of a power $< 2^{\aleph_0}$. In fact, suppose that the set of all real numbers c is the set of all individual constants in $\mathfrak{L}$, and that $\mathfrak{L}$ contains the sign of equality $\mathfrak{e}$. Let formulas α_n and γ_n be defined by (11) and (13), i.e. they state that there exist at least n elements, or exactly n elements, respectively. Let $\{w_n\}$ be the sequence formed of all rational numbers. Let $\mathcal{A}$ be the set composed of the axioms for equality V, § 12, (e_1), (e_2), (e_3), of all the formulas

$$-\gamma_n \quad \text{where} \quad n \notin N_0$$

and, for every positive integer n, of all the formulas of the form

$$(a_{n+1} \Rightarrow -\mathfrak{e}(c_1\, c_2))\,,$$

where c_1, c_2 are any real numbers (i.e. individual constants) such that $c_1 < w_n < c_2$. The theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$ has the required properties [1].

§ 20. Non-enumerable arithmetic and enumerable set theory. The theorems and estimations given in the previous section sometimes lead to conclusions which, at the first glance, seem to be paradoxes.

Consider, as the first example, the formalized arithmetic described in detail in V, § 13, D). The formalized arithmetic has, of course, an enumerable ordinary semantic model, viz. the natural model in the set of all positive integers. By theorem 19.4, the formalized arithmetic also has an ordinary semantic model in every non-enumerable set J! [2]. Roughly speaking, in every non-enumerable set J it is possible to define addition and multiplication which have the same properties as addition and multiplication in the set of all positive integers. More precisely, there exists an interpretation $\mathfrak{I}$ of the language of the formalized arithmetic in the set J which turns J into an algebra with two binary operations $+$ and $\cdot$ which have the properties of the addition and multiplication of positive integers, i.e. the corresponding realization $\mathfrak{I}^*$ (for notation, see § 4 (1), p. 289, and VI, § 6, p. 228) is an ordinary semantic model for the formalized arithmetic in the set J. An element is the interpretation of the arithmetical unit 1 (i.e. the constant 1 in the formalized arithmetic). We shall denote it by the same symbol 1. The element $1+1$ corresponds to the number 2; the element $1+1+1$ corresponds to the number 3, etc. If J is non-enumerable, then the elements

$$1,\ 1+1,\ 1+1+1,\ \ldots \tag{1}$$

do not exhaust the whole set J. However, every element $j \in J$ has its successor $j+1$ and the addition and multiplication satisfy the induction schemes

$$j_1 + (j_2+1) = (j_1+j_2)+1\,,$$

$$j_1 \cdot (j_2+1) = j_1 \cdot j_2 + j_1$$

stated by axioms V, § 12, D_3) and D_5). This result is the more paradoxical since the set of all axioms for the formalized arithmetic contains the induction principle (see V, § 13, D_6)) which is consequently true in the

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[1] This example is due to A. Ehrenfeucht.

[2] Cf. footnote [1] on p. 345.

algebra J. Roughly speaking, the induction principle enables us to conclude that

$$\beta(j) \quad \text{holds for every} \quad j \in J$$

provided that

$$\beta(1) \quad \text{holds}$$

and

$$\text{the implication} \quad \beta(j) \Rightarrow \beta(j+1) \quad \text{holds for every} \quad j \in J.$$

The precise formulation of the induction principle imposes some restrictions: (i) the above induction principle is true only for propositional functions $\beta(x)$ of the form $\alpha_{\mathfrak{J}}(x)$ where $\alpha(x)$ is a formula in the formalized arithmetic, i.e. only for those propositional functions which are interpretations of formulas in the formalized arithmetic; (ii) the word "holds" in the above induction principle applied to the interpretation $\gamma_{\mathfrak{J}}$ of a formula γ should mean: γ is a theorem of the formalized arithmetic.

The first restriction is essential. Propositional functions in J which are interpretations of formulas do not exhaust the set of all propositional functions in J, i.e. there are propositional functions which are not formulated in the formalized language of the arithmetic. For instance, if J contains elements different from those in (1), then the propositional function $\beta(x)$ stating that x is one of the elements (1) is not of the form $\alpha_{\mathfrak{J}}(x)$. Indeed, the induction principle fails for $\beta(x)$ since the sentences

$$\beta(1),$$

$$\text{for every } j \in J, \text{ if } \beta(j) \text{ then } \beta(j+1)$$

are true, but the sentence

$$\beta(j) \quad \text{for every} \quad j \in J$$

is false.

The second restriction is also essential. The word "holds" in the above induction scheme cannot be interpreted as "is valid". In fact, the formalized arithmetic is not a maximal theory, and consequently none of its semantic models is adequate (see 12.3 and p. 320).

Note that there also exist ordinary semantic models for an arithmetic in an enumerable set J so that the elements (1) do not exhaust the whole set J. For instance, add to the formalized language of the arithmetic a new individual constant c, and add to the set of axioms for arithmetic all the formulas

$$(2) \qquad \begin{aligned} &-(c = 1), \\ &-(c = (1+1)), \\ &-(c = ((1+1)+1)), \end{aligned}$$

The formalized theory just obtained has a model since every ordinary semantic model R_0 for arithmetic in a non-enumerable set J_0 can be extended to a model of this theory by assuming that c_{R_0} is an element in J but not in (1). By 19.3, (6), this theory has an ordinary semantic model R in an enumerable set J. This model, restricted to functors and predicates in the formalized arithmetic, is also a model for the arithmetic. By (2), the element $c_R \in J$ is not equal to any of the elements (1). Thus the elements (1) do not exhaust the whole set J.

Another construction of an enumerable semantic model for the arithmetic, which does not coincide with the natural model of intuitive arithmetic, can be based on the fact that the formalized arithmetic is not a maximal theory, i.e. there exists a closed formula a such that neither a nor $-a$ is a theorem of the formalized arithmetic. One of the formulas a and $-a$, say a, is valid in the natural model of the formalized arithmetic in intuitive mathematics. Add the formula $-a$ to the axioms of the formalized arithmetic. The formalized theory just obtained is consistent by 11.2. By 19.3, (6), it has an ordinary semantic model R in an enumerable set J. The set J contains all the elements (1), i.e. all the substitutes for positive integers, and the operations of addition and multiplication on elements (1) in model R coincide with the ordinary ones. Since $-a$ is valid in R but is not valid in the natural intuitive model, i.e. in the model R restricted to the elements (1), J must also have elements different from the elements (1).

We see that the existence of many essentially different models for the formalized arithmetic is caused by at least two reasons: by the non-maximality and by general theorems on the existence of models of arbitrarily high power. The formalized arithmetic is not a complete description of the set of all positive integers. This incompleteness is not a special property of the formalized arithmetic. It is a general phenomenon in the metamathematics of formalized theories of the first order, and this phenomenon is caused by the same reasons as in the case of arithmetic. We have stated in § 12 (p. 319) that, in practice, formalized mathematical theories are of two categories: formalized descriptions of some classes of models from intuitive mathematics (like the formalized theory of Boolean algebras or of groups), and formalized descriptions of some special models from intuitive mathematics (like arithmetic). It follows from the aim of the formalization that theories of the first category have many essentially different models. But in the case of formalized theories of the second category it is to be expected that they will have only one semantic ordinary model (up to isomorphism), viz. the model which was the starting point of the considered formalized theory. We have seen that, in general, this does not hold.

The non-maximality of the theory, as a source for existence of essentially different models, can always be removed, at least theoretically, when we pass to its maximal extensions through 12.6. But the phenomenon caused by theorems presented in the former sections will always appear: if a theory has an infinite ordinary semantic model, then it has many essentially different models of arbitrarily high powers. Thus, in some sense, formalized theories of the first order are never a complete description of a fixed infinite model from intuitive mathematics.

The reason for this incompleteness lies in the poverty of formalized languages of the first order. A better description can be obtained by means of some non-elementary languages mentioned in V, § 3, which are not investigated in this book.

We shall now discuss another paradox, of a different kind. This paradox appears in the application of the estimation 19.3, (6), to the formalized set theory described in V, § 13, E). We recall that this theory contains only two specific constants, viz. two binary predicates: ε and the equality sign. The set of all axioms for the formalized set theory is enumerable. By 18.3, (6) (or by theorem 5.3) there exists an enumerable semantic model for the formalized set theory! (1).

It does not follow directly from 19.3, (6) and 5.3 but it can be proved in another way that there exists an ordinary semantic model R for the formalized set theory in a set J such that:

1) J is enumerable;

2) all elements j in J are enumerable or finite sets,

3) ε_R is the characteristic of the set-theoretical relation $j_1 \in j_2$ between elements (sets) j_1, j_2 in J.

The last result can also be formulated as follows. Let $\mathfrak{I}$ be the interpretation which assigns:

to the equality sign—the relation of equality in the set J,

to the sign ε—the set-theoretical relation $\in$ in the set J.

Since $R = \mathfrak{I}^*$ (for notation, see § 4, p. 289, or VI, § 6, p. 228), all the axioms, and consequently all theorems of the formalized set theory are true in the interpretation $\mathfrak{I}$ of the formalized language of set theory in the set J.

Thus J is an enumerable set of sets satisfying all the axioms of formalized set theory. Moreover all sets $j \in J$ are at most enumerable!

The last statement seems to be a paradox. We have stated in V, § 13, E) that we can reproduce in the formalized set theory the whole

(1) Skolem [2].

of intuitive set theory. In particular we can prove, in the formalized set theory, the existence of non-enumerable sets! In fact, the axiom of infinity (V, § 13, (E_4)) states the existence of an infinite set A. The axiom of the power-set (V, § 13, (E_3)) states the existence of the set B of all subsets of A. We can also prove in the formalized set theory that there exists no one-to-one mapping from A onto B (see V, § 13, (7)), i.e. that B has a higher power than A and, consequently, B is non-enumerable.

The paradox is only apparent. In the enumerable model R for the formalized set theory, B is not the class of all subsets of the set $A \in J$; B is only the class of all subsets of A which belong to J. Since the sets $A, B \in J$ are enumerable, there exists, of course, a one-to-one mapping (i.e. a set C of ordered pairs—see V, § 13, E)) from A onto B. This mapping (i.e. this set C) does not belong to J! It does not belong to J because we have a general theorem in the formalized set theory (see V, § 13, (7)) that there exists no set of ordered pairs which is a one-to-one mapping from A onto B, and this theorem is valid in every model for the formalized set theory (see 2.2). Thus, in the model R, the set B has a higher power than A and, in the model R, the set B is non-enumerable.

We see that, in the above example, we have to distinguish two different notions: the (absolute) power of sets $A, B \in J$, and the power of $A, B \in J$ in the model R. A similar relativization is also necessary for other set-theoretical notions.

If we remember that R is a model for the formalized set theory only under the relativization of all notions to the class J, the paradox discussed above ceases to be a paradox at all. It remains only a fascinating theorem that the formalized set theory has an enumerable model.

§ 21. Effectivity problems. Consider the following three theorems proved in § 9, § 15 and I, § 8.

(F) *Every non-degenerate Boolean algebra has a maximal filter.*

(M) *Every consistent theory of the first order has a semantic model.*

(M^0) *Every consistent open theory of the first order has a semantic model.*

As we have explained in VII, § 9, theorem (F) is not effective since its proof is based on the axiom of choice. Theorem (M) was deduced in § 9 from (F) without using the axiom of choice (see 9.1 and 9.2). More exactly, we deduced (M) from (F) and 8.3 which is a consequence of 8.1, applied to enumerable sets J. Theorem 8.1 is not effective in general, but the case in which J is enumerable is effective! Thus we have effectively deduced (M) from (F).

Of course, (M) effectively implies (M^0).

We shall now prove that (M^0) effectively implies (F), i.e. that all the statements (F), (M), and (M^0) are effectively equivalent [1].

By VII, § 11, it suffices to prove that (M^0) effectively implies the analogous statement for theories of zero order:

(M_0^0) *Every consistent theory of zero order has a semantic model.*

Let $\mathfrak{T}_0 = \{\mathfrak{L}_0, C, \mathcal{A}_0\}$ be a consistent theory of zero order, V_0 being the set of all propositional variables in $\mathfrak{L}_0$. Let $\mathfrak{L}$ be any formalized language of the first order such that the set E_0 of all elementary formulas in $\mathfrak{L}$ without individual variables has the power $\bar{\bar{V}}_0$ (e.g. we can take as $\mathfrak{L}$ a formalized language containing $\bar{\bar{V}}_0$ individual constants, a predicate and no other specific signs). Let s be any one-to-one mapping from V_0 onto E_0. By definition, s is a substitution from $\mathfrak{L}_0$ into $\mathfrak{L}$. For any formula δ in $\mathfrak{L}_0$, denote by $s\delta$ the result of the substitution s in δ. By definition, $s\delta$ is both an open and a closed formula in $\mathfrak{L}$.

Let $\mathcal{A}$ be the set of all the formulas $s\delta$ where δ is in $\mathcal{A}_0$.

For every formula δ in $\mathfrak{L}_0$, if $s\delta$ is a theorem in the open theory $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$, then δ is a theorem in $\mathfrak{T}_0$. In fact, by the deduction theorem 10.2, there exists a conjunction α of axioms in $\mathcal{A}$ such that $(\alpha \Rightarrow s\delta)$ is a predicate tautology. The formula α is of the form $s\gamma$ where γ is a conjunction of some of the axioms of $\mathfrak{T}_0$. Since the formula $(s\gamma \Rightarrow s\alpha)$, i.e. the formula $s(\gamma \Rightarrow \alpha)$, is a predicate tautology, the formula $(\gamma \Rightarrow \alpha)$ is a propositional tautology, by 6.2. This proves, by the deduction theorem VII, 8.2, that δ is a theorem in $\mathfrak{T}_0$.

The open theory $\mathfrak{T}$ is consistent, for if δ is a formula in $\mathfrak{L}_0$ which is not a theorem in $\mathfrak{T}_0$, $s\delta$ is a formula in $\mathfrak{L}$ which is not a theorem in $\mathfrak{T}$.

By (M^0), there exists a model R for $\mathfrak{T}$ in a set $J \neq 0$ and the two-element Boolean algebra A_0. The formula

$$v_a^0 = \begin{cases} V & \text{if} \quad sa_R = V, \\ \Lambda & \text{if} \quad sa_R = \Lambda, \end{cases} \qquad (a \text{ in } V_0)$$

defines a valuation $v^0 = \{v_a^0\}_{a \in V_0}$. By VI, 6.9,

$$\delta_{A_0}(v^0) = s\delta_R = V$$

for every formula δ in $\mathcal{A}_0$. Thus v^0 is a semantic model for $\mathfrak{T}_0$.

§ 22. Canonical semantic models. Representation problems for Q-algebras of theories. We have seen that all models in the existential theorems proved above were constructed in the set of all terms of a theory. More precisely, they were canonical realizations of terms or homomorphic images of canonical realizations of terms, i.e. they were obtained from

[1] Łoś [3].

canonical realizations of terms by the identification of terms modulo a congruence in algebras of terms. This points to the significance of models of the type just mentioned in the metamathematics of formalized theories of the first order.

If a semantic model for a theory $\mathfrak{C}$ in the set T of all terms in $\mathfrak{C}$ is the canonical realization of terms (see § 3 and VI, § 7), then it is said to be a *canonical semantic model.* This terminology agrees with that in § 3 and VI, § 7. In fact, let A_0 be the two-element Boolean algebra. If R is a canonical semantic model for $\mathfrak{C}$ in the sense just defined, then by VI, 6.2, VI, § 6, (4) and VI, 7.1, (3), (3′) the formula

$$h(\|\alpha\|) = \alpha_R(\mathfrak{i})$$

where $\mathfrak{i}$ is the identity valuation, defines a Q-homomorphism from $\bar{F} = F/\sim$ into A_0 where $F, \bar{F}, \sim$ have meanings as in VI, § 4 (p. 219 and p. 220). R coincides with the canonical realization (in the sense defined in VI, § 7, p. 239, where $\approx$ should be replaced by $\sim$) determined by the Q-homomorphism h.

The problem of the existence of canonical semantic models is the subject of this section. Canonical semantic models do not always exist. In fact, in § 19, p. 344, we have given an example of a theory $\mathfrak{C}$ such that $\bar{\bar{T}} < \mathrm{card}_2\, \mathfrak{C}$. This theory does not have any semantic model in the set T and consequently it does not have any canonical semantic model.

We recall (see VI, § 3) that the algebra of terms is, by definition, the algebra

$$\{T, \{\varphi\}_{\varphi \in \Phi}\}$$

where Φ is the set of all functors, and every m-argument functor φ is interpreted as the m-argument operation

$$\varphi(\tau_1, \ldots, \tau_m) = \varphi(\tau_1 \ldots \tau_m) \tag{1}$$

($m = 0, 1, 2, \ldots$). If R is a realization in a set $J \neq 0$, then J_R denotes the algebra

$$\{J, \{\varphi_R\}_{\varphi \in \Phi}\}.$$

22.1. *If a theory $\mathfrak{C}$ has a semantic model R in a set $J \neq 0$ and if there exists a homomorphism f of the algebra T of all terms in $\mathfrak{C}$ onto the algebra J_R, then $\mathfrak{C}$ has a canonical semantic model R_0 such that, for every formula α, α is satisfiable in R if and only if α is satisfiable in R_0. Viz.*

$$\alpha_R(f v) = V \quad \text{if and only if} \quad \alpha_{R_0}(v) = V$$

for every valuation v: $V \to T$.

Let R_0 be the canonical realization of terms, i.e., for every m-argument functor φ, the operation φ_{R_0} coincides with the mapping defined by (1). For every m-argument predicate ϱ $(m = 1, 2, ...)$, let

$$\varrho_{R_0}(\tau_1, ..., \tau_m) = \varrho_R(f(\tau_1), ..., f(\tau_m)) . \tag{2}$$

Since f: $T \to J_R$ is a homomorphism, we have

$$f(\varphi_{R_0}(\tau_1, ..., \tau_m)) = \varphi_R(f(\tau_1), ..., f(\tau_m)) \tag{3}$$

for every m-argument functor φ $(m = 0, 1, 2, ...)$. Identities (2) and (3) are the hypothesis of theorem 2.8 which immediately implies 22.1.

If there exists a homomorphism f from T onto J_R, then $\overline{\overline{J}} \leqslant \overline{\overline{T}}$. Note that the existence of the homomorphism f cannot be replaced in theorem 22.1 by the weaker condition $\overline{\overline{J}} \leqslant \overline{\overline{T}}$: the existence of a semantic model in a set J of the power $\overline{\overline{T}}$ does not imply, in general, the existence of a canonical semantic model in T. For instance, suppose that a language $\mathfrak{L}$ of the first order contains an enumerable set of free individual variables, a non-enumerable set of individual constants (and no other functors), the sign $\mathfrak{e}$ of equality, a unary predicate ϱ and a binary predicate π. The set $\mathcal{A}$ of mathematical axioms of the theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ consists of axioms of equality (see V, § 12), of the formulas

$$\bigcap_\xi \bigcup_\eta \pi(\xi\, \eta) ,$$

$$(\pi(x\, y) \Rightarrow -\varrho(y)) ,$$

$$((-\mathfrak{e}(x_1\, x_2) \cap (\pi(x_1\, y_1) \cap \pi(x_2\, y_2)) \Rightarrow -\mathfrak{e}(y_1\, y_2)) ,$$

of all the formulas

$$-\mathfrak{e}(c_1\, c_2)$$

where c_1, c_2 are any distinct individual constants, and of all the formulas

$$\varrho(c)$$

where c is any individual constant. The set T of all terms is composed of the enumerable set V of free individual variables and the non-enumerable set of individual constants. Roughly speaking, the axioms of $\mathfrak{T}$ assert the existence of a one-to-one function (which assigns to every element x an element y such that the relation $\pi(x\, y)$ holds) whose values are different from the elements corresponding to individual constants. Thus it is easy to see that $\mathfrak{T}$ has a semantic model in every set J of the power $\overline{\overline{T}}$ (and even in the set T). But $\mathfrak{T}$ does not have any canonical semantic model R in T. For suppose that such a model R exists. Since R is canonical, we have $c_R = c$, i.e. individual constants have to be realized in the model R by individual constants. But there exists no

one-to-one function from T into T whose values are not individual constants, i.e. are free individual variables, because

$$\overline{\overline{V}} < \overline{\overline{T}} .$$

For any theory $\mathcal{C} = \{\mathfrak{L}, C, \mathcal{A}\}$, denote by $\mathfrak{m}_{\mathcal{C}}$ the cardinal of the union of the set V of all free individual variables in $\mathfrak{L}$ and the set Φ_1 of all functors which do not appear in axioms in $\mathcal{A}$.

22.2. *If a theory $\mathcal{C}$ has a semantic model R in a set $J \neq 0$ of a power $\leqslant \mathfrak{m}_{\mathcal{C}}$, then $\mathcal{C}$ has a canonical semantic model R_0. Moreover, for every given formula α satisfiable in R, we can choose R_0 in such a way that $\alpha_{R_0}(\mathfrak{i}) = V$ where $\mathfrak{i}$ is the identity valuation* [1].

Let $x_1, \ldots, x_n$ be all free individual variables in α and let $j_1, \ldots, j_n$ be elements in J such that

$$\alpha_R(j_1, \ldots, j_n) = V . \tag{4}$$

Let Φ_1' be the set of all functors occurring neither in α nor in any axiom in $\mathcal{A}$. Let T_1 be the set of all terms in the formalized language $\mathfrak{L}_1$ obtained from the language $\mathfrak{L}$ by replacing the set Φ of all functors by the subset Φ_1'. In other words, T_1 is the set of all terms in $\mathfrak{L}$ which do not contain any functor in $\Phi - \Phi_1'$.

Without any restriction of generality we may assume that $\overline{\overline{J}} = \mathfrak{m}_{\mathcal{C}}$ (see 19.1). Thus $\overline{\overline{T}}_1 = \mathfrak{m}_{\mathcal{C}} = \overline{\overline{J}}$ and, consequently, there is a one-to-one mapping f_0 from T_1 onto J. Moreover we may assume that

$$f_0(x_k) = j_k \quad \text{for} \quad k = 1, \ldots, n .$$

Thus, by (4),

$$\alpha_R(f_0 \mathfrak{i}) = V .$$

Let R' be the following modification of the realization R:

$$\varrho_{R'} = \varrho_R \quad \text{for every predicate } \varrho ,$$

$$\varphi_{R'} = \varphi_R \quad \text{for every functor } \varphi \text{ in } \Phi - \Phi_1' ,$$

and

$$\varphi_{R'}(f_0(\tau_1), \ldots, f_0(\tau_m)) = f_0(\varphi(\tau_1 \ldots \tau_m)) \tag{5}$$

for every m-argument functor φ in Φ_1' $(m = 0, 1, 2, \ldots)$.

Since the realization R' differs from R only for functors which appear neither in axioms in $\mathcal{A}$ nor in the formula α, the realization R' is also a semantic model for $\mathcal{C}$, and

$$\alpha_{R'}(f_0 \mathfrak{i}) = V .$$

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[1] For theorems 22.2, 22.3, 22.4, 22.5, 22.6 see Rasiowa and Sikorski [6] and Sikorski [12], [13].

By (5), the mapping f_0 is a homomorphism from the algebra $\{T_1, \{\varphi\}_{\varphi\in\Phi_1'}\}$ onto the algebra $\{J, \{\varphi_{R'}\}_{\varphi\in\Phi_1'}\}$. The mapping f_0 can be extended to a homomorphism f from the algebra $\{T, \{\varphi\}_{\varphi\in\Phi}\}$ onto the algebra $J_{R'} = \{J, \{\varphi_{R'}\}_{\varphi\in\Phi}\}$. In fact, the restriction of f_0 to the set V of all free individual variables can be extended to a homomorphism f from the algebra of terms $\{T, \{\varphi\}_{\varphi\in\Phi}\}$ into J_R by VI, 3.2. Since the mapping f restricted to T_1 is a homomorphism from $\{T_1, \{\varphi\}_{\varphi\in\Phi_1}\}$ into $\{J, \{\varphi_{R'}\}_{\varphi\in\Phi_1}\}$ and coincides with f_0 on V, f coincides with f_0 on T_1, i.e. f is an extension of f_0. Since f_0 maps T_1 onto J, f maps T onto J.

Since f is an extension of f_0, we have

$$a_{R'}(f\mathfrak{i}) = V .$$

Now theorem 22.2 follows directly from 22.1 (where R is replaced by R').

It follows from 22.2 and 9.3 that by adding to a theory $\mathfrak{T}$ a set of free individual variables or a set of functors we can obtain a theory $\mathfrak{T}'$ which is an inessential extension of $\mathfrak{T}$ and which has canonical semantic models. In this way, we can obtain a theory $\mathfrak{T}'$ which has so many canonical semantic models that for every irrefutable formula α there exists a canonical semantic model in which α is satisfiable.

There is a natural one-to-one correspondence between canonical semantic models and Q-filters in the Q-algebra $\mathfrak{A}(\mathfrak{T})$ (see § 1, p. 281). The correspondence is established in the following theorem.

22.3. *For every canonical semantic model R, the set ∇_R of all $\|\alpha\| \in \mathfrak{A}(\mathfrak{T})$ such that*

$$a_R(\mathfrak{i}) = V ,$$

where $\mathfrak{i}$ is the identity valuation, is a Q-filter in $\mathfrak{A}(\mathfrak{T})$. Conversely, for every Q-filter ∇ in $\mathfrak{A}(\mathfrak{T})$, the formula

$$\varrho_{R_\nabla}(\tau_1, \ldots, \tau_m) = \begin{cases} V & \text{if} \quad \|\varrho(\tau_1 \ldots \tau_m)\| \in \nabla, \\ \Lambda & \text{if} \quad \|\varrho(\tau_1 \ldots \tau_m)\| \notin \nabla \end{cases}$$

determines a canonical semantic model R_∇ for $\mathfrak{T}$. Moreover, for any canonical semantic model R and any Q-filter ∇

$$\nabla = \nabla_R \quad \text{if and only if} \quad R = R_\nabla .$$

The proof of the first part is by an easy verification (see VI, § 6, (4) and VI, 7.1, (3), (3')).

The second part follows immediately from VI, 7.3, where h is the natural homomorphism from $\mathfrak{A}(\mathfrak{T})$ onto the two-element Boolean algebra $\mathfrak{A}(\mathfrak{T})/\nabla$. In fact, the realization R^0 mentioned in this theorem coincides with R. By VI, 7.3, (7), all axioms in $\mathfrak{T}$ are valid in R, i.e. R is a model for $\mathfrak{T}$.

To prove the last part of 22.3, let us first suppose that $\mathcal{V} = \mathcal{V}_R$. Then each of the following conditions is equivalent to the next one:

$$\varrho_{R_\Delta}(\tau_1, \ldots, \tau_m) = V\,,$$

$$\|\varrho(\tau_1 \ldots \tau_m)\| \in \mathcal{V}\,,$$

$$\varrho(\tau_1 \ldots \tau_m)_R(\mathfrak{i}) = V\,,$$

$$\varrho_R(\tau_1, \ldots, \tau_m) = V\,,$$

which proves that $R_{\mathcal{V}} = R$. Conversely, suppose that $R = R_{\mathcal{V}}$. Then each of the following conditions is equivalent to the next one:

$$\|\varrho(\tau_1 \ldots \tau_m)\| \in \mathcal{V}_R\,,$$

$$\varrho(\tau_1 \ldots \tau_m)_R(\mathfrak{i}) = V\,,$$

$$\varrho(\tau_1 \ldots \tau_m)_{R_{\mathcal{V}}}(\mathfrak{i}) = V\,,$$

$$\|\varrho(\tau_1 \ldots \tau_m)\| \in \mathcal{V}\,.$$

This proves that the natural Q-homomorphisms from $\mathfrak{A}(\mathfrak{C})$ onto $\mathfrak{A}(\mathfrak{C})/\mathcal{V}_R$ and onto $\mathfrak{A}(\mathfrak{C})/\mathcal{V}$ coincide on the set of elements $\|\alpha\|$ where α is an elementary formula. Since elements of this form generate the Q-algebra $\mathfrak{A}(\mathfrak{C})$, the two Q-homomorphisms coincide. This implies that $\mathcal{V}_R = \mathcal{V}$.

By $\mathfrak{F}(\mathfrak{C})$ we shall denote the set of all Q-filters in $\mathfrak{A}(\mathfrak{C})$. In other words, applying the notation from II, § 9, p. 86,

$$\mathfrak{F}(\mathfrak{C}) = \mathfrak{F}_{\mathrm{Q}}(\mathfrak{A}(\mathfrak{C}))$$

where (Q) denotes, as usually, the set of all infinite joins and meets in $\mathfrak{A}(\mathfrak{C})$, corresponding to logical quantifiers (see § 1, p. 280, and VI, § 11, p. 252). For every element $\|\alpha\| \in \mathfrak{A}(\mathfrak{C})$ we shall denote by $|||\alpha|||$ (or, more precisely, by $|||\alpha|||_{\mathfrak{C}}$) the set of all $\mathcal{V} \in \mathfrak{F}(\mathfrak{C})$ such that $\|\alpha\| \in \mathcal{V}$. By definition, for every Q-filter $\mathcal{V}$ in $\mathfrak{A}(\mathfrak{C})$,

$$\mathcal{V} \in |||\alpha||| \quad \text{if and only if} \quad \|\alpha\| \in \mathcal{V}\,.$$

By $\boldsymbol{P}(\mathfrak{C})$ we shall denote the class of all sets $|||\alpha|||$ where $\|\alpha\| \in \mathfrak{A}(\mathfrak{C})$. It follows from the definition and II 9.1 that $\boldsymbol{P}(\mathfrak{C})$ is a field of subsets of $\mathfrak{F}(\mathfrak{C})$, and that the mapping $H_{\mathfrak{C}}$:

$$H_{\mathfrak{C}}(\|\alpha\|) = |||\alpha||| \tag{6}$$

is a Q-homomorphism from $\mathfrak{A}(\mathfrak{C})$ into the field of all subsets of $\mathfrak{F}(\mathfrak{C})$, and onto $\boldsymbol{P}(\mathfrak{C})$. Thus $H_{\mathfrak{C}}$ transforms infinite joins and meets corresponding to logical quantifiers onto set-theoretical unions and intersections respectively.

22.4. *The following conditions are equivalent*:

(i) *there exists a* Q*-isomorphism from* $\mathfrak{A}(\mathcal{T})$ *into the field of all subsets of a set*;

(ii) $H_{\mathcal{T}}$ *is an isomorphism*;

(iii) *for every* $\|\alpha\| \neq \Lambda$ *in* $\mathfrak{A}(\mathcal{T})$ *there exists a* Q*-filter* ∇ *such that* $\|\alpha\| \in \nabla$;

(iv) *for every irrefutable formula* α *there exists a canonical semantic model* R *for* $\mathcal{T}$ *in which the identity valuation satisfies the formula* α.

The equivalence of (i), (ii), (iii) follows from II, 9.2. The equivalence of (iii) and (iv) follows directly from 22.3.

We shall complement theorem 22.4 by the following theorems.

22.5. *If* $\mathfrak{m}_{\mathcal{T}} = \overline{\overline{T}} = \mathfrak{m}$, *then each of the conditions* (i)-(iv) *is equivalent to the condition*:

(v) *for every formula* α *irrefutable in* $\mathcal{T}$ *there exists a semantic model* R *of a power* $\leqslant \mathfrak{m}$ *such that* α *is satisfiable in* R.

(iv) directly implies (v). The converse implication follows from 22.2.

22.6. *If the set of axioms of the theory* $\mathcal{T}$ *has a power* $\leqslant \mathfrak{m}_{\mathcal{T}}$, *then* $H_{\mathcal{T}}$ *is an isomorphism.*

In particular, if the set of axioms of $\mathcal{T}$ *is at most enumerable, then* $H_{\mathcal{T}}$ *is an isomorphism.*

This follows from the equivalence of (ii) and (v) and from 9.3.

By $\boldsymbol{P}_0(\mathcal{T})$ we shall denote the class of all sets $|||\alpha|||$ where α is an open formula. Obviously $\boldsymbol{P}_0(\mathcal{T})$ is a field of subsets of $\mathfrak{F}(\mathcal{T})$.

The set $\mathfrak{F}(\mathcal{T})$ will always be considered a topological space, $\boldsymbol{P}_0(\mathcal{T})$ being assumed as an open basis in $\mathfrak{F}(\mathcal{T})$. By definition, all sets in $\boldsymbol{P}_0(\mathcal{T})$ are both open and closed in $\mathfrak{F}(\mathcal{T})$. A set $S \subset \mathfrak{F}(\mathcal{T})$ is open if and only if it is a union of some sets in $\boldsymbol{P}_0(\mathcal{T})$.

22.7. *The space* $\mathfrak{F}(\mathcal{T})$ *is totally disconnected. More precisely, for any pair of distinct filters* $\nabla_1, \nabla_2 \in \mathfrak{F}(\mathcal{T})$ *there exists an elementary formula* α *such that the both open and closed set* $|||\alpha|||$ *contains* ∇_1 *but does not contain* ∇_2 [1].

Let h_1 and h_2 be the natural Q-homomorphisms from the Q-algebra $\mathfrak{A}(\mathcal{T})$ onto the two-element Boolean algebra $\boldsymbol{A}_0 = \mathfrak{A}(\mathcal{T})/\nabla_1 = \mathfrak{A}(\mathcal{T})/\nabla_2$ respectively. Suppose that, for every elementary formula α, either $\|\alpha\| \in \nabla_1$ and $\|\alpha\| \in \nabla_2$, or $\|\alpha\| \notin \nabla_1$ and $\|\alpha\| \notin \nabla_2$. Then $h_1(\|\alpha\|) = h_2(\|\alpha\|)$ for every elementary formula α. Since the set of all elements $\|\alpha\|$ where α is an elementary formula generates the Q-algebra $\mathfrak{A}(\mathcal{T})$, the homomorphisms h_1 and h_2 coincide. This implies that $\nabla_1 = \nabla_2$.

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[1] Theorems 22.7, 22.8, 22.9 were proved by Sikorski [12], [13].

Suppose the theory $\mathcal{C}$ is a linguistically invariant extension of a theory $\mathcal{C}'$. According to VI, § 10, elements of $\mathfrak{A}(\mathcal{C})$ and $\mathfrak{A}(\mathcal{C}')$, determined by a formula α, will be denoted by $\|\alpha\|_{\mathcal{C}}$ and $\|\alpha\|_{\mathcal{C}'}$ respectively. The mapping which, to any $\|\alpha\|_{\mathcal{C}'}$, assigns the element $\|\alpha\|_{\mathcal{C}}$ is a Q-homomorphism h from $\mathfrak{A}(\mathcal{C}')$ onto $\mathfrak{A}(\mathcal{C})$. Hence it follows that, for every filter $F \in \mathfrak{F}(\mathcal{C})$, its counter-image $F' = h^{-1}(F)$ (i.e. the set of all $\|\alpha\|_{\mathcal{C}'}$ such that $\|\alpha\|_{\mathcal{C}} \in F$) is a prime Q-filter in $\mathfrak{A}(\mathcal{C}')$, i.e. $F' \in \mathfrak{F}(\mathcal{C}')$. It is easy to see that the mapping which assigns F' to F is one-to-one. This enables us to identify F with F'. After this identification

$$\mathfrak{F}(\mathcal{C}) \subset \mathfrak{F}(\mathcal{C}') . \tag{7}$$

Moreover, for every formula α

$$|||\alpha|||_{\mathcal{C}} = |||\alpha|||_{\mathcal{C}'} \cap \mathfrak{F}(\mathcal{C}) . \tag{8}$$

Taking as α any open formula, we infer that sets in $\boldsymbol{P}_0(\mathcal{C})$ are intersections of sets in $\boldsymbol{P}_0(\mathcal{C}')$ with the set $\mathfrak{F}(\mathcal{C})$, and conversely. Consequently open sets in $\mathfrak{F}(\mathcal{C})$ are intersections of open sets in $\mathfrak{F}(\mathcal{C}')$ with the set $\mathfrak{F}(\mathcal{C})$, and conversely.

Thus, by I, § 2, (11) (see also III, § 2, p. 97), we have

22.8. *If $\mathcal{C}$ is a linguistically invariant extension of $\mathcal{C}'$ then $\mathfrak{F}(\mathcal{C})$ is a topological subspace of the space $\mathfrak{F}(\mathcal{C}')$.*

We shall complete this theorem by the following one.

22.9. *Let $\mathcal{C}$ be a linguistically invariant extension of $\mathcal{C}'$.*

If $H_{\mathcal{C}'}$ is an isomorphism and $\mathfrak{F}(\mathcal{C})$ is a dense subset of $\mathfrak{F}(\mathcal{C}')$, then every open formula which is a theorem in $\mathcal{C}$ is also a theorem in $\mathcal{C}'$.

Conversely, if $H_{\mathcal{C}}$ is an isomorphism and if every open formula which is a theorem in $\mathcal{C}$ is also a theorem in $\mathcal{C}'$, then $\mathfrak{F}(\mathcal{C})$ is a dense subset of $\mathfrak{F}(\mathcal{C}')$.

To prove the first part, let us assume that an open formula α is a theorem in $\mathcal{C}$, i.e. that $\|\alpha\|_{\mathcal{C}}$ is the unit element of $\mathfrak{A}(\mathcal{C})$. Thus $|||\alpha|||_{\mathcal{C}} = \mathfrak{F}(\mathcal{C})$, i.e. $\mathfrak{F}(\mathcal{C}) \subset |||\alpha|||_{\mathcal{C}'}$ by (8). Consequently

$$\mathfrak{F}(\mathcal{C}') = \mathbf{C}\mathfrak{F}(\mathcal{C}) \subset \mathbf{C}|||\alpha|||_{\mathcal{C}'} = |||\alpha|||_{\mathcal{C}'} \subset \mathfrak{F}(\mathcal{C}')$$

since $\mathfrak{F}(\mathcal{C})$ is dense in $\mathfrak{F}(\mathcal{C}')$ and $|||\alpha|||_{\mathcal{C}'}$ is both open and closed (here $\mathbf{C}$ denotes the closure operation in $\mathfrak{F}(\mathcal{C}')$). Since $H_{\mathcal{C}'}$ is an isomorphism, the identity $H_{\mathcal{C}'}(\|\alpha\|_{\mathcal{C}'}) = |||\alpha|||_{\mathcal{C}'} = \mathfrak{F}(\mathcal{C}')$ just proved implies that $\|\alpha\|_{\mathcal{C}'}$ is the unit element of $\mathfrak{A}(\mathcal{C}')$, i.e. that α is a theorem in $\mathcal{C}'$.

To prove the second part, suppose that $\mathfrak{F}(\mathcal{C})$ is not dense in $\mathfrak{F}(\mathcal{C}')$. It follows from the definition of the topology in $\mathfrak{F}(\mathcal{C}')$ that there exists an open formula α such that

$$\mathfrak{F}(\mathcal{C}) \subset |||\alpha|||_{\mathcal{C}'} \neq \mathfrak{F}(\mathcal{C}') .$$

Thus a is not a theorem in $\mathfrak{T}'$. By (8),

$$H_{\mathfrak{T}}(\|a\|_{\mathfrak{T}}) = |||a|||_{\mathfrak{T}} = \mathfrak{F}(\mathfrak{T}) .$$

Since $H_{\mathfrak{T}}$ is an isomorphism, $\|a\|_{\mathfrak{T}}$ is the unit of $\mathfrak{A}(\mathfrak{T})$, i.e. a is a theorem in $\mathfrak{T}$.

§ 23. A topological characterization of open theories. Let $\mathfrak{T}$ be a theory of the first order. Denote by $\mathfrak{A}^0(\mathfrak{T})$ the subalgebra of $\mathfrak{A}(\mathfrak{T})$ consisting of all elements $\|a\|$ where a is an open formula. Let $\mathfrak{F}^0(\mathfrak{T})$ be the Stone space of the Boolean algebra $\mathfrak{A}^0(\mathfrak{T})$, i.e. using the notation from II, § 8,

$$\mathfrak{F}^0(\mathfrak{T}) = \mathfrak{F}(\mathfrak{A}^0(\mathfrak{T})) .$$

By definition, $\mathfrak{F}^0(\mathfrak{T})$ is the compact space of all prime (i.e. maximal) filters $\mathcal{V}_0$ in the Boolean algebra $\mathfrak{A}^0(\mathfrak{T})$. For every open formula a, let $|||a|||^0$ (or, more precisely, $|||a|||^0_{\mathfrak{T}}$) denote the set of all $\mathcal{V}_0 \in \mathfrak{F}^0(\mathfrak{T})$ such that $\|a\| \in \mathcal{V}_0$. By definition, for every open formula a and every prime filter $\mathcal{V}_0$ in $\mathfrak{A}^0(\mathfrak{T})$,

(1) $\qquad \|a\| \in \mathcal{V}_0 \quad$ if and only if $\quad \mathcal{V}_0 \in |||a|||^0 .$

The mapping which, to every element $\|a\| \in \mathfrak{A}^0(\mathfrak{T})$ (a open), assigns the set $|||a|||^0$ is the Stone isomorphism of the Boolean algebra $\mathfrak{A}^0(\mathfrak{T})$ onto the field $\boldsymbol{P}^0(\mathfrak{T})$ of all both open and closed subsets of the Stone space $\mathfrak{F}^0(\mathfrak{T})$.

For every Q-filter $\mathcal{V} \in \mathfrak{F}(\mathfrak{T})$, the intersection of $\mathcal{V}$ with the subalgebra $\mathfrak{A}^0(\mathfrak{T})$ is a prime filter $\mathcal{V}_0$ in $\mathfrak{A}^0(\mathfrak{T})$, i. e. an element of $\mathfrak{F}^0(\mathfrak{T})$. The mapping which assigns to $\mathcal{V} \in \mathfrak{F}(\mathfrak{T})$ the filter $\mathcal{V}_0 \in \mathfrak{F}^0(\mathfrak{T})$ just defined is one-to-one (this follows easily from the proof of 22.7). This allows us to identify $\mathcal{V}$ with $\mathcal{V}_0$. By this identification,

(2) $\qquad \mathfrak{F}(\mathfrak{T}) \subset \mathfrak{F}^0(\mathfrak{T}) .$

It follows from (1) that, for every open formula a,

(3) $\qquad |||a||| = |||a|||^0 \cap \mathfrak{F}(\mathfrak{T}) .$

Thus, by I, § 2, (11) (see also III, § 2, p. 97), we have

23.1. *$\mathfrak{F}(\mathfrak{T})$ is a topological subspace of the compact Stone space $\mathfrak{F}^0(\mathfrak{T})$.*

In general, the sign of inclusion in (2) cannot be replaced by the sign of equality. It is possible in some special cases. For instance,

23.2. *If $\mathfrak{T}$ is an open theory, then $\mathfrak{F}(\mathfrak{T}) = \mathfrak{F}^0(\mathfrak{T})$. Consequently the space $\mathfrak{F}(\mathfrak{T})$ is compact for every open theory $\mathfrak{T}$.*

It suffices to show that for every $\mathcal{V}_0 \in \mathfrak{F}^0(\mathfrak{T})$ there exists a $\mathcal{V} \in \mathfrak{F}(\mathfrak{T})$ such that the filter $\mathcal{V}_0$ is the intersection of the filter $\mathcal{V}$ and the subalgebra $\mathfrak{A}^0(\mathfrak{T})$ of $\mathfrak{A}(\mathfrak{T})$.

Let R be a semantic realization of the language of the theory $\mathfrak{T}$ such that R is a canonical realization of terms and

$$\varrho_R(\tau_1, \ldots, \tau_m) = \begin{cases} V & \text{if} \quad \|\varrho(\tau_1 \ldots \tau_m)\| \in V_0, \\ \Lambda & \text{if} \quad \|\varrho(\tau_1 \ldots \tau_m)\| \notin V_0 \end{cases}$$

for every m-argument predicate ϱ and arbitrary terms $\tau_1, \ldots, \tau_m$. Since $\mathfrak{A}^0(\mathfrak{T})$ can be identified with the algebra $\mathfrak{A}(\mathfrak{T}^0)$ defined in § 15 (see 15.6), R is the canonical model for $\mathfrak{T}$, determined by the natural homomorphism from $\mathfrak{A}(\mathfrak{T}^0)$ onto the two-element Boolean algebra $A_0 = \mathfrak{A}(\mathfrak{T}^0)/V_0$ in the sense defined in § 15, p. 329 (see also 15.3).

Thus R is a semantic model for the open theory $\mathfrak{T}$. By 22.3 the set V of all $\|\alpha\| \in \mathfrak{A}(\mathfrak{T})$ such that $\alpha_R(\mathfrak{i}) = V$ is a prime Q-filter, i.e. an element of $P(\mathfrak{T})$. This filter has the required properties.

We recall that $\mathfrak{m}_{\mathfrak{T}}$ denotes the cardinal of the union of the set V of all individual variables and the set Φ_1 of all functors which do not appear in axioms of the theory $\mathfrak{T}$. As usual the letter T denotes the set of all terms in $\mathfrak{T}$, and $H_{\mathfrak{T}}$ is the mapping defined in § 22, (6).

23.3. *If $\mathfrak{T}$ is an open theory and $\overline{\overline{T}} = \mathfrak{m}_{\mathfrak{T}}$, then the homomorphism $H_{\mathfrak{T}}$ is an isomorphism.*

By 17.5 and 22.2, for every irrefutable formula β there exists a canonical semantic model R for $\mathfrak{T}$ such that $\beta_R(\mathfrak{i}) = V$.

This proves that the homomorphism $H_{\mathfrak{T}}$ is an isomorphism by 22.4.

The hypothesis $\overline{\overline{T}} = \mathfrak{m}_{\mathfrak{T}}$ cannot be omitted in 23.3. If $\mathfrak{T}$ is an open theory, then $\mathfrak{T}$ has many canonical semantic models. Namely, for every irrefutable open formula α in $\mathfrak{T}$ there is a canonical semantic model in which the identity valuation satisfies α (this result can be obtained, e.g., by a slight modification of the proof of 15.5). On the other hand, for every irrefutable formula α in $\mathfrak{T}$ there exists a semantic model in a set of power $\overline{\overline{T}}$ in which α is satisfiable (see 17.5, p. 338). However, it is not true, in general, that for every irrefutable formula α there is a canonical semantic model in which α is satisfiable. In that case $H_{\mathfrak{T}}$ is not an isomorphism because of the equivalence of (ii) and (iv) in 22.4.

The required example is a slight modification of the example of a theory from § 22, p. 356. Namely, let $\mathfrak{L}$ be a language of the first order such that $\mathfrak{L}$ contains an enumerable set of free individual variables, a non-enumerable set of individual constants (and no other functors), the sign $\mathfrak{e}$ of equality, a unary predicate ϱ and a binary predicate π. The set $\mathcal{A}$ of mathematical axioms of the theory $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ consists of axioms of equality (see V, § 12), of the open formulas

$$(\pi(x\,y) \Rightarrow -\varrho(y)),$$

$$((-\mathfrak{e}(x_1\,x_2) \frown (\pi(x_1\,y_i) \frown \pi(x_2\,y_2)) \Rightarrow -\mathfrak{e}(y_1\,y_2)),$$

of all the open formulas

$$-\varrho(c_1\, c_2)$$

where c_1, c_2 are any distinct individual constants, and of all open formulas

$$\varrho(c)$$

where c is any individual constant. Let a be the formula

$$\bigcap_\xi \bigcup_\eta \pi(\xi\ \eta)\,.$$

In the notation just adopted, the theory examined in the example from § 22, p. 356, is the theory $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A} \cup [a]\}$. Since the last theory has semantic models, it is consistent. By 11.2, the closed formula a is irrefutable in the theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$. There is no canonical semantic model for $\mathfrak{T}$ in which a is satisfiable. For suppose such a canonical semantic model R exists. Since a is closed, the mapping a_R is constant. Since a is satisfiable in R, a_R assumes the value V. Thus a_R is identically equal to V, i.e. R is a model for a. This implies that R is a canonical semantic model for the theory $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A} \cup [a]\}$. On the other hand, we have shown in § 22, p. 356, that the theory $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A} \cup [a]\}$ has no canonical semantic model. Contradiction!

We recall that two theories $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ and $\mathfrak{T}' = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}'\}$ based on the same language $\mathfrak{L}$ are said to be equivalent if they have the same set of theorems: $\mathfrak{C}(\mathcal{A}) = \mathfrak{C}(\mathcal{A}')$.

23.4. *Suppose that the sets of all free individual variables and the set of all terms in the theory $\mathfrak{T}$ have the same power. Then in order that $\mathfrak{T}$ be equivalent to an open theory it is necessary and sufficient that $\mathfrak{F}(\mathfrak{T})$ be compact and $H_{\mathfrak{T}}$ be an isomorphism* (1).

Suppose that $\mathfrak{T}$ is equivalent to an open theory $\mathfrak{T}'$ based on the same language. The hypothesis $\overline{\overline{V}} = \overline{\overline{T}}$ implies that $\mathfrak{m}_{\mathfrak{T}'} = \overline{\overline{T}}$. By 23.2 and 23.3 the space $\mathfrak{F}(\mathfrak{T}')$ is compact and $H_{\mathfrak{T}'}$ is an isomorphism. Since $\mathfrak{T}$ and $\mathfrak{T}'$ are equivalent, we have $\mathfrak{A}(\mathfrak{T}) = \mathfrak{A}(\mathfrak{T}')$. Consequently $\mathfrak{F}(\mathfrak{T}) = \mathfrak{F}(\mathfrak{T}')$ and $H_{\mathfrak{T}} = H_{\mathfrak{T}'}$. This proves the necessity.

To prove the sufficiency, let $\mathfrak{T}'$ be the theory based on the same language as $\mathfrak{T}$, such that the set $\mathcal{A}'$ of all axioms for $\mathfrak{T}'$ consists of all open formulas which are theorems in $\mathfrak{T}$. By definition, $\mathfrak{T}'$ is open and $\mathfrak{C}(\mathcal{A}') \subset \mathfrak{C}(\mathcal{A})$, i.e. $\mathfrak{T}$ is a linguistically invariant extension of $\mathfrak{T}'$. By § 22, (7), $\mathfrak{F}(\mathfrak{T})$ is a subset of $\mathfrak{F}(\mathfrak{T}')$. Since all open theorems in $\mathfrak{T}$ are theorems in $\mathfrak{T}'$, the set $\mathfrak{F}(\mathfrak{T})$ is dense in $\mathfrak{F}(\mathfrak{T}')$ by the second part of 22.9

(1) Theorems 2.1-2.4 were proved by Sikorski [12], [13], [10]. See also Beth [4] and Ehrenfeucht and Mostowski [1]. Another characterization of open theories by means of submodels was given by Łoś [5], [6] (it was also known to Tarski—see footnote 31 in Łoś [5]).

and the hypothesis that $H_{\mathfrak{C}}$ is an isomorphism. Since $\mathfrak{F}(\mathfrak{C})$ is compact, $\mathfrak{F}(\mathfrak{C})$ is a closed subset of $\mathfrak{F}(\mathfrak{C}')$ by I, 2.3. This proves that

$$\mathfrak{F}(\mathfrak{C}) = \mathfrak{F}(\mathfrak{C}') .$$

Suppose that a is a theorem in $\mathfrak{C}$, i.e. $|||a|||_{\mathfrak{C}} = \mathfrak{F}(\mathfrak{C}) = \mathfrak{F}(\mathfrak{C}')$. Hence it follows by § 22, (8), that $|||a|||_{\mathfrak{C}'} = \mathfrak{F}(\mathfrak{C}')$. Since $\mathfrak{C}'$ is open and $\mathfrak{m}_{\mathfrak{C}'} = \bar{\bar{T}}$ (which is a consequence of $\bar{\bar{V}} = \bar{\bar{T}}$), the homomorphism $H_{\mathfrak{C}'}$ is an isomorphism by 23.3. Thus $H_{\mathfrak{C}'}(\|a\|_{\mathfrak{C}'}) = |||a|||_{\mathfrak{C}'} = \mathfrak{F}(\mathfrak{C}')$ implies that $\|a\|_{\mathfrak{C}'}$ is the unit element of $\mathfrak{A}(\mathfrak{C}')$, i.e. that a is a theorem in $\mathfrak{C}'$. This proves that $\mathfrak{C}(\mathcal{A}) \subset \mathfrak{C}(\mathcal{A}')$. The converse inclusion having been established earlier, we infer that $\mathfrak{C}(\mathcal{A}) = \mathfrak{C}(\mathcal{A}')$, i.e. that $\mathfrak{C}$ is equivalent to the open theory $\mathfrak{C}'$.

§ 24. The algebra of the two-valued predicate calculus. Let $\mathfrak{S} = \{\mathfrak{L}, \mathfrak{C}\} = \{\mathfrak{L}, \mathfrak{C}, 0\}$ be a predicate calculus with a set F of formulas. The letter E will denote the set of all elementary formulas in $\mathfrak{S}$.

24.1. *The* Q*-algebra* $\mathfrak{A}(\mathfrak{S})$ *is a generalized free algebra in the class of all complete Boolean algebras, and the set of all elements* $\|a\|$, *where* a *is an elementary formula, is a set of free generators* [1].

The proof is similar to that of the analogous theorem VII 4.1 on the algebra of a propositional calculus.

Let f be a mapping defined on the set of all $\|a\| \in \mathfrak{A}(\mathfrak{S})$, where a is in E, with values in a complete Boolean algebra $\boldsymbol{A}$. The mapping f induces a mapping f_0 defined as follows:

$$f_0(|a|) = f(\|a\|) \quad \text{for} \quad a \text{ in } E.$$

We recall that $\bar{F}$ denotes the Q-algebra of the language $\mathfrak{L}$, and $|a|$ denotes the element of $\bar{F}$ determined by a formula a (see VI, § 4, p. 220). Since $\bar{F}$ is a generalized free algebra (VI, 4.1), the mapping f_0 can be extended to a homomorphism g from the Q-algebra $\bar{F}$ into the Boolean algebra $\boldsymbol{A}$.

Let R be the realization of $\mathfrak{L}$ in the set T of all terms and in the complete Boolean algebra $\boldsymbol{A}$, such that R is the canonical realization of terms and, for every m-argument predicate ϱ ($m = 1, 2, \ldots$) and for any terms $\tau_1, \ldots, \tau_m$,

$$\varrho_R(\tau_1, \ldots, \tau_m) = g(|\varrho(\tau_1 \ldots \tau_m)|) \in \boldsymbol{A} .$$

By VI, 7.3, (8), for every formula γ

$$\gamma_R(\mathfrak{i}) = g(|\gamma|) , \tag{1}$$

$\mathfrak{i}$ being the identity valuation.

Suppose that a and β are formulas such that $\|a\| = \|\beta\|$, i.e. both $(a \Rightarrow \beta)$ and $(\beta \Rightarrow a)$ are predicate tautologies. By 6.1, we have

$$(a \Rightarrow \beta)_R(\mathfrak{i}) = V = (\beta \Rightarrow a)_R(\mathfrak{i}) .$$

[1] Rieger [4].

Hence it follows, in virtue of (1), that

$$g(|\alpha|) \Rightarrow g(|\beta|) = (\alpha \Rightarrow \beta)_R(\mathfrak{i}) = V$$

and similarly

$$g(|\beta|) \Rightarrow g(|\alpha|) = (\beta \Rightarrow \alpha)_R(\mathfrak{i}) = V .$$

This proves that

$$\|\alpha\| = \|\beta\| \quad \text{implies} \quad g(|\alpha|) = g(|\beta|) .$$

Consequently the formula

$$h(\|\alpha\|) = g(|\alpha|) \quad \text{for } \alpha \text{ in } F$$

defines a mapping h: $\mathfrak{A}(\mathcal{S}) \to \boldsymbol{A}$. It is easy to verify that the mapping h is a Q-homomorphism and that it is an extension of the mapping f.

We have proved that every mapping f defined on the set of all $\|\alpha\|$ with α in E, with values in a complete Boolean algebra $\boldsymbol{A}$, can be extended to a homomorphism h: $\mathfrak{A}(\mathcal{S}) \to \boldsymbol{A}$. Thus $\mathfrak{A}(\mathcal{S})$ is a generalized free algebra in the class of all complete Boolean algebras, and the set of all $\|\alpha\|$ where α is in E is a set of free generators.

We shall now apply the main results from §§ 22-23 to the case in which $\mathfrak{S} = \mathcal{S} = \{\mathfrak{L}, \mathfrak{C}, 0\}$. By the general definition, $\mathfrak{T}(\mathcal{S})$ is the space of all prime Q-filters in the Boolean algebra $\mathfrak{A}(\mathcal{S})$. For every formula α, the symbol $|||\alpha|||$ and the symbol $H_{\mathcal{S}}(\|\alpha\|)$ denote the set of all filters $V \in \mathfrak{T}(\mathcal{S})$ such that $\|\alpha\| \in V$. Directly from 23.3 follows

24.2. *The* Q*-homomorphism* $H_{\mathcal{S}}$ *is an isomorphism.*

$H_{\mathcal{S}}$ is an isomorphism from $\mathfrak{A}(\mathcal{S})$ onto the field $\boldsymbol{P}(\mathcal{S})$ of all sets $|||\alpha|||$ where α is any formula. As in § 22, let the symbol $\boldsymbol{P}_0(\mathcal{S})$ denote the class of all sets $|||\alpha|||$ where α is an open formula. According to § 22, $\mathfrak{T}(\mathcal{S})$ is considered as a topological space, $\boldsymbol{P}_0(\mathcal{S})$ being assumed as a basis in $\mathfrak{T}(\mathcal{S})$. Since $\mathcal{S}$ can be considered as the open theory $\{\mathfrak{L}, \mathfrak{C}, 0\}$ with the empty set of axioms, we infer from 23.2 that the space $\mathfrak{T}(\mathcal{S})$ is compact. We shall now describe in greater detail the topological structure of the space $\mathfrak{T}(\mathcal{S})$.

Let $\mathfrak{D}$ be the Cantor discontinuum

$$\mathfrak{D} = U^E \tag{2}$$

where U is the set consisting of the integers 0 and 1 only. In other words, $\mathfrak{D}$ is the set of all $u = \{u_\alpha\}_{\alpha \in E}$ such that $u_\alpha = 0$ or 1 for every $\alpha \in E$ where, as previously, E denotes the set of all elementary formulas.

24.3. *The space* $\mathfrak{T}(\mathcal{S})$ *is homeomorphic with the Cantor discontinuum* (2) [1].

[1] Rieger [4], Rasiowa and Sikorski [6], Sikorski [13].

Let us recall that (see § 23) $\mathfrak{A}^0(\mathcal{S})$ denotes the subalgebra of $\mathfrak{A}(\mathcal{S})$ consisting of all elements a where a is an open formula, and $\mathfrak{T}^0(\mathcal{S})$ denotes the Stone space of the Boolean algebra $\mathfrak{A}^0(\mathcal{S})$. By 23.2, $\mathfrak{T}(\mathcal{S}) = \mathfrak{T}^0(\mathcal{S})$. Thus it suffices to show that $\mathfrak{T}^0(\mathcal{S})$ is homeomorphic with $\mathfrak{D}$.

Let $\mathcal{S}_0$ be a propositional calculus such that the set V_0 of all propositional variables in $\mathcal{S}_0$ and the set E have the same power. Let s be a one-to-one mapping from V_0 onto E. For every formula δ in $\mathcal{S}_0$, let $s\delta$ denote the result of the substitution s in δ. By definition, $s\delta$ is an open formula in $\mathcal{S}$. Since s maps V_0 onto E, every open formula in $\mathcal{S}$ is of the form $s\delta$ for a formula δ in $\mathcal{S}_0$. By theorem VIII, 7.2, δ is a propositional tautology if and only if $s\delta$ is a predicate tautology. Replacing δ by $(\delta_1 \Rightarrow \delta_2)$ and by $(\delta_2 \Rightarrow \delta_1)$ we infer that, for any formulas δ_1, δ_2 is $\mathcal{S}_0$, if $\|\delta_1\| = \|\delta_2\| \in \mathfrak{A}(\mathcal{S}_0)$ then $\|s\delta_1\| = \|s\delta_2\| \in \mathfrak{A}^0(\mathcal{S})$, and conversely. Thus the equality

$$h(\|\delta\|) = \|s\delta\|$$

defines an isomorphism from $\mathfrak{A}(\mathcal{S}_0)$ onto $\mathfrak{A}^0(\mathcal{S})$. This allows us to identify both the Boolean algebras and, consequently, their Stone spaces $\mathfrak{T}(\mathcal{S}_0)$ and $\mathfrak{T}^0(\mathcal{S})$. By VII, 4.3, $\mathfrak{T}(\mathcal{S}_0)$ is homeomorphic with the Cantor discontinuum U^{V_0}, i.e. with the Cantor discontinuum $\mathfrak{D} = U^E$ provided any $a \in V_0$ is identified with the elementary formula sa, and consequently V_0 is identified with E. This proves that $\mathfrak{T}^0(\mathcal{S})$ is homeomorphic with $\mathfrak{D}$.

§ 25. The deduction theorem for open theories. As a simple application of the topological investigations in the last sections we shall quote a topological proof of the following deduction theorem for open theories.

25.1. *An open formula β is a theorem of an open theory $\mathcal{C} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ if and only if there exists a conjunction α of open substitutions of axioms in $\mathcal{A}$ such that the implication $(\alpha \Rightarrow \beta)$ is a predicate tautology.*

If there exists a formula α such that α is a conjunction of substitutions in axioms in $\mathcal{A}$ and $(\alpha \Rightarrow \beta)$ is a predicate tautology, then α is a theorem in $\mathcal{C}$ (by the rule (r_2) of substitution for individual variables and by VI 10.8) and consequently β is a theorem in $\mathcal{C}$ by *modus ponens*.

Conversely, suppose that β is a theorem in $\mathcal{C}$. By the deduction theorem 10.2, there is a finite set of axioms in $\mathcal{A}$:

$$a_i(x_1, \ldots, x_{m_i}) \quad (i = 1, \ldots, n) \tag{1}$$

such that the implication $(\alpha_0 \Rightarrow \beta)$ is a predicate tautology where α_0 is the conjunction of closures

$$\bigcap_{\xi_1} \cdots \bigcap_{\xi_{m_i}} a_i(\xi_1, \ldots, \xi_{m_i})$$

of formulas (1). This implies that the corresponding elements in the algebra $\mathfrak{A}(\mathcal{S})$ of the predicate calculus $\mathcal{S} = \{\mathfrak{L}, \mathfrak{C}, 0\}$ satisfy the inequality

$\|\alpha_0\| \leqslant \|\beta\|$. Hence it follows that, in the space $\mathfrak{T}(\mathfrak{S})$, the inclusion $|||\alpha_0||| \subset |||\beta|||$ holds, i. e.

$$(3) \qquad \bigcap_{i=1}^{n} \bigcap_{\tau_1 \in T} \dots \bigcap_{\tau_{m_i} \in T} |||\alpha_i(\tau_1, \dots, \tau_{m_i})||| \subset |||\beta|||$$

where $\bigcap$ denotes the set-theoretical intersection. Since all the formulas in (3) are open, all the sets $|||\ |||$ in (3) are both open and closed by the definition of topology in $\mathfrak{T}(\mathfrak{S})$. Since the sets $|||\alpha_i(\tau_1, \dots, \tau_{m_i})|||$ are closed, the set $|||\beta|||$ is open and the space $\mathfrak{T}(\mathfrak{S})$ is compact there exist a finite number of sets on the left-hand side of (3) such that their intersection is contained in the set $|||\beta|||$ (see I, § 2, p. 16). In other words, there exists a finite conjunction α of formulas appearing on the left-hand side of inclusion (3), such that $|||\alpha||| \subset |||\beta|||$. This inclusion is equivalent to $\|\alpha\| \leqslant \|\beta\|$ (see 24.2). Thus the implication $(\alpha \Rightarrow \beta)$ is a predicate tautology (see 1.1, (3)).

Theorem 25.1 can also be proved in another way, without using topological notions. For instance, it can be proved in the same way as the analogous theorem X, 17.1.

§ 26. Herbrand disjunctions. In the case of a propositional calculus we have a simple method to verify whether a given formula is a propositional tautology or not. This is the truth-table method described in V, § 5 (see also VII, § 2). By 7.2, this method can easily be extended to the case of open formulas in a predicate calculus $\mathfrak{S} = \{\mathfrak{L}, C\}$. No such simple method exists in the case of arbitrary formulas in $\mathfrak{S}$. In this section we shall prove a theorem stating that, to every formula α in $\mathfrak{S}$, we can effectively assign (by some simple operations on formulas) a set A of open formulas such that α is a tautology if and only if one of the formulas in A is a tautology. Since for open formulas we can apply the truth-table method, this theorem can be considered as a method of verification whether a given (in general, non-open) formula α in $\mathfrak{S}$ is a predicate tautology or not. However, this method has no practical meaning because if α is not open, the corresponding set A of open formulas is infinite (we may always suppose that A is then enumerable). In spite of this fact the theorem mentioned is interesting from the theoretical point of view. We shal prove it in a more general form, viz. for arbitrary open theories.

In this section we shall consider a fixed formalized language $\mathfrak{L}$ of the first order and a fixed formula α in $\mathfrak{L}$. Without less of generality we may assume that α is in the prenex form. For if α is not, we can replace α by an equivalent formula α' in the prenex form. By 16.1, we can find α' effectively, by means of some simple operations on signs in α.

Thus we assume hereafter that α is in the prenex form

$$(1) \qquad \bigcup_{\xi_1} \bigcap_{\eta_1} \dots \bigcup_{\xi_k} \bigcap_{\eta_k} \beta(x_0, \xi_1, \eta_1, \dots, \xi_k, \eta_k)$$

where $\beta(x_0, \ldots)$ is an open formula. We use here the same abbreviations as in § 16, p. 330, viz.

$$x_0 = (x_{01}, \ldots, x_{0n}) . \tag{2}$$
$$\xi_i = (\xi_{i1}, \ldots, \xi_{im_i}) , \quad \eta_i = (\eta_{i1}, \ldots, \eta_{in_i}) \quad \text{for} \quad i = 1, \ldots, k ,$$

where $x_{01}, \ldots, x_{0n}$ are all free individual variables occurring in α, and ξ_{ij}, η_{ij} are bound variables. Hereafter we shall also use the notation

$$x_i = (x_{i1}, \ldots, x_{in_i}) \tag{3}$$

where x_{ij} are any free individual variables, and

$$\tau_i = (\tau_{i1}, \ldots, \tau_{im_i}) \tag{4}$$

where τ_{ij} are any terms ($i = 1, \ldots, k$).

Denote by Z_0 the set consisting of the formula α only. For $r = 1, \ldots, k$, let Z_r be the set of all formulas

$$\bigcup_{\xi_{r+1}} \bigcap_{\eta_{r+1}} \ldots \bigcup_{\xi_k} \bigcap_{\eta_k} \beta(x_0, \tau_1, x_1, \ldots, \tau_r, x_r, \xi_{r+1}, \eta_{r+1}, \ldots, \xi_k, \eta_k) . \tag{5}$$

In particular, Z_k is the set of all open formulas of the form

$$\beta(x_0, \tau_1, x_1, \ldots, \tau_k, x_k) . \tag{6}$$

Let Z be the union of the sets $Z_0, Z_1, \ldots, Z_k$.

By an *Herbrand disjunction for* α we shall understand any disjunction δ of distinct formulas

$$\alpha_1, \ldots, \alpha_l \tag{7}$$

belonging to Z.

Suppose that the disjunction δ of formulas (7) is an Herbrand disjunction for α. Suppose also that α_j is of the form (5) with $r > 0$. Let α_j' be the formula

$$\bigcup_{\xi_r} \bigcap_{\eta_r} \ldots \bigcup_{\xi_k} \bigcap_{\eta_k} \beta(x_0, \tau_1, x_1, \ldots, \tau_{r-1}, x_{r-1}, \xi_r, \eta_r, \ldots, \xi_k, \eta_k) . \tag{8}$$

If α_j' is not identical with one of the formulas $\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_l$, then the disjunction δ' of formulas

$$\alpha_1, \ldots, \alpha_{j-1}, \alpha_j', \alpha_{j+1}, \ldots, \alpha_l$$

is said to be a *direct derivative of* δ. If α_j' coincides with one of the formulas $\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_l$, then the disjunction δ' of formulas

$$\alpha_1, \ldots, \alpha_{j-1}, \alpha_{j+1}, \ldots, \alpha_l$$

is said to be a *direct derivative of* δ. Note that every direct derivative of an Herbrand disjunction for α is also an Herbrand disjunction for α.

A formula δ' is said to be a *derivative* of an Herbrand disjunction δ for a if there exists a sequence $\delta_1, \ldots, \delta_q$ of formulas, such that δ_1 is identical with δ, δ_q is identical with δ', and δ_{j+1} is a direct derivative of δ_j for $j = 1, \ldots, q-1$. Then δ' is also an Herbrand disjunction for a.

Let the disjunction δ of formulas (7) be an Herbrand disjunction for a. The Herbrand disjunction δ is said to be *reducible* provided there exists an integer j such that a_j is of the form (5) with $r > 0$, and

(a) all the individual variables $x_{r1}, \ldots, x_{rn_r}$ are distinct each from another,

(b) all the individual variables $x_{r1}, \ldots, x_{rn_r}$ are distinct from all the individual variables x_{ij} where $j = 1, \ldots, n_i$ and $i = 0, \ldots, r-1$ (see (3)), and are distinct from all the free individual variables occurring in all the formulas $a_1, \ldots, a_{j-1}, a_{j+1}, \ldots, a_l$ and terms $\tau_1, \ldots, \tau_r$.

26.1. *If γ is a closed formula, δ is a reducible Herbrand disjunction for a, and $(\gamma \Rightarrow \delta)$ is a tautology, then there exists a direct derivative δ' of δ such that $(\gamma \Rightarrow \delta')$ is a tautology.*

Suppose that δ is a disjunction of formulas (7) and that a_j satisfies conditions (a) and (b). Since $(\gamma \Rightarrow \delta)$ is a tautology, in the algebra $\mathfrak{A}(S)$ we have

$$\|\gamma\| \leqslant \|a_1\| \cup \ldots \cup \|a_l\| .$$

Hence it follows that

$$\|\gamma\| \leqslant \|a_1\| \cup \ldots \cup \|a_{j-1}\| \cup \|a'_j\| \cup \|a_{j+1}\| \cup \ldots \cup \|a_l\|$$

where a'_j is formula (8). Let δ' be the direct derivative of δ, described in the definition on p. 369. Since $\|\gamma\| \leqslant \|\delta'\|$, the formula $(\gamma \Rightarrow \delta')$ is a tautology.

An Herbrand disjunction δ for a is said to be *hereditarily reducible* if δ and all of its derivatives are reducible or coincide with a.

26.2. *If γ is a closed formula, δ is a hereditarily reducible Herbrand disjunction for a, and $(\gamma \Rightarrow \delta)$ is a tautology, then the implication $(\gamma \Rightarrow a)$ is also a tautology.*

By 26.1 we can define by induction a sequence $\{\delta_q\}$ of Herbrand disjunctions for a such that δ_1 is identical with δ, δ_{q+1} is a direct derivative of δ_q, and $(\gamma \Rightarrow \delta_q)$ is a tautology $(q = 1, 2, \ldots)$. We can always define δ_{q+1} provided δ_q is not identical with a. On the other hand, every sequence $\{\delta_q\}$ such that δ_{q+1} is a direct derivative of δ_q must be finite. Therefore our inductive definition has to stop at an integer, say q_0. Hence it follows that δ_{q_0} is identical with a and, consequently, $(\gamma \Rightarrow a)$ is a tautology.

An Herbrand disjunction δ for a is said to be *proper* provided it is hereditarily reducible and is an open formula. If δ is a disjunction o

formulas (7), the last condition means that all the formulas (7) are open, i.e. they are of the form (6).

26.3. *If δ is a proper Herbrand disjunction for a, and $\bar{\delta}$ is the closure of δ, then $(\bar{\delta} \Rightarrow a)$ is a tautology.*

Theorem 26.3 follows immediately from 26.2 where γ is the closed formula $\bar{\delta}$.

We recall that, for any set S, S^m denotes the Cartesian product $S \times \ldots \times S$ m times. Thus, by (3) and (4),

$$x_i \in V^{m_i} \quad (i = 0, 1, \ldots, k), \quad \tau_i \in T^{m_i} \quad (i = 1, \ldots, k)$$

where V and T denote, as usual, the set of all free variables and the set of all terms, respectively.

The symbol $\mathfrak{F}_i$ will denote the set of all mappings

$$f_i\colon\ T^{m_1} \times \ldots \times T^{m_i} \to V^{m_i},$$

and the symbol $\mathfrak{F}_i^*$ will denote the set of all mappings

$$f_i\colon\ T^{m_1} \times \ldots \times T^{m_i} \to T^{m_i}$$

$(i = 1, \ldots, k)$. By definition,

$$\mathfrak{F}_i \subset \mathfrak{F}_i^* .$$

Let

$$f_i \in \mathfrak{F}_i^* \quad (i = 1, \ldots, k) \tag{9}$$

be fixed functions. By an *Herbrand $(f_1, \ldots, f_k)$-disjunction for a* we shall understand any disjunction of a finite number of distinct formulas of the form

$$\beta\big(x_0, \tau_1, f_1(\tau_1), \tau_2, f_2(\tau_1, \tau_2), \ldots, \tau_k, f_k(\tau_1, \ldots, \tau_k)\big) . \tag{10}$$

By definition, every Herbrand $(f_1, \ldots, f_k)$-disjunction for a is an Herbrand disjunction for a in the sense defined on p. 369 since all formulas (10) are of the form (6).

26.4. *If a is a theorem in an open theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}\}$, then for any given functions* (9) *there exists an Herbrand $(f_1, \ldots, f_k)$-disjunction which is a theorem in $\mathfrak{T}$.*

To prove 26.4, let us consider the algebra $\mathfrak{A}(\mathfrak{T}^0)$ of the auxiliary theory $\mathfrak{T}^0 = \{\mathfrak{L}^0, \mathfrak{C}^0, \mathfrak{A}\}$ associated with the theory $\mathfrak{T}$ in § 15, p. 327. Suppose that a is a theorem in $\mathfrak{T}$, but no Herbrand $(f_1, \ldots, f_k)$-disjunction is a theorem in $\mathfrak{T}$, and consequently in $\mathfrak{T}^0$ (see § 15, (1)). This means that the unit element of $\mathfrak{A}(\mathfrak{T}^0)$ does not belong to the ideal Δ_0 generated by all the elements $|\gamma| \in \mathfrak{A}(\mathfrak{T}^0)$ where γ is any formula (10). Thus the ideal Δ_0 is proper. By I, 8.4, the ideal Δ_0 is contained in a maximal ideal Δ.

Let h be the natural homomorphism from $\mathfrak{A}(\mathfrak{T}^0)$ onto the two-element Boolean algebra $A_0 = \mathfrak{A}(\mathfrak{T}^0)/\Delta$, and let R^0 be the canonical semantic realization (in the set T of all terms) determined by the homomorphism h (see § 15, p. 327). We recall that by 15.2, (8)

$$\gamma_{R^0}(v) = h(|v'\gamma|) \tag{11}$$

where $v'\gamma$ is the result of a substitution v (i.e. valuation v in T) in the formula γ.

Since R^0 is a model for T by 15.3, and α is a theorem in T, the formula α is valid in R^0 (see 2.2). By the inductive definition of α_{R^0}, we have

$$\alpha_R(\mathrm{i}) = \bigcup_{\tau_1 \in T^{m_1}} \bigcap_{\sigma_1 \in T^{n_1}} \dots \bigcup_{\tau_k \in T^{m_k}} \bigcap_{\sigma_k \in T^{n_k}} \beta_{R_0}(x_0, \tau_1, \sigma_1, \dots, \tau_k, \sigma_k)$$

where i is the identity valuation, $\sigma_i = (\sigma_{i1}, \dots, \sigma_{in_i}) \in T^{n_i}$ and $\beta_{R^0}(x_0, \dots)$ is the obvious abbreviation for the value of β_{R^0} (written in the form VI, § 6, (2)) at the mentioned point of $T \times \dots \times T$. Since α is valid in R^0, we have $\alpha_{R^0}(\mathrm{i}) = V$. Thus there exists a term τ_1 such that

$$\bigcap_{\sigma_1 \in T^{n_1}} \bigcup_{\tau_2 \in T^{m_2}} \bigcap_{\sigma_2 \in T^{n_2}} \dots \bigcup_{\tau_k \in T^{m_k}} \bigcap_{\sigma_k \in T^{n_k}} \beta_R(x_0, \tau_1, \sigma_1, \dots, \tau_k, \sigma_k) = V .$$

Hence

$$\bigcup_{\tau_2 \in T^{m_2}} \bigcap_{\sigma_2 \in T^{n_2}} \dots \bigcup_{\tau_k \in T^{m_k}} \bigcap_{\sigma_k \in T^{n_k}} \beta_{R^0}(x_0, \tau_1, f_1(\tau_1), \tau_2, \sigma_2, \dots, \tau_k, \sigma_k) = V .$$

Repeating this procedure k times, we assert that there exist terms $\tau_1, \dots, \tau_k$ such that

$$\beta_{R^0}\big(x_0, \tau_1, f_1(\tau_1), \tau_2, f_2(\tau_1, \tau_2), \dots, \tau_k, f_k(\tau_1, \dots, \tau_k)\big) = V . \tag{12}$$

Let γ be the formula $\beta\big(x_0, \tau_1, f_1(\tau_1), \dots, \tau_k, f_k(\tau_1, \dots, \tau_k)\big)$. We have $h(|\gamma|) = V$ since $h(|\gamma|)$ is equal to the element on the left-hand side of (12) because of (11). On the other hand, $|\gamma| \in A_0 \subset \Delta$ and consequently $h(|\gamma|) = \Lambda$. Contradiction! (1)

In the case where $\overline{\overline{T}} = \overline{\overline{V}}$ we can give another shorter topological proof (2) of 26.4 which shows the close connection with the compactness of the space $\mathfrak{F}(\mathfrak{T})$ defined in § 22.

Let $|||\ |||$ have the same meaning as in §§ 22 and 23 (see p. 359). Since α is a theorem in $\mathfrak{T}$, we have $|||\alpha||| = \mathfrak{F}(\mathfrak{T})$. Thus

$$\begin{aligned}\mathfrak{F}(\mathfrak{T}) &= \bigcup_{\tau_1 \in T^{m_1}} \bigcap_{\sigma_1 \in T^{n_1}} \dots \bigcup_{\tau_k \in T^{m_k}} \bigcap_{\sigma_k \in T^{n_k}} |||\beta(x_0, \tau_1, \sigma_1, \dots, \tau_k, \sigma_k)||| \\ &= \bigcap_{f_1 \in \mathfrak{F}_1^*} \dots \bigcap_{f_k \in \mathfrak{F}_k^*} \bigcup_{\tau_1 \in T^{m_1}} \dots \\ &\qquad \dots \bigcup_{\tau_k \in T^{m_k}} |||\beta\big(x_0, \tau_1, f_1(\tau_1), \dots, \tau_k, f_k(\tau_1, \dots, \tau_k)\big)|||\end{aligned}$$

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(1) This proof is due to Łoś, Rasiowa, Mostowski [1], [2].

(2) This proof is due to Sikorski [6], [12].

by the distributive laws for sets. Hence it follows that, for the functions $f_1, \ldots, f_k$ mentioned in 26.4,

$$(13)\quad \mathfrak{F}(\mathfrak{C}) = \bigcup_{\tau_1 \in T^{m_1}} \ldots \bigcup_{\tau_k \in T^{m_k}} |||\beta(x_0, \tau_1, f_1(\tau_1), \ldots, \tau_k, f_k(\tau_1, \ldots, \tau_k))|||\,.$$

Since the space $\mathfrak{F}(\mathfrak{C})$ is compact by 23.2, and the sets on the right-hand side of (13) are open, $\mathfrak{F}(\mathfrak{C})$ is equal to the union of a finite number of them. In other words, there exists a Herbrand $(f_1, \ldots, f_k)$-disjunction δ for α such that $\mathfrak{F}(\mathfrak{C}) = |||\delta||| = H_{\mathfrak{C}}(\|\delta\|_{\mathfrak{C}})$ (see p. 359). By 23.3, $H_{\mathfrak{C}}$ is an isomorphism. Thus $\|\delta\|_{\mathfrak{C}}$ is the unit element of the algebra $\mathfrak{A}(\mathfrak{C})$, i.e. δ is a theorem in $\mathfrak{C}$.

26.5. *If the sets V and T have the same power, then there exist functions*

$$f_i \in \mathfrak{F}_i \qquad (i = 1, \ldots, k)$$

such that every $(f_1, \ldots, f_k)$-disjunction for α is proper.

Let $\mathfrak{m}$ be the common cardinal of V and T, and let V_{ip} $(p = 1, \ldots, n_i$, $i = 1, \ldots, k)$ be disjoint subsets of the set $V - (x_{01}, \ldots, x_{0n_0})$, each of which has the power $\mathfrak{m}$. Let $<$ be a well-ordering relation in V such that the ordinal of $\{V, <\}$ is the least ordinal of the power $\mathfrak{m}$. By transfinite induction we define a one-to-one mapping

$$f_{ip}\colon\ T^{m_1} \times \ldots \times T^{m_i} \to V_{ip}$$

such that, for all $\tau_1, \ldots, \tau_i$, the free individual variable $f_i(\tau_1, \ldots, \tau_i)$ is greater (in the ordering $<$) than all free individual variables occurring in $\tau_1, \ldots, \tau_i$.

The mappings

$$f_i(\tau_1, \ldots, \tau_i) = \big(f_{i1}(\tau_1, \ldots, \tau_i), \ldots, f_{in_i}(\tau_1, \ldots, \tau_i)\big)$$

(where $i = 1, \ldots, k$) have the required property. In fact, $f_i \in \mathfrak{F}_i$. Every derivative of an Herbrand $(f_1, \ldots, f_k)$-disjunction for α is a disjunction of some formulas (7), each of which is of the form

$$(12)\quad \bigcup_{\xi_{r+1}} \bigcap_{\eta_{r+1}} \ldots$$
$$\ldots \bigcup_{\xi_k} \bigcap_{\eta_k} \beta(x_0, \tau_1, f_1(\tau_1), \ldots, \tau_r, f_r(\tau_1, \ldots, \tau_r), \xi_{r+1}, \eta_{r+1}, \ldots, \xi_k, \eta_k)\,.$$

Thus, to complete the proof it suffices to show that every disjunction δ of formulas (7) of the form (12) is reducible (except for the case in which δ is the formula α). Consider the set of all free individual variables which occur in all $\tau_1, \ldots, \tau_k$ in the formula δ. Take the greatest element in this set, say x. Among all formulas (7) which contain x in some τ_i, take one of them, viz. the α_j which is represented in the form (12) with the greatest possible r. Let

$$f_r(\tau_1, \ldots, \tau_r) = (x_{r1}, \ldots, x_{rn_r})\,,$$

i.e.

$$x_{rp} = f_{rp}(\tau_1, \ldots, \tau_r) \quad \text{for} \quad p = 1, \ldots, n_r .$$

It follows directly from the definition of f_{tp} that the formula a_j and the variables $x_{r1}, \ldots, x_{rn_r}$ satisfy conditions (a) and (b) on p. 370. Thus δ is reducible.

26.6. *If a is a theorem in an open theory $\mathfrak{T}$, then a proper Herbrand disjunction for a is also a theorem in $\mathfrak{T}$.*

Consider first the case in which the set of all terms and the set of all free individual variables are enumerable. Let (9) be functions such that every Herbrand $(f_1, \ldots, f_k)$-disjunction for a is proper. These functions (9) exist on account of 26.5. Apply theorem 26.4 to these functions $f_1, \ldots, f_k$: By 26.4, there exists an Herbrand $(f_1, \ldots, f_k)$-disjunction δ which is a theorem in $\mathfrak{T}$. The proper Herbrand disjunction δ has all the required properties.

Suppose now that the cardinals of V and T are arbitrary. If a is a theorem in $\mathfrak{T}$, there exists a finite set $\mathcal{A}'$ of open axioms of $\mathfrak{T}$ such that a belongs to $\mathfrak{C}(\mathcal{A}')$. Let $\mathfrak{L}'$ be a formalized language such that $\mathfrak{L}$ is an extension of $\mathfrak{L}'$, $\mathfrak{L}'$ contains all the signs occurring in a and in all the formulas in $\mathcal{A}'$, and the set of all signs in $\mathfrak{L}'$ is enumerable. By definition, a is a theorem in the open theory $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}, \mathcal{A}'\}$. Applying the part of 26.6 just proved to the theory $\mathfrak{T}'$ we infer that a proper Herbrand disjunction δ for a is a theorem in $\mathfrak{T}'$, and consequently also in $\mathfrak{T}$.

We combine the main results of this section in the form of the following theorem.

26.7. *Let $\mathfrak{T}$ be an open theory. A formula a is a theorem in $\mathfrak{T}$ if and only if a proper Herbrand disjunction for a is a theorem in $\mathfrak{T}$* (1).

In fact, if a proper Herbrand disjunction δ for a is a theorem in $\mathfrak{T}$, then its closure $\bar{\delta}$ is also a theorem in $\mathfrak{T}$. By 26.3, $(\bar{\delta} \Rightarrow a)$ is a tautology and consequently a theorem in $\mathfrak{T}$. By *modus ponens*, a is a theorem in $\mathfrak{T}$.

The remaining part of 26.7 follows immediately from 26.6.

26.8. *In order that a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ be equivalent to an open theory it is necessary and sufficient that, for every formula a (in the prenex form), a is a theorem in $\mathfrak{T}$ if and only if a proper Herbrand disjunction for a is a theorem in $\mathfrak{T}$.*

It is easy to see that theorem 26.7 is also true for every theory equivalent to an open theory. This proves the necessity. To prove the sufficiency, let us associate, with every theorem a in $\mathfrak{T}$, its proper Herbrand disjunction δ_a which is a theorem in $\mathfrak{T}$. Let $\mathcal{A}'$ be the set of all the open

(1) Herbrand [2].

formulas δ_α. It follows easily from 26.3 that $\mathfrak{T}$ is equivalent to the open theory $\mathfrak{T}' = \{\mathfrak{L}, \mathfrak{C}, \mathfrak{A}'\}$.

Applying theorem 26.7 to the open theory $\{\mathfrak{L}, \mathfrak{C}, 0\}$, we obtain the theorem

26.9. *A formula α is a predicate tautology if and only if a proper Herbrand disjunctions for α is a tautology.*

Given any formula α, let $\mathfrak{L}'$ be a sublanguage of $\mathfrak{L}$ such that $\mathfrak{L}'$ contains all the signs occurring in α, and the set of all signs in $\mathfrak{L}'$ is enumerable. Let A be the set of all proper Herbrand disjunctions for α from the language $\mathfrak{L}'$. By definition, the set A is at most enumerable. Since α is a predicate tautology if and only if it is a theorem in $\{\mathfrak{L}', \mathfrak{C}, 0\}$, we infer that α is a predicate tautology if and only if one of the open formulas in A is a predicate tautology. Note that theorem 26.5 can be proved effectively if the formalized language in question contains only an enumerable set of signs. Thus the set A mentioned above can be constructed effectively for every fixed α. This proves the remarks formulated at the beginning of this section.

PART THREE

NON-CLASSICAL LOGICS

CHAPTER IX

INTUITIONISTIC PROPOSITIONAL CALCULI

§ 1. Introduction. In V § 11 we introduced a general notion of logic: a mapping C which, to every formalized language $\mathfrak{L}$ of the zero or first order, assigns a consequence operation $C_{\mathfrak{L}}$ in $\mathfrak{L}$. We also introduced a general scheme (L) for defining logics C: we chose a set I of tautologies in a formalized language of zero order. For every formalized language $\mathfrak{L}$, we assume the set $\mathcal{A}_{\mathfrak{L}}$ of all substitutions of formulas in I as the set of logical axioms in $\mathfrak{L}$, and we define $C_{\mathfrak{L}}(S)$ as the least set containing $\mathcal{A}_{\mathfrak{L}}$ and S and closed under *modus ponens* if $\mathfrak{L}$ is of zero order, or closed with respect to the rules V, § 8, (r_1)-(r_6), if $\mathfrak{L}$ is of the first order.

In Chapters VII and VIII we examined the case in which I consisted of tautologies (t_1)-(t_{12}) from V, § 6 (p. 168). It was shown that the consequence operation C so defined, called the *classical* or *two-valued logic*, satisfies condition (C) from V, § 10, i.e. that it is a good formalization of logical means applied in intuitive mathematics. On the other hand, it is evident that condition (C) in V, § 10, uniquely determines the logic C, so that the classical logic is the only logic satisfying this condition. Thus the classical logic is the only logic which is used in mathematics.

However, from the purely theoretical point of view, we can also examine other logics obtained by choosing other sets I of tautologies in the general scheme (L). If they differ from classical logic, then they are called *non-classical logics* or *non-classical consequence operations*.

The examination of non-classical logics can be useful and interesting for methodological reasons. For instance, if we take as I a proper subset of the set composed of (t_1)-(t_{12}), then we obtain a non-classical logic, say C′. An examination of C′ can explain the role played by the logical axioms not included in I, etc.

The examination of non-classical logics can also be caused by assuming another interpretation of the meaning of the propositional con-

nectives $\cup$, $\cap$, $\Rightarrow$, $-$ and the quantifiers $\bigcup$, $\bigcap$ and by assuming another point of view on the problem of truth and falsity of sentences, etc.

Although no mathematical theories based on non-classical logics have an important meaning for mathematics, the metatheory of some non-classical formalized theories is interesting from a mathematical point of view because of its connection with topology and lattice theory.

In Chapters IX and X we shall examine a non-classical logic called *intuitionistic logic*. In Chapter XI we shall discuss some other non-classical logics.

The intuitionistic logic has developed as a result of certain philosophical views on the foundations of mathematics, known as *intuitionism* [1]. Intuitionists' view-point on the meaning of the basic logical and set-theoretical concepts used in mathematics is different from that of most mathematicians in their research.

The basic difference lies in the interpretation of the word "exists". For example, let $\alpha(x)$ be a propositional function in the arithmetic of positive integers. For the mathematician the sentence

$$\bigcup_\xi \alpha(\xi) \tag{1}$$

is true if it is a theorem of arithmetic, i.e. if it can be deduced from the axioms of arithmetic by means of classical logic. If the mathematician proves sentence (1), this does not mean that he is able to indicate a method of the construction of a positive integer n such that $\alpha(n)$ holds. On the contrary, for the intuitionist sentence (1) is true only if he is able to construct a number n such that $\alpha(n)$ is true.

The mathematician often obtains the proof of the existential sentence (1) by proving first the sentence

$$-\bigcap_\xi -\alpha(\xi)\,. \tag{2}$$

Next he makes use of the tautology

$$(-\bigcap_\xi -\alpha(\xi) \Rightarrow \bigcup_\xi \alpha(\xi))\,. \tag{3}$$

By applying *modus ponens* to (2) and (3) he obtains (1). For the intuitionist such a method of reasoning is inacceptable, for it does not give any method of constructing a number n such that $\alpha(n)$ holds. For this reason intuitionists do not accept tautology (3) as true.

This last remark is connected also with the fact that intuitionists interpret differently the meaning of propositional connectives. The implication $(\beta \Rightarrow \gamma)$ is considered true if there exists a method by which

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[1] Intuitionism was originated by L. E. J. Brouwer [1]. For an exposition of principles of intuitionism, see e.g. Heyting [3], Kleene [4].

a proof of γ can be deduced from a proof of β. In the case of implication (3) there is no general method which, from a proof of sentence (2), permits us to obtain an intuitionistic proof of sentence (1), i.e. to construct a number n such that $\alpha(n)$ holds.

Also negation and disjunction are understood differently by intuitionists. The sentence $-\alpha$ is considered true if the acceptance of the sentence α leads to an absurdity. With this conception of negation and implication, the tautology

$$(\alpha \Rightarrow --\alpha) \tag{4}$$

is accepted as true by intuitionists but the tautology

$$(--\alpha \Rightarrow \alpha) \tag{5}$$

is not intuitionistically true.

The intuitionist regards a disjunction $(\alpha \cup \beta)$ as true if one of the sentences α, β is true and there exists a method by which it is possible to find out which of them is true. With this conception of the truth of disjunction, the tautology

$$(\alpha \cup -\alpha) \tag{6}$$

is not accepted by the intuitionist as true since there is no general method of finding out, for any given sentence α, whether α or $-\alpha$ is true. Intuitionists thus reject the tautology *tertium non datur*.

Intuitionists' view of the concept of infinite set also differs from that of most mathematicians. Intuitionists reject the idea of infinite set as a closed whole. They look upon an infinite set as something which is constantly in a state of formation. Thus, for example, the set of all positive integers is not looked upon as a closed whole. It is infinite in the sense that to any given finite set of positive integers it is always possible to add one more positive integer. The notion of the set of all subsets of the set of all positive integers is not regarded meaningful. Thus intuitionists reject the general idea of set and the whole modern set theory.

An exact exposition of the basic ideas of intuitionism is outside the range of this book. Moreover, it is difficult for mathematicians to understand exactly the ideas of intuitionists since the degree of precision in the formulation of intuitionistic ideas is far from the degree of precision to which mathematicians are accustomed in their daily work.

The subject of our studies will not be intuitionism itself but intuitionistic logic (1), which is a sort of reflection of intuitionistic ideas for-

(1) The formalization of the intuitionistic logic is due to A. Heyting [1], [2].

mulated in formalized deductive systems. A precise definition of intuitionistic logic offers no difficulty. The definition will be given according to the general scheme (L) in V, § 11, as in the case of classical logic with the single difference that the set I of accepted tautologies will be different from that adopted for classical logic.

Of course the set I of accepted tautologies must be chosen in such a way that, for every language $\mathfrak{L}$ of the zero or first order, the corresponding consequence operation $C_{i\mathfrak{L}}$ in $\mathfrak{L}$ has the following property:

(I) The set $C_{i\mathfrak{L}}(0)$ coincides with the set of all intuitionistically true tautologies in $\mathfrak{L}$.

It follows from the above consideration that the set $C_{i\mathfrak{L}}(0)$ is a proper subset of the set $C_{\mathfrak{L}}(0)$ of all tautologies, C denoting the classical logic.

We assume the following definition of *intuitionistic logic* C_i (also called the *intuitionistic consequence operation*): it is the consequence operation defined by means of the scheme (L) from V, § 11, where the set I is composed of all the tautologies (t_1)-(t_{11}) from V, § 6 (p. 168).

Thus the only difference between the definition of the intuitionistic and the classical logic is that the tautology (t_{12}) is omitted because it is not intuitionistically true (see (6)).

By definition, for every language $\mathfrak{L}$ of the zero or first order, the set $\mathscr{A}_{\mathfrak{L}}$ of logical axioms for the intuitionistic logic [1] consists of all formulas of the forms

(T_1) $((\alpha \Rightarrow \beta) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow \gamma)))$,

(T_2) $(\alpha \Rightarrow (\alpha \cup \beta))$,

(T_3) $(\beta \Rightarrow (\alpha \cup \beta))$,

(T_4) $((\alpha \Rightarrow \gamma) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow ((\alpha \cup \beta) \Rightarrow \gamma)))$,

(T_5) $((\alpha \cap \beta) \Rightarrow \alpha)$,

(T_6) $((\alpha \cap \beta) \Rightarrow \beta)$,

(T_7) $((\gamma \Rightarrow \alpha) \Rightarrow ((\gamma \Rightarrow \beta) \Rightarrow (\gamma \Rightarrow (\alpha \cap \beta))))$,

(T_8) $((\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \cap \beta) \Rightarrow \gamma))$,

(T_9) $(((\alpha \cap \beta) \Rightarrow \gamma) \Rightarrow (\alpha \Rightarrow (\beta \Rightarrow \gamma)))$,

(T_{10}) $((\alpha \cap -\alpha) \Rightarrow \beta)$,

(T_{11}) $((\alpha \Rightarrow (\alpha \cap -\alpha)) \Rightarrow -\alpha)$,

where α, β, γ are any formulas in $\mathfrak{L}$. If $\mathfrak{L}$ is a formalized language of zero order, then for every set S of formulas in $\mathfrak{L}$ the set $C_{i\mathfrak{L}}(S)$ of all intuitionistic consequences of S is the least set containing $\mathscr{A}_{\mathfrak{L}}$ and S

[1] This set of axioms differs from Heting's original set of axioms but it is equivalent.

and closed under *modus ponens*. If $\mathfrak{L}$ is a formalized language of the first order, then for every set S of formulas in $\mathfrak{L}$ the set $C_{i\mathfrak{L}}(S)$ of all intuitionistic consequences of S is the least set containing $\mathfrak{A}_1$ and S and closed under all the rules of inference V, § 8, (r_1)-(r_6).

According to the convention in V, § 11 (p. 188) we shall usually avoid the subscript $_{\mathfrak{L}}$ at C_i. We shall always write

$$\{\mathfrak{L}, C_i\} \quad \text{instead of} \quad \{\mathfrak{L}, C_{i\mathfrak{L}}\},$$
$$\{\mathfrak{L}, C_i, \mathfrak{A}\} \quad \text{instead of} \quad \{\mathfrak{L}, C_{i\mathfrak{L}}, \mathfrak{A}\}.$$

We shall also write

$$C_i(S) \quad \text{instead of} \quad C_{\mathfrak{L}_i}(S)$$

if only one language $\mathfrak{L}$ is under consideration.

To emphasize that deductive systems $\{\mathfrak{L}, C_i\}$ and theories $\{\mathfrak{L}, C_i, \mathfrak{A}\}$ are based on the intuitionistic logic C_i, we shall often call them *intuitionistic deductive systems* and *intuitionistic theories* respectively.

The intuitionistic logic C_i just defined has the property (I) but we shall not prove it here. We are not able to prove it because we have not defined precisely what we understand by the words "intuitionistically true tautology" [1]. We shall also not examine the problem of the intuitionistic interpretation of propositional connectives and quantifiers because the precision of the interpretation given above is not sufficient. We shall not investigate the philosophical aspects of the intuitionistic logic at all.

Only mathematical aspects of the intuitionistic logic will be the subject of Chapters IX and X. The philosophical origin of intuitionistic logic will not play any part in our investigations. We shall see later that, from the mathematical point of view, the metatheory of the intuitionistic logic C_i coincides with the theory of pseudo-Boolean algebras in the same sense as the metatheory of classical logic coincides with the theory of Boolean algebras. It follows from the representation theorems proved in Chapter IV that the theory of pseudo-Boolean algebras is the theory of lattices of open subsets of topological spaces. Consequently the investigation of intuitionistic logic consists of an examination of lattices of open subsets of topological spaces, and it is very surprising that some philosophical ideas have led to the formulation of a logic whose mathematical contents coincide with the theory of the lattices of open subsets of topological spaces! [2].

[1] For an investigation of this problem, see Beth [3], [4], Kreisel [1], [2], [3], and Dyson and Kreisel [1].

[2] The first results concerning the relationship between the intuitionistic logic and algebras of open subsets of topological spaces were stated by Stone [4] and Tarski [8].

§ 2. Preliminaries. In this Chapter we shall deal with a fixed formalized language $\mathfrak{L}_0 = \{A_0, F_0\}$ of zero order. The alphabet A_0 of $\mathfrak{L}_0$ and the set F_0 of all formulas in $\mathfrak{L}_0$ were described in V, § 6. The letter V_0 will always denote the set of all propositional variables $a, b, c, \ldots$ in $\mathfrak{L}_0$. By a *formula* we shall mean, in this Chapter, a formula from $\mathfrak{L}_0$, unless the contrary is explicitly stated.

In this Chapter we shall examine, in a detailed way, the deductive system

$$\mathfrak{S}_{0\iota} = \{\mathfrak{L}_0, \mathfrak{C}_\iota\}$$

called the *intuitionistic propositional calculus based on the language* $\mathfrak{L}_0$. According to § 1, the symbol $\mathfrak{C}_\iota$ always denotes the intuitionistic consequence operation. Sometimes we shall use the classical consequence operation. It will always be denoted by $\mathfrak{C}$, according to V, § 11 and VII and VIII. Sometimes we shall also deal with the classical propositional calculus based on $\mathfrak{L}_0$. It will be denoted by $\mathfrak{S}_0$, according to VII, § 1.

The following definitions are intuitionistic analogues of definitions from VII, § 1, for the classical propositional calculus.

A finite sequence $a_1, \ldots, a_n$ of formulas is said to be a *formal proof of a formula* a *in* $\mathfrak{S}_{0\iota}$ *from a set* S *of formulas* (or: an *intuitionistic formal proof of* a *from* S) provided a_n is the formula a, and for every $j \leqslant n$ either a_j is one of the logical axioms (T_1)-(T_{11}) from § 1, or it is a formula in S, or it is an immediate consequence of some formulas a_{j_1}, a_{j_2} with $j_1, j_2 < j$ by *modus ponens*. A formula a is said to be *derivable in* $\mathfrak{S}_{0\iota}$ *from* S (or: *intuitionistically derivable from* S) if there exists a formal proof of a from S in $\mathfrak{S}_{0\iota}$.

By this definition and the definition of $\mathfrak{C}_\iota$ from § 1, the set $\mathfrak{C}_\iota(S)$ of all *intuitionistic consequences* of a set S of formulas is the set of all formulas derivable from S in $\mathfrak{C}_{0\iota}$.

A formula a is said to be *derivable in* $\mathfrak{S}_{0\iota}$ or *intuitionistically derivable* provided it is in $\mathfrak{C}_\iota(0)$. By this definition we have

2.1. *Every formula derivable in the intuitionistic propositional calculus* $\mathfrak{S}_{0\iota} = \{\mathfrak{L}_0, \mathfrak{C}_\iota\}$ *is also derivable in the classical propositional calculus* $\mathfrak{S}_0 = \{\mathfrak{L}_0, \mathfrak{C}\}$ *based on the same language* $\mathfrak{L}_0$ *of zero order.*

In other words,

Every formula derivable intuitionistically is classically derivable.

Sometimes in this Chapter we shall consider formalized *intuitionistic theories*

$$\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathfrak{A}\}$$

where $\mathfrak{A}$ is a set of formulas (the set of the axioms of the theory $\mathfrak{T}$). Formulas in $\mathfrak{C}_\iota(\mathfrak{A})$ are called *theorems* of the theory $\mathfrak{T}$. According to

the convention adopted on p. 187, the formalized theory $\{\mathfrak{L}_0, \mathfrak{C}_i, 0\}$ with the empty set of axioms will be identified with the intuitionistic propositional calculus $\mathfrak{S}_{0i} = \{\mathfrak{L}_0, \mathfrak{C}_i\}$.

By *theory* we shall alwyas understand in this Chapter a theory of zero order based on the language $\mathfrak{L}_0$, unless the contrary is explicitly stated.

Let $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_i, \mathfrak{A}\}$ be an intuitionistic theory and let $\{F_0, \cup, \cap, \Rightarrow, -\}$ be the algebra of formulas of the language L_0. By VI, § 10, the relation $\approx$ defined as follows:

(1) $\alpha \approx \beta$ if and only if both formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are theorems in $\mathfrak{T}$;

is a congruence relation in the algebra $\{F_0, \cup, \cap, \Rightarrow, -\}$. We recall that the quotient algebra $F_0/\approx$ is called the algebra of the theory $\mathfrak{T}$ and is denoted by $\mathfrak{A}(\mathfrak{T})$. In particular $\mathfrak{A}(\mathfrak{S}_{0i})$ is the algebra $F_0/\approx$ where $\approx$ is the congruence relation in $\{F_0, \cup, \cap, \Rightarrow, -\}$ defined as follows:

(2) $\alpha \approx \beta$ if and only if both the formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are derivable in $\mathfrak{S}_{0i}$.

From VI, 10.2-10.5, 10.7, 10.10 follows

2.2. *The algebra $\mathfrak{A}(\mathfrak{T})$ is a pseudo-Boolean algebra. Moreover, for any formulas α, β*

$$\|\alpha\| \cup \|\beta\| = \|(\alpha \cup \beta)\|, \quad \|\alpha\| \cap \|\beta\| = \|(\alpha \cap \beta)\|. \tag{3}$$
$$\|\alpha\| \Rightarrow \|\beta\| = \|(\alpha \Rightarrow \beta)\|, \quad -\|\alpha\| = \|-\alpha\|.$$

The relation

$$\|\alpha\| \leqslant \|\beta\| \tag{4}$$

holds if and only if $(\alpha \Rightarrow \beta)$ is a theorem in $\mathfrak{T}$. A formula α is a theorem in $\mathfrak{T}$ if and only if the element $\|\alpha\|$ is the unit V of $\mathfrak{A}(\mathfrak{T})$. A formula α is irrefutable in $\mathfrak{T}$ if and only if $\|\alpha\| \neq \Lambda$. The theory $\mathfrak{T}$ is consistent if and only if the pseudo-Boolean algebra $\mathfrak{A}(\mathfrak{T})$ is non-degenerate.

Let $\{\boldsymbol{A}, \cup, \cap, \Rightarrow, -\}$ be an abstract algebra with three binary operations $\cup, \cap, \Rightarrow$ and one unary operation $-$. As was explained in VI, § 2, every formula α in F_0 uniquely determines a mapping

$$\alpha_{\boldsymbol{A}}: \boldsymbol{A}^{V_0} \to \boldsymbol{A}. \tag{5}$$

Every element $v = \{v_a\}_{a \in V_0}$ of the Cartesian product $\boldsymbol{A}^{V_0}$, i.e. every mapping

$$v: V_0 \to \boldsymbol{A} \tag{6}$$

is called a *valuation* in $\boldsymbol{A}$.

In this Chapter we shall always assume that

$\{\boldsymbol{A}, \cup, \cap, \Rightarrow, -\}$ *is a non-degenerate pseudo-Boolean algebra.*

In particular, we shall sometimes take as $\boldsymbol{A}$ the algebra $\mathfrak{A}(\mathfrak{C})$ of a formalized theory $\mathfrak{C} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$. Then the valuation

$$v^0 = \{\|a\|\}_{a \in V_0} \tag{7}$$

in $\mathfrak{A}(\mathfrak{C})$ is called the *canonical valuation* for $\mathfrak{C}$. The following theorem is a particular case of VI, 2.6.

2.3. *If v^0 is the canonical valuation for $\mathfrak{C}$, then for every formula a*

$$a_{\mathfrak{A}(\mathfrak{C})}(v^0) = \|a\| . \tag{8}$$

A valuation $v \in \boldsymbol{A}^{V_0}$ is said to be a *model* for a set $\mathcal{A}$ of formulas if

$$a_{\boldsymbol{A}}(v) = \mathit{V} \quad \text{for every formula} \quad a \text{ in } \mathcal{A} ,$$

In particular (the case in which $\mathcal{A}$ consists of one formula only), the valuation v is said to be a *model* for a formula a if $a_{\boldsymbol{A}}(v) = \mathit{V}$. A valuation v is said to be a *model* for a theory $\mathfrak{C} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$ provided v is the model for the set $\mathcal{A}$ of axioms of $\mathfrak{C}$. Every model in the two-element Boolean algebra is said to be *semantic*.

A formula a is said to be *valid* in a pseudo-Boolean algebra $\boldsymbol{A}$ provided every valuation $v \in \boldsymbol{A}^{V_0}$ is a model for a, i.e. if the mapping $a_{\boldsymbol{A}}$ is identically equal to the unit V of $\boldsymbol{A}$. We then write

$$a_{\boldsymbol{A}} \equiv \mathit{V} . \tag{9}$$

A formula is said to be *intuitionistically valid* [1] or to be an *intuitionistic propositional tautology* if it is valid in every pseudo-Boolean algebra.

Since the two-element Boolean algebra is also a pseudo-Boolean algebra, there follows immediately from the last definition

2.4. *Every intuitionistic tautology is a tautology in the sense defined in* V, § 6.

The following theorem is an intuitionistic analogue of VII, 1.3.

2.5. *Every intuitionistically derivable formula is an intuitionistic tautology.*

First we shall verify that all the logical axioms $(T)_1$-(T_{11}) are intuitionistically valid. Let us denote, for brevity,

$$\boldsymbol{a} = a_{\boldsymbol{A}}(v) , \quad \boldsymbol{b} = \beta_{\boldsymbol{A}}(v) , \quad \boldsymbol{c} = \gamma_{\boldsymbol{A}}(v) .$$

[1] For another interpretation of validity see Beth [3]. There is also an approach to this question connected with Kleene's notion of realizability. See Rose [1].

Then

$$((a \Rightarrow \beta) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow (a \Rightarrow \gamma)))_A(v) = (\boldsymbol{a} \Rightarrow \boldsymbol{b}) \Rightarrow ((\boldsymbol{b} \Rightarrow \boldsymbol{c}) \Rightarrow (\boldsymbol{a} \Rightarrow \boldsymbol{c})) = V$$

by I, 12.2, (21) and (4). Thus (T_1) is intuitionistically valid. (T_2) is also intuitionistically valid since

$$(a \Rightarrow (a \cup \beta))_A(v) = \boldsymbol{a} \Rightarrow (\boldsymbol{a} \cup \boldsymbol{b}) = V$$

by I, 12.2, (4). Similarly we verify that (T_3), (T_5), (T_6) are intuitionistically valid. (T_4) is also intuitionistically valid since

$$\begin{aligned}
&((a \Rightarrow \gamma) \Rightarrow ((\beta \Rightarrow \gamma) \Rightarrow ((a \cup \beta) \Rightarrow \gamma)))_A(v) \\
&= (\boldsymbol{a} \Rightarrow \boldsymbol{c}) \Rightarrow \big((\boldsymbol{b} \Rightarrow \boldsymbol{c}) \Rightarrow \big((\boldsymbol{a} \cup \boldsymbol{b}) \Rightarrow \boldsymbol{c}\big)\big) \\
&= \big((\boldsymbol{a} \Rightarrow \boldsymbol{c}) \cap (\boldsymbol{b} \Rightarrow \boldsymbol{c})\big) \Rightarrow \big((\boldsymbol{a} \cup \boldsymbol{b}) \Rightarrow \boldsymbol{c}\big) \\
&= \big((\boldsymbol{a} \cup \boldsymbol{b}) \Rightarrow \boldsymbol{c}\big) \Rightarrow \big((\boldsymbol{a} \cup \boldsymbol{b}) \Rightarrow \boldsymbol{c}\big) = V
\end{aligned}$$

because of I, 12.2, (18), (17), (6). (T_7) is intuitionistically valid since

$$\begin{aligned}
&((\gamma \Rightarrow a) \Rightarrow ((\gamma \Rightarrow \beta) \Rightarrow (\gamma \Rightarrow (a \cap \beta))))_A(v) \\
&= (\boldsymbol{c} \Rightarrow \boldsymbol{a}) \Rightarrow \big((\boldsymbol{c} \Rightarrow \boldsymbol{b}) \Rightarrow \big(\boldsymbol{c} \Rightarrow (\boldsymbol{a} \cap \boldsymbol{b})\big)\big) \\
&= \big((\boldsymbol{c} \Rightarrow \boldsymbol{a}) \cap (\boldsymbol{c} \Rightarrow \boldsymbol{b})\big) \Rightarrow \big(\boldsymbol{c} \Rightarrow (\boldsymbol{a} \cap \boldsymbol{b})\big) \\
&= \big(\boldsymbol{c} \Rightarrow (\boldsymbol{a} \cap \boldsymbol{b})\big) \Rightarrow \big(\boldsymbol{c} \Rightarrow (\boldsymbol{a} \cap \boldsymbol{b})\big) = V
\end{aligned}$$

because of I, 12.2, (18), (16), (6). The intuitionistic validity of (T_8) and (T_9) follows similarly from I, 12.2, (18), (6). (T_{10}) is intuitionistically valid since

$$((a \cap -a) \Rightarrow \beta)_A(v) = (\boldsymbol{a} \cap -\boldsymbol{a}) \Rightarrow \boldsymbol{b} = \Lambda \Rightarrow \boldsymbol{b} = V$$

by I, § 12, (27), (4). (T_{11}) is intuitionistically valid since

$$\begin{aligned}
&((a \Rightarrow (a \cap -a)) \Rightarrow -a)_A(v) \\
&\qquad = \big(\boldsymbol{a} \Rightarrow (\boldsymbol{a} \cap -\boldsymbol{a})\big) \Rightarrow -\boldsymbol{a} = (\boldsymbol{a} \Rightarrow \Lambda) \Rightarrow -\boldsymbol{a} = -\boldsymbol{a} \Rightarrow -\boldsymbol{a} = V
\end{aligned}$$

by I, § 12, (27), (2), (6).

The set of all intuitionistically valid formulas is closed under ***modus ponens*** for

$$\text{if } a_A(v) = V \text{ and } (a \Rightarrow \beta)_A(v) = V, \text{ then } \beta_A(v) = V.$$

In fact, we then have

$$\beta_A(v) = V \Rightarrow \beta_A(v) = a_A(v) \Rightarrow \beta_A(v) = (a \Rightarrow \beta)_A(v) = V$$

by I, 12.2, (8).

This proves that the set of all intuitionistically valid formulas contains all intuitionistically derivable formulas.

A theorem converse to 2.5 will be proved in § 3.

2.6. *The intuitionistic propositional calculus* $\mathcal{S}_{0\iota}$ *is consistent.*

Theorem 2.6 can be deduced from 2.5 by the same method as the theorem VII, 1.4, was deduced from the analogous statement VII, 1.3. Theorem 2.6 can also be deduced directly from VII, 1.4. In fact, consider the classical propositional calculus $\mathcal{S}_0 = \{\mathfrak{L}_0, \mathfrak{C}\}$ based on the same language $\mathfrak{L}_0$. By 2.1, $\mathfrak{C}_\iota(0) \subset \mathfrak{C}(0)$. By VII, 1.4, $\mathfrak{C}(0) \neq F_0$. Thus $\mathfrak{C}_\iota(0) \neq F_0$, i.e. $\mathcal{S}_{0\iota}$ is consistent.

The following theorem is an analogue to theorem V, 6.2, for classical propositional calculi.

2.7. *If* a *is an intuitionistic tautology and* s *is a substitution from* $\mathfrak{L}_0$ *into* $\mathfrak{L}_0$, *then the formula* sa (*i.e. the result of the substitution* s *in* a) *is also an intuitionistic tautology.*

In fact, for every valuation v in any pseudo-Boolean algebra $\boldsymbol{A}$, we have, by VI, 2.5

$$sa_A(v) = a_A(s_A v) = V$$

where $s_A v$ is the valuation induced by s (see VI, § 2, p. 214).

§ 3. The completeness theorem. Theorem 2.5 states that every intuitionistically derivable formula is an intuitionistic tautology. The converse statement is also true. In fact, we can prove the following theorem called the *completeness theorem for intuitionistic propositional calculi* because it is the intuitionistic analogue of the completeness theorem VII, 2.1, for classical propositional calculi.

3.1. *A formula in* $\mathfrak{L}_0$ *is an intuitionistic tautology if and only if it is derivable in the intuitionistic propositional calculus* $\mathcal{S}_{0\iota} = \{\mathfrak{L}_0, \mathfrak{C}_\iota\}$.

Theorem 3.1 is part of the following theorem.

3.2 [1]. *For every formula* a *in* $\mathfrak{L}_0$, *the following conditions are equivalent*:

(i) a *is derivable in* $\mathcal{S}_{0\iota}$,

(ii) a *is intuitionistically valid*,

(iii) a *is valid in every pseudo-Boolean algebra of open subsets of a topological space* X,

(iv) $a_{\mathfrak{A}(\mathcal{S}_{0\iota})}(v^0) = V$, *where* v^0 *is the canonical valuation for* $\mathcal{S}_{0\iota}$,

(v) a *is valid in every finite pseudo-Boolean algebra*,

(vi) a *is valid in every pseudo-Boolean algebra with at most* 2^{2^r} *elements, where* r *is the number of all subformulas of the formula* a,

---

[1] Tarski [8], McKinsey and Tarski [2], [3]. See also Rieger [1] for some partial results, and Stone [4].

(vii) *α is valid in the pseudo-Boolean algebra of all open subsets of a dense-in-itself metric space $X \neq 0$.*

Condition (i) implies (ii) by 2.5. Clearly (ii) implies (iii). Since every pseudo-Boolean algebra (in particular, the algebra $\mathfrak{A}(\mathfrak{S}_{0\iota})$) is isomorphic with a subalgebra of the algebra of all open subsets of a topological space (see IV, 9.2), condition (iii) implies (iv). Now we shall prove that (iv) implies (i). Suppose that a formula α is not derivable in $\mathfrak{S}_{0\iota}$. By 2.2 the element $\|\alpha\|$ of $\mathfrak{A}(\mathfrak{S}_{0\iota})$ is not the unit element. Hence, by 2.3, $\alpha_{\mathfrak{A}(\mathfrak{S}_{0\iota})}(v^0) \neq V$. Thus α is not any intuitionistic tautology. We have proved that the conditions (i)-(iv) are equivalent.

Clearly (ii) implies (v) and (v) implies (vi). We shall prove that (vi) implies (iv). Suppose that (iv) does not hold for a formula α, i.e. by 2.3 $\alpha_{\mathfrak{A}(\mathfrak{S}_{0\iota})}(v^0) = \|\alpha\| \neq V$. Suppose that α contains r subformulas. Let A_0 be the set consisting of all $\|\beta\|$ such that β is a subformula of α. It follows from IV, 9.3, that there exists a finite pseudo-Boolean algebra A containing at most 2^{2^r} elements such that $V \in A$, $A_0 \subset A$ and the operations in A are extensions of operations in $A_0' = A_0 \cup (V)$, i.e. for any $\|\beta\|, \|\gamma\|, \|\delta\|$ in A_0':

(a) if $\|\beta\|$ is the join of $\|\gamma\|, \|\delta\|$ in $\mathfrak{A}(\mathfrak{S}_{0\iota})$ then $\|\beta\|$ is also the join of $\|\gamma\|$ and $\|\delta\|$ in A,

(b) if $\|\beta\|$ is the meet of $\|\gamma\|, \|\delta\|$ in $\mathfrak{A}(\mathfrak{S}_{0\iota})$ then $\|\beta\|$ is also the meet of $\|\gamma\|$ and $\|\delta\|$ in A,

(c) if $\|\beta\|$ is the pseudo-complement of $\|\gamma\|$ relative to $\|\delta\|$, in $\mathfrak{A}(\mathfrak{S}_{0\iota})$ then $\|\beta\|$ is also the pseudo-complement of $\|\gamma\|$ relative to $\|\delta\|$ in A,

(d) if $\|\beta\|$ is the pseudo-complement of $\|\gamma\|$ in $\mathfrak{A}(\mathfrak{S}_{0\iota})$ then $\|\beta\|$ is also the pseudo-complement of $\|\gamma\|$ in A.

Let v be the valuation of $\mathfrak{S}_{0\iota}$ in A defined as follows:

$$v_a = \begin{cases} v_a^0 = \|a\| & \text{for any propositional variable } a \text{ in } \alpha, \\ V & \text{for any propositional variable } a \text{ which does not occur in } \alpha. \end{cases}$$

Note that for any propositional variable a which occurs in α v_a is an element of A_0. We shall prove that for any subformula β of the formula α

$$\beta_A(v) = \|\beta\| \in A_0' . \tag{1}$$

The proof is by induction on the length of β. If β is a propositional variable occurring in α, then (1) follows from the definition of v. Suppose that β is the disjunction $(\gamma \cup \delta)$ (the conjunction $(\gamma \cap \delta)$, the implication $(\gamma \Rightarrow \delta)$, the negation $-\gamma$) and

$$\gamma_A(v) = \|\gamma\| \in A_0' , \qquad \delta_A(v) = \|\delta\| \in A_0' . \tag{2}$$

Then by (2), (a) ((b), (c), (d)) and 2.2, (3), we have

$$(\gamma \cup \delta)_A(v) = \gamma_A(v) \cup \delta_A(v) = \|\gamma\| \cup \|\delta\| = \|(\gamma \cup \delta)\| \in A'_0,$$
$$(\gamma \cap \delta)_A(v) = \gamma_A(v) \cup \delta_A(v) = \|\gamma\| \cap \|\delta\| = \|(\gamma \cap \delta)\| \in A'_0,$$
$$(\gamma \Rightarrow \delta)_A(v) = \gamma_A(v) \Rightarrow \delta_A(v) = \|\gamma\| \Rightarrow \|\delta\| = \|(\gamma \Rightarrow \delta)\| \in A'_0,$$
$$(-\gamma)_A(v) = -(\gamma_A(v)) = -\|\gamma\| = \|-\gamma\| \in A'_0).$$

Thus in any case equation (1) holds for β. In particular we have

$$a_A(v) = \|a\| \neq V$$

i.e. (vi) does not hold for a. Thus condition (vi) implies (iv). In consequence, conditions (i)-(vi) are equivalent.

Note that (iii) implies (vii). To complete the proof of 3.2 it is sufficient to show that (vii) implies (vi). Suppose that

$$a_A(v) \neq V \tag{3}$$

for a valuation v in a finite pseudo-Boolean algebra A. Let $X \neq 0$ be a dense-in-itself metric space. By IV, 4.1, there exists a dense open set $G \subset X$ and an isomorphism h_0 from A into the algebra $\mathfrak{G}(G)$ of all open subsets of G. It follows from VI 2.4 and from (3) that

$$a_{\mathfrak{G}(G)}(h_0 v) = h_0 a_A(v) \neq G \tag{4}$$

where the valuation $h_0 v$: $V_0 \to \mathfrak{G}(G)$ is the superposition of v: $V_0 \to A$ and h_0: $A \to \mathfrak{G}(G)$.

On the other hand, by IV, 8.2, the mapping

$$h(A) = A \cap G \qquad (A \in \mathfrak{G}(X))$$

is a homomorphism from the pseudo-Boolean algebra $\mathfrak{G}(X)$ of all open subsets of X onto $\mathfrak{G}(G)$. For every propositional variable a in V_0, let A_a be an open subset of X such that

$$h(A_a) = h_0(v_a).$$

Then $v' = \{A_a\}_{a \in V_0}$ is a valuation in $\mathfrak{G}(X)$ and the valuation hv' (i.e. the superposition of v': $V_0 \to \mathfrak{G}(X)$ and h: $\mathfrak{G}(X) \to \mathfrak{G}(G)$) is identical with the valuation $h_0 v$. Thus by (4) and VI, 2.4, we obtain

$$h(a_{\mathfrak{G}(X)}(v')) = a_{\mathfrak{G}(G)}(hv') = a_{\mathfrak{G}(G)}(h_0 v) \neq G.$$

Consequently $a_{\mathfrak{G}(X)}(v') \neq X$, i.e. a is not valid in $\mathfrak{G}(X)$. This completes the proof of 3.2.

It follows directly from the equivalence of conditions (i) and (vi) that in order to verify whether a formula a is derivable in $\mathfrak{S}_{0\alpha}$ it is sufficient

to calculate $a_A(v)$ in all finite pseudo-Boolean algebras $\boldsymbol{A}$ with the cardinal $\leqslant 2^{2^r}$, where r is the number of all subformulas of a. Thus we have a general decision procedure for determining whether or not a formula a in S_{0i} is derivable (1). In fact, this method consists in verifying whether $a_A(v)$ is identically equal to 1 in every pseudo-Boolean algebra $\boldsymbol{A}$ with the cardinal $\leqslant 2^{2^r}$. When a is a formula constructed of n propositional variables $a_1, \ldots, a_n$ and $\boldsymbol{A}$ is composed of m elements we have to calculate the value of $a_A(v) = a_A(v_{a_1}, \ldots, v_{a_n})$ at m^n points.

By 3.1 the phrases "intuitionistic tautology" and "formula derivable in S_{0i}" have the same meaning. In practice we shall almost exclusively use the first phrase.

§ 4. Examples of intuitionistic propositional tautologies. To illustrate what tautologies are intuitionistic tautologies we shall prove the following theorem. The enumeration of tautologies is the same as in VII, 3.1.

4.1. *For any formulas a, β, γ, the following formulas are intuitionistic tautologies*:

(T_0) $(a \Rightarrow a)$,

(T_{13}) $(a \Rightarrow (\beta \Rightarrow a))$,

(T_{14}) $(a \Rightarrow (\beta \Rightarrow (a \cap \beta)))$,

(T_{15}) $((a \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow (\beta \Rightarrow (a \Rightarrow \gamma)))$,

(T_{16}) $((a \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((a \Rightarrow \beta) \Rightarrow (a \Rightarrow \gamma)))$,

(T_{17}) $(a \Rightarrow --a)$,

(T_{19}) $-(a \cap -a)$,

(T_{21}) $((-a \cup \beta) \Rightarrow (a \Rightarrow \beta))$,

(T_{22}) $(-(a \cup \beta) \Rightarrow (-a \cap -\beta))$,

(T_{23}) $((-a \cap -\beta) \Rightarrow -(a \cup \beta))$,

(T_{25}) $((-a \cup -\beta) \Rightarrow -(a \cap \beta))$,

(T_{26}) $((a \Rightarrow \beta) \Rightarrow (-\beta \Rightarrow -a))$,

(T_{27}) $((a \Rightarrow -\beta) \Rightarrow (\beta \Rightarrow -a))$,

(T_{65}) $(---a \Rightarrow -a)$,

(T_{66}) $(-a \Rightarrow ---a)$,

(T_{67}) $(--(a \Rightarrow \beta) \Rightarrow (a \Rightarrow --\beta))$,

(T_{68}) $((\gamma \Rightarrow a) \Rightarrow ((\gamma \Rightarrow (a \Rightarrow \beta)) \Rightarrow (\gamma \Rightarrow \beta)))$.

---

(1) Jaśkowski [1]. This follows also from results of Gentzen [1]. See also Wajsberg [1], McKinsey and Tarski [2], Rieger [1], Пилчак [1], [2].

We shall indicate only the number of statements in I § 12 from which the intuitionistic validity of the formulas follows (the exact proof is similar to the first part of the proof of 2.5): (T_0) (6); (T_{13}) (13); (T_{14}) (22); (T_{15}) (18); (T_{16}) (23); (T_{17}) (29); (T_{19}) (28); (T_{21}) (33); (T_{22}) (31); (T_{23}) (31); (T_{25}) (32); (T_{26}) (34); (T_{27}) (35); (T_{65}) (30); (T_{66}) (30); (T_{67}) (37); (T_{68}) (19).

We shall complete theorem 4.1 by the following remark.

4.2. *The following tautologies are not intuitionistic tautologies:*

(t_{12}) $(a \cup -a)$,

(t_{18}) $(--a \Rightarrow a)$,

(t_{20}) $((a \Rightarrow b) \Rightarrow (-a \cup b))$,

(t_{24}) $(-(a \cap b) \Rightarrow (-a \cup -b))$,

(t_{28}) $((-a \Rightarrow b) \Rightarrow (-b \Rightarrow a))$,

(t_{29}) $((-a \Rightarrow -b) \Rightarrow (b \Rightarrow a))$,

(t_{30}) $(((a \Rightarrow b) \Rightarrow a) \Rightarrow a)$,

where a, b are different propositional variables.

All these formulas are propositional tautologies by VII, 3.1.

In order to prove that a formula α from among those mentioned in 4.2 is not a intuitionistic tautology it suffices to find a pseudo-Boolean algebra $\boldsymbol{A}$ and a valuation v in $\boldsymbol{A}$ such that $\alpha_{\boldsymbol{A}}(v) \neq V$. As the algebra $\boldsymbol{A}$ we shall take here the algebra of all open sets of real numbers. The formula α contains either one propositional variable a or two propositional variables a and b. Consequently $\alpha_{\boldsymbol{A}}(v)$ depends only on the value v_a or on the values v_a and v_b of the mapping v: $V_0 \to \boldsymbol{A}$, respectively. Hence it suffices to define only v_a, or v_a and v_b. We shall quote below the values of v_a and v_b such that $\alpha_{\boldsymbol{A}}(v) \neq V$. The easy calculation is left to the reader. For brevity, A denotes the infinite open interval $(0, \infty)$, B denotes the infinite open interval $(-\infty, 0)$, and C denotes the set of all real numbers:

(t_{12}): $v_a = A$;
(t_{18}): $v_a = A \cup B$;
(t_{20}): $v_a = A$, $v_b = A$;
(t_{24}): $v_a = A$, $v_b = B$;
(t_{28}): $v_a = A \cup B$, $v_b = 0$;
(t_{29}): $v_a = A \cup B$, $v_b = C$;
(t_{30}): $v_a = A \cup B$, $v_b = 0$.

By the completeness theorem 3.1, all the formulas quoted in 4.1 are derivable in the intuitionistic propositional calculus $\mathfrak{S}_{0\alpha}$, i.e. they have formal proofs from axioms (T_1)-(T_{11}) by means of *modus ponens*. We shall not quote here any formal proofs for them. We mention only that there

is a general method of finding intuitionistic proofs for intuitionistic tautologies (1). The method is similar to that discussed in VII, § 6, for classical calculi.

By 3.1, all the formulas quoted in 4.2 are not derivable in $\mathfrak{S}_{0\alpha}$.

§ 5. Connection between tautologies and intuitionistic tautologies. We recall that an element a of a pseudo-Boolean algebra $\boldsymbol{A}$ is said to be dense if $-a = \wedge$ or, equivalently, if $--a = \vee$ (see IV, § 5).

5.1 (2). *The following conditions are equivalent for every formula* α:

(i) α *is a propositional tautology*;

(ii) *for every pseudo-Boolean algebra* $\boldsymbol{A}$, *the element* $\alpha_{\boldsymbol{A}}(v)$ *is dense for every valuation* v *in* $\boldsymbol{A}$;

(iii) *in a non-degenerate pseudo-Boolean algebra* $\boldsymbol{A}$, *the element* $\alpha_{\boldsymbol{A}}(v)$ *is dense for every valuation* v *in* $\boldsymbol{A}$.

(i) implies (ii). In fact, suppose that α is a propositional tautology and v is a valuation in a pseudo-Boolean algebra $\boldsymbol{A}$. Let ∇ be the filter of all dense elements in $\boldsymbol{A}$ and let h be the natural homomorphism from $\boldsymbol{A}$ onto the quotient algebra $\boldsymbol{A}/\nabla$. Since $\boldsymbol{A}/\nabla$ is a Boolean algebra (see IV, 5.8), we have

$$\alpha_{\boldsymbol{A}/\nabla}(hv) = \vee_{\boldsymbol{A}/\nabla}.$$

Hence, by VI, 2.4,

$$h\alpha_{\boldsymbol{A}}(v) = \alpha_{\boldsymbol{A}/\nabla}(hv) = \vee_{\boldsymbol{A}/\nabla},$$

i.e. $\alpha_{\boldsymbol{A}}(v) \in \nabla$, q. e. d.

(ii) implies (iii). This is obvious.

(iii) implies (i). Suppose that (iii) holds. Let ∇ be the filter of all dense elements in $\boldsymbol{A}$ and let h be the natural homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{A}/\nabla$. Since h is a mapping onto, every valuation in $\boldsymbol{A}/\nabla$ is of the form hv where v is a valuation in $\boldsymbol{A}$. By VI, 2.4

$$\alpha_{\boldsymbol{A}}(hv) = h\alpha_{\boldsymbol{A}}(v) = \vee_{\boldsymbol{A}/\nabla},$$

i.e. α is valid in the Boolean algebra $\boldsymbol{A}/\nabla$. Since $\boldsymbol{A}$ is non-degenerate, the zero element of $\boldsymbol{A}$ is not dense. Thus ∇ is a proper filter and consequently $\boldsymbol{A}/\nabla$ is non-degenerate. Since α is valid in a non-degenerate Boolean algebra, α is valid in the two-element Boolean algebra, i.e. α is a propositional tautology.

5.2. *A formula* α *is a propositional tautology if and only if the element* $\|\alpha\| \in \mathfrak{A}(\mathfrak{S}_{0\alpha})$ *is dense.*

If α is a propositional tautology, α is dense by 5.1, (i), (ii) since (see 2.3)

$$\|\alpha\| = \alpha_{\mathfrak{A}(\mathfrak{S}_{0\alpha})}(v^0),$$

(1) Gentzen [1], Beth [3].

(2) Tarski [8].

v^0 being the canonical valuation. Conversely, if $\|a\|$ is dense, then $\|--a\| = --\|a\| = 1$. Thus $--a$ is an intuitionistic tautology by 3.2 and 2.2 and consequently a tautology by 2.4. Hence it follows by VI, 10.9, that a is a propositional tautology.

5.3. *A formula a is a propositional tautology if and only if the formula $--a$ is an intuitionistic propositional tautology* (1).

By 5.2, a is a propositional tautology if and only if the element $\|a\| \in \mathfrak{A}(\mathfrak{S}_\alpha)$ is dense, i.e. if $\|--a\| = --\|a\| = 1$. By 2.2 and 3.1 the last condition holds if and only if $--a$ is an intuitionistic tautology.

5.4. *A formula $-a$ is a propositional tautology if and only if $-a$ is an intuitionistic propositional tautology.*

The element $\|-a\| = -\|a\| \in \mathfrak{A}(\mathfrak{S}_\alpha)$ is regular, by IV, 6.3. By IV, 6.6, the regular element $\|-a\|$ is dense if and only if $\|-a\| = 1$, i.e. if $-a$ is an intuitionistic tautology (see 2.2 and 3.1). This directly implies 5.4 on account of 5.2.

5.5. *A formula $(a \Rightarrow -\beta)$ is a propositional tautology if and only if it is an intuitionistic propositional tautology.*

The proof of 5.5 is similar to that of 5.4.

The element $\|(a \Rightarrow -\beta)\| = \|a\| \Rightarrow -\|\beta\| \in \mathfrak{A}(\mathfrak{S}_\alpha)$ is regular by IV, 6.3. Thus, by IV, 6.6, the element $\|(a \Rightarrow -\beta)\|$ is dense if and only if $\|(a \Rightarrow -\beta)\| = 1$, i.e. if $(a \Rightarrow -\beta)$ is an intuitionistic tautology. This implies 5.5 by 5.2.

5.6. *If a formula a contains no connectives except $\cap$ and $-$, then a is a propositional tautology if and only if it is an intuitionistic propositional tautology* (2).

The proof is by induction on the length of a. If a is a propositional variable, then theorem 5.6 holds because then a is neither a tautology nor an intuitionistic tautology. Suppose that a is not a propositional variable. If a does not contain $\cup$ and $\Rightarrow$, a must be of the form $(\beta \cap \gamma)$ or $-\delta$. In the first case, β and γ do not contain $\cup$ and $\Rightarrow$ and, by the induction hypothesis, β and γ are tautologies if and only if they are intuitionistic tautologies. On the other hand, $(\beta \cap \gamma)$ is a tautology or an intuitionistic tautology if and only if both β and γ are tautologies or intuitionistic tautologies, respectively (see the general theorem VI 10.8). Thus $(\beta \cap \gamma)$ is a tautology if and only if $(\beta \cap \gamma)$ is an intuitionistic tautology. In the second case, i.e. when a is of the form $-\delta$, the formula $-\delta$ is a tautology if and only if $-\delta$ is an intuitionistic tautology, by 5.4.

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(1) Glivenko [1] and Tarski [8].

(2) Gödel [4].

For any formula a in $\mathfrak{L}_0$, let a' be the formula defined by induction as follows:

1) if a is a propositional variable, then a' is a;

2) if a is a disjunction $(\beta \cup \gamma)$, then a' is the formula $-(-\beta' \cap -\gamma')$;

3) if a is a conjunction $(\beta \cap \gamma)$, then a' is the formula $(\beta' \cap \gamma')$;

4) if a is an implication $(\beta \Rightarrow \gamma)$, then a' is the formula $-(\beta' \cap -\gamma')$;

5) if a is a negation $-\beta$, then a' is the formula $-\beta'$.

The formula a' associated with a has the following properties:

(a) a' does not contain the connectives $\cup$ and $\Rightarrow$;

(b) the formulas $(a \Rightarrow a')$ and $(a' \Rightarrow a)$ are propositional tautologies.

(a) follows immediately from the definition of a'. To prove (b) it suffices to show that for every valuation v in a Boolean algebra $\boldsymbol{A}$,

$$a_{\boldsymbol{A}}(v) = a'_{\boldsymbol{A}}(v) .$$

The easy proof of this equation, by induction on the length of a, is left to the reader.

5.7. *A formula a is a propositional tautology if and only if the formula a' is an intuitionistic tautology* [1].

If a is a propositional tautology, then a' is also a propositional tautology by (b) and *modus ponens*. Hence it follows, by (a) and 5.6, that a' is an intuitionistic propositional tautology.

The converse statement follows directly from (b) and 2.4.

We recall (see IV, § 6) that, for any pseudo-Boolean algebra $\boldsymbol{A}$, the symbol $\mathfrak{R}(\boldsymbol{A})$ denotes the Boolean algebra of all regular elements in $\boldsymbol{A}$ (i.e. of all elements $\boldsymbol{a} \in \boldsymbol{A}$ such that $\boldsymbol{a} = --\boldsymbol{a}$). The operations $\cup$, $\Rightarrow$, $-$ in the Boolean algebra $\mathfrak{R}(\boldsymbol{A})$ are the same as in $\boldsymbol{A}$. The join $\cup^*$ in $\mathfrak{R}(\boldsymbol{A})$ is defined by the identity,

$$\boldsymbol{a} \cup^* \boldsymbol{b} = --(\boldsymbol{a} \cup \boldsymbol{b}) = -(-\boldsymbol{a} \cap -\boldsymbol{b}) \qquad (\boldsymbol{a}, \boldsymbol{b} \in \mathfrak{R}(\boldsymbol{A}))$$

where $\cup$, as usual, denotes the join in $\boldsymbol{A}$.

A valuation $v = \{v_a\}_{a \in V_0}$ in $\boldsymbol{A}$ is said to be *regular* if $v_a \in \mathfrak{R}(\boldsymbol{A})$ for every propositional variable a, i.e. if v is a valuation in $\mathfrak{R}(\boldsymbol{A})$.

5.8. *If a formula a does not contain the connective $\cup$, then*

$$a_{\boldsymbol{A}}(v) = a_{\mathfrak{R}(\boldsymbol{A})}(v)$$

for every regular valuation v in a pseudo-Boolean algebra $\boldsymbol{A}$.

The proof, by induction on the length of a, is left to the reader.

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[1] Gödel [4].

For every formula a in $\mathfrak{L}_0$, let a^0 be the formula defined by induction as follows:

1° if a is a propositional variable a, then a^0 is a;

2° if a is a disjunction $(\alpha \cup \beta)$, then a^0 is the formula $-(-\beta^0 \cap -\gamma^0)$;

3° if a is a conjunction $(\beta \cap \gamma)$, then a^0 is the formula $(\beta^0 \cap \gamma^0)$;

4° if a is an implication $(\beta \Rightarrow \gamma)$, then a^0 is the formula $(\beta^0 \Rightarrow \gamma^0)$;

5° if a is a negation $-\beta$, then a^0 is the formula $-\beta^0$.

The formula a^0 associated with a has the following properties:

(c) a^0 does not contain the connective $\cup$;

(d) the formulas $(a \Rightarrow a^0)$ and $(a^0 \Rightarrow a)$ are propositional tautologies.

(c) follows immediately from the definition. The proof of (d) is similar to that of (b) on p. 392.

Let s be the substitution from $\mathfrak{L}_0$ into $\mathfrak{L}_0$ defined by the equality:

$$sa = --a \quad \text{for every propositional variable } a.$$

According to VI, § 2, p. 210, for every formula a, the symbol sa denotes the result of the substitution s in a. The substitution s has the following property which follows immediately from VI, 2.5 and II, 1.3, (5):

(e) the formulas $(a \Rightarrow sa)$ and $(sa \Rightarrow a)$ are propositional tautologies.

For every formula a, we shall denote by a^* the formula sa^0, i.e. the result of the substitution s in the formula a^0. Of course, a^* can also be defined directly by induction on the length of a, in a way similar, to 1°-5°. For this purpose it suffices to replace everywhere ^{0} by * in 2°-5°, and to replace 1° by the sentence:

1*. if a is a propositional variable, then a^* is $--a$.

It follows from (c), (d) and (e) that a^* has the following properties:

(f) a^* does not contains the connective $\cup$;

(g) the formulas $(a \Rightarrow a^*)$ and $(a^* \Rightarrow a)$ are propositional tautologies.

5.9. *A formula a is a propositional tautology if and only if the formula a^* is an intuitionistic propositional tautology* [1].

Suppose that a is a propositional tautology. By (d), a^0 is also a propositional tautology. Let $\boldsymbol{A}$ be a pseudo-Boolean algebra and let v be any valuation in $\boldsymbol{A}$. By VI, 2.5,

$$a^*_{\boldsymbol{A}}(v) = sa^0_{\boldsymbol{A}}(v) = a^0_{\boldsymbol{A}}(s_{\boldsymbol{A}} v)$$

where $s_{\boldsymbol{A}} v$ is defined as in VI, § 2, p. 214, i.e.

$$(s_{\boldsymbol{A}} v)_a = (sa)_{\boldsymbol{A}}(v) = --v_a \quad \text{for every propositional variable } a.$$

[1] Cf. Kleene [3], p. 495.

It follows from the last identity that the valuation $s_A v$ is regular (see IV, 6.3). Thus, by 5.8 and (c),

$$a^0_A(s_A v) = a^0_{\mathfrak{R}(A)}(s_A v) = V$$

since $\mathfrak{R}(A)$ is a Boolean algebra and a^0 is a propositional tautology. This proves that

$$a^*_A(v) = V$$

for every valuation v, i.e. that a^* is an intuitionistic tautology.

Conversely, if a^* is an intuitionistic tautology, then it is also a tautology by 2.4. Hence, by (g), a is a tautology.

§ 6. A theorem on intuitionistically derivable disjunctions. In a classical propositional calculus, it can happen that a disjunction $(a \cup \beta)$ is a tautology, but neither a nor β is a tautology (the simplest example is given by the tautology $(a \cup -a)$). This does not hold for an intuitionistic propositional calculus.

6.1. *A disjunction $(a \cup \beta)$ is an intuitionistic tautology if and only if either a or β is an intuitionistic tautology* [1].

If at least one of the formulas a, β is intuitionistically derivable, so is $(a \cup \beta)$ by *modus ponens* and (T_2) or (T_3) (this part of 6.1 also holds, of course, for classical propositional calculi).

Suppose $(a \cup \beta)$ is an intuitionistic tautology. By IV, 9.2, there exists an isomorphism h from $\mathfrak{A}(\mathfrak{S}_{0\alpha})$ into the algebra $\mathfrak{G}(X)$ of all open subsets of a topological space X. By VI, 2.4, we have, for any formula,

$$(1) \qquad h a_{\mathfrak{A}(\mathfrak{S}_{0\alpha})}(v^0) = a_{\mathfrak{G}(X)}(hv^0)$$

where v^0 is the canonical valuation in $\mathfrak{A}(\mathfrak{S}_{0\alpha})$ (see § 2, p. 383), and the valuation hv is the superposition of v^0: $V_0 \to \mathfrak{A}(\mathfrak{S}_{0\alpha})$ and h: $\mathfrak{A}(\mathfrak{S}_{0\alpha}) \to \mathfrak{G}(X)$.

By III, 5.1, X is an open subset of a strongly compact space X_0. By IV, 8.2, the mapping

$$h_0(A) = X \cap A \quad \text{for} \quad A \in \mathfrak{G}(X_0)$$

is a pseudo-Boolean homomorphism from $\mathfrak{G}(X_0)$ onto $\mathfrak{G}(X)$.

For every propositional variable a, let G_a be an open subset of X_0 such that

$$h_0(G_a) = hv^0_a = h(\|a\|) .$$

The valuation $v = \{G_a\}_{a \in V_0}$ in $\mathfrak{G}(X_0)$ satisfies the equation

$$(2) \qquad h_0 v = hv^0 .$$

[1] This theorem was announced without proof by Gödel [5]. It follows also from Gentzen [1]. The quoted proof is a slight modification of a proof given by McKinsey and Tarski [3], [2]. See also Rieger [1], Kleene [6] and references in Kleene [6].

Since $(\alpha \cup \beta)$ is an intuitionistic tautology, we have

$$a_{\mathfrak{G}(X_0)}(v) \cup \beta_{\mathfrak{G}(X_0)}(v) = (\alpha \cup \beta)_{\mathfrak{G}(X_0)}(v) = X_0 .$$

Since X_0 is strongly compact, at least one of the summands is equal to X_0. Assume, e.g., that

$$a_{\mathfrak{G}(X_0)}(v) = X_0 .$$

Hence, by (1) and (2), we obtain

$$h(a_{\mathfrak{A}(\mathfrak{S}_{0\iota})})(v^0) = a_{\mathfrak{G}(X)}(hv^0) = a_{\mathfrak{G}(X)}(h_0 v) = h_0 a_{\mathfrak{G}(X_0)}(v) = h_0(X_0) = X .$$

Since h is an isomorphism, we infer from the last identity that

$$a_{\mathfrak{A}(\mathfrak{S}_{0\iota})}(v^0) = V .$$

This proves, by 3.2, that α is an intuitionistic tautology.

§ 7. The algebra of the intuitionistic propositional calculus. The following theorem analogous to VII, 4.1, characterizes the algebra $\mathfrak{A}(\mathfrak{S}_{0\iota})$ in the class of all pseudo-Boolean algebras.

7.1. *$\mathfrak{A}(\mathfrak{S}_{0\iota})$ is a free algebra in the class of all pseudo-Boolean algebras, the set of all elements $\|a\|$ for a in V_0 being the set of free generators* (1).

The proof is the same as that of VII, 4.1.

Let f be an arbitrary mapping defined on the set of all elements $\|a\|$, where a is in V_0, with values in a pseudo-Boolean algebra $\boldsymbol{A}$. The mapping f induces a mapping v: $V_0 \to \boldsymbol{A}$, viz. v is defined as follows:

$$v_a = f\|a\| \quad \text{for any } a \text{ in } V_0 .$$

The mapping v is a valuation in the algebra $\boldsymbol{A}$. By VI, 2.1, v can be extended to a homomorphism v_A from F_0 into $\boldsymbol{A}$.

Let α, β be any formulas in F_0 such that $\|\alpha\| = \|\beta\|$. Then the formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are derivable in $\mathfrak{S}_{0\iota}$. By 2.5 they are intuitionistically valid. In particular they are valid in $\boldsymbol{A}$. Consequently for the valuation v

$$a_A(v) \Rightarrow \beta_A(v) = (\alpha \Rightarrow \beta)_A(v) = V ,$$

$$\beta_A(v) \Rightarrow a_A(v) = (\beta \Rightarrow \alpha)_A(v) = V .$$

Hence $a_A(v) = \beta_A(v)$ on account of I, 12.2, (5). Since, for every formula γ in F_0, $\gamma_A(v) = v_A(\gamma)$ (see VI, § 2), we get the equation $v_A(\alpha) = v_A(\beta)$. Thus the condition $\|\alpha\| = \|\beta\|$ implies that $v_A(\alpha) = v_A(\beta)$. This proves that the equation

$$h\|\alpha\| = v_A(\alpha) \quad \text{for any} \quad \|\alpha\| \text{ in } \mathfrak{A}(\mathfrak{S}_{0\iota})$$

(1) See e.g. Rieger [1].

uniquely defines a mapping h: $\mathfrak{A}(\mathcal{S}_0) \to \boldsymbol{A}$. The mapping h is an extension of the mapping f. In fact we have

$$h\|a\| = v_{\boldsymbol{A}}(a) = f\|a\| \quad \text{for every } a \text{ in } V_0 .$$

Since $v_{\boldsymbol{A}}$ is a homomorphism from $\boldsymbol{F}_0$ into $\boldsymbol{A}$, the mapping h is a homomorphism from $\mathfrak{A}(\mathcal{S}_{0\iota}) = \boldsymbol{F}_0/\approx$ into $\boldsymbol{A}$ (see § 2, (1)).

7.2. *For any different propositional variables* a, b

$$\|a\| \neq \|b\| .$$

This follows from the fact that $(a \Rightarrow b)$ is not an intuitionistic tautology (see the similar statement VII, 4.2).

The following theorem is another formulation of 6.1.

7.3. *The unit filter of* $\mathfrak{A}(\mathcal{S}_{0\iota})$ *is prime.*

In other words (see I, 8.7):

7.4. *The set of all elements different from the unit element is a maximal ideal in the pseudo-Boolean algebra* $\mathfrak{A}(\mathcal{S}_{0\iota})$ [1].

Now let us consider, besides the pseudo-Boolean algebra $\mathfrak{A}(\mathcal{S}_{0\iota})$, the Boolean algebra $\mathfrak{A}(\mathcal{S}_0)$ where $\mathcal{S}_0 = \{\mathfrak{L}_0, C\}$ is the classical propositional calculus based on the language $\mathfrak{L}_0$ of $\mathcal{S}_{0\iota}$. According to the more precise notation from VI, § 10, p. 245, for any formula a, $\|a\|_{\mathcal{S}_{0\iota}}$ will denote the corresponding element in $\mathfrak{A}(\mathcal{S}_{0\iota})$, and $\|a\|_{\mathcal{S}_0}$ will denote the corresponding element in $\mathfrak{A}(\mathcal{S}_0)$.

If $(a \Rightarrow \beta)$ and $(\beta \Rightarrow a)$ are derivable in $\mathcal{S}_{0\iota}$, then they are also derivable in $\mathcal{S}_0$ (see 2.1). Hence it follows that the equation

$$h(\|a\|_{\mathcal{S}_{0\iota}}) = \|a\|_{\mathcal{S}_0}$$

defines a mapping h from $\mathfrak{A}(\mathcal{S}_{0\iota})$ onto $\mathfrak{A}(\mathcal{S}_0)$. It is easy to verify that h is a pseudo-Boolean homomorphism (this follows directly from 2.2, (3), and the analogous identities in VII, 1.1, (3)). By the definition of h, $h(\|a\|_{\mathcal{S}_{0\iota}})$ is the unit element of $\mathfrak{A}(\mathcal{S}_0)$ if and only if a is a tautology, i.e. if and only if $\|a\|_{\mathcal{S}_{0\iota}}$ is a dense element of $\mathfrak{A}(\mathcal{S}_{0\iota})$ (see 5.2). Hence follows, on account of I 13.4:

7.5. *The Boolean algebra* $\mathfrak{A}(\mathcal{S}_0)$ *is isomorphic with* $\mathfrak{A}(\mathcal{S}_{0\iota})/V$ *where* V *is the filter of all dense elements in* $\mathfrak{A}(\mathcal{S}_{0\iota})$.

From 7.5 and IV, 6.7, follows

7.6. *The Boolean algebra* $\mathfrak{A}(\mathcal{S}_0)$ *is isomorphic with the Boolean algebra* $\mathfrak{R}(\mathfrak{A}(\mathcal{S}_{0\iota}))$ *of all regular elements in* $\mathfrak{A}(\mathcal{S}_{0\iota})$.

---

[1] See Rieger [1].

§ 8. Consistency and the existence of models. The section contains theorems concerning connections between the consistency of a theory of zero order based on intuitionistic logic and the existence of models (see § 2) in pseudo-Boolean algebras.

8.1. *If a is a theorem of an intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$, then every model for $\mathfrak{T}$ in any pseudo-Boolean algebra $\boldsymbol{A}$ is a model for a.*

Let $v \in \boldsymbol{A}^{V_0}$ (where V_0 denotes the set of all propositional variables in $\mathfrak{L}_0$) be a model for $\mathfrak{T}$ in a pseudo-Boolean algebra $\boldsymbol{A}$. By 2.5 every logical axiom is intuitionistically valid; in particular it is valid in $\boldsymbol{A}$. Hence v is a model for every logical axiom. Moreover, if v is a model for a formula a and for a formula $(a \Rightarrow \beta)$, then v is a model for β. In fact, if $a_A(v) = V$ and $V = (a \Rightarrow \beta)_A(v) = a_A(v) \Rightarrow \beta_A(v)$, then $V = a_A(v) \leqslant \beta_A(v)$ and consequently $\beta_A(v) = V$. Consequently the set of all formulas for which v is a model contains all logical axioms, all formulas in $\mathcal{A}$ and is closed under *modus ponens*. Thus it contains $\mathfrak{C}_\iota(\mathcal{A})$, i.e. the set of all theorems in $\mathfrak{T}$.

8.2. *If there exists a model $v \in \boldsymbol{A}^{V_0}$ in a pseudo-Boolean algebra $\boldsymbol{A}$ for an intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$, then $\mathfrak{T}$ is consistent.*

The proof is the same as that of the analogous theorem VII, 7.2. If both formulas a and $-a$ are theorems in $\mathfrak{T}$, then by 8.1 $a_A(v) = V$ and $(-a)_A(v) = -a_A(v) = V$. Hence $\Lambda = V$, which does not hold in any non-degenerate pseudo-Boolean algebra.

A model $v \in \boldsymbol{A}^{V_0}$ of a theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$ is said to be *adequate* for $\mathfrak{T}$ provided, for every formula a in $\mathfrak{L}_0$, a is a theorem in $\mathfrak{T}$ if and only if v is a model for a.

8.3. *Every consistent intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}_\mathfrak{c}, \mathfrak{C}_\iota, \mathcal{A}\}$ has an adequate model. Viz., the canonical valuation is an adequate model.*

In fact, if $\mathfrak{T}$ is consistent, then by 2.2 the algebra $\mathfrak{A}(\mathfrak{T})$ is a non-degenerate pseudo-Boolean algebra. Let v^0 be the canonical valuation, i.e. $v_a = \|a\| \in \mathfrak{A}(\mathfrak{T})$ for $a \in V_0$. By 2.3 we have

$$a_{\mathfrak{A}(\mathfrak{T})}(v^0) = \|a\| \quad \text{for any } a \text{ in } F_0 .$$

Hence it follows by 2.2 that

$$a_{\mathfrak{A}(\mathfrak{T})}(v^0) = V \quad \text{if and only if} \quad a \in \mathfrak{C}_\iota(\mathcal{A}) .$$

This proves that v^0 is an adequate model for $\mathfrak{T}$.

The next theorem is analogous to VII, 7.7, concerning classical theories of zero order.

8.4. *For any intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$ the following conditions are equivalent:*

(i) $\mathcal{T}$ *is consistent,*

(ii) *there exists a model for* $\mathcal{T}$,

(iii) *there exists an adequate model for* $\mathcal{T}$,

(iv) *there exists an adequate model for* $\mathcal{T}$ *in a pseudo-Boolean algebra of all open subsets of a topological space* X,

(v) *there exists a semantic model for* $\mathcal{T}$,

(vi) *the classical theory* $\mathcal{T}_0 = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ *is consistent.*

In fact, condition (i) implies (iii) by 8.3. Clearly (iii) implies (ii). Condition (ii) implies (i) by 8.2. Thus (i), (ii), (iii) are equivalent. It follows from IV, 9.2, that (iii) is equivalent to (iv).

Suppose that $\mathcal{T}$ is consistent. By 2.2 the pseudo-Boolean algebra $\mathfrak{A}(\mathcal{T})$ is non-degenerate. By I, 8.6, there exists a maximal filter ∇ in $\mathfrak{A}(\mathcal{T})$. Let h be the natural homomorphism from $\mathfrak{A}(\mathcal{T})$ onto the two-element Boolean algebra $\boldsymbol{A}_0 = \mathfrak{A}(\mathcal{T})/\nabla$ (see I, 13.10 and II, § 4). The valuation $v = hv^0$, where v^0 is the canonical valuation in $\mathfrak{A}(\mathcal{T})$, i.e. the valuation

$$v = \{h(\|a\|)\}_{a \in V_0}$$

is a semantic model. In fact, for every formula a

$$a_{\boldsymbol{A}_0}(v) = h(a_{\mathfrak{A}(\mathcal{T})})(v^0) = h(\|a\|)$$

by 2.3 and VI, 2.4. Hence by 2.2

$$a_{\boldsymbol{A}_0}(v) = h(\|a\|) = h(\nabla) = \nabla$$

for every a in $\mathcal{A}$, i.e. v is a semantic model for $\mathcal{T}$. This proves that (i) implies (v).

By definition, any semantic model for $\mathcal{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$ is also a semantic model for $\mathcal{T}_0 = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$. Hence, by VII, 7.2, $\mathcal{T}_0$ is consistent, i.e. (v) implies (vi). Since every theorem in $\mathcal{T}$ is also a theorem in $\mathcal{T}_0$, we infer that (vi) implies (i), which completes the proof of 8.4.

We now formulate conditions which are necessary and sufficient for any formula to be a theorem of a theory $\mathcal{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$.

8.5. *For any formula* a *in a consistent intuitionistic theory* $\mathcal{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$ *of zero order the following conditions are equivalent:*

(i) a *is a theorem of* $\mathcal{T}$,

(ii) *every model for* $\mathcal{T}$ *is a model for* a,

(iii) *every model for* $\mathcal{T}$ *in the pseudo-Boolean algebra of all open subsets of every topological space is a model for* a,

(iv) $a_{\mathfrak{A}(\mathcal{T})}(v^0) = \nabla$ *for the canonical valuation* v^0.

Condition (i) implies (ii) by 8.1. Clearly (ii) implies (iii). Condition (iii) implies (iv) by IV, 9.2. It follows from 8.3 that (iv) implies (i).

8.6. *Every linguistic extension of a theory* $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathfrak{A}\}$ *is inessential.*

Let a language $\mathfrak{L}_1$ be an extension of $\mathfrak{L}_0$, and $\mathfrak{T}_1 = \{\mathfrak{L}_1, \mathfrak{C}_\iota, \mathfrak{A}\}$. We have to prove that if a formula α in $\mathfrak{L}_0$ is not a theorem in $\mathfrak{T}$, it is not a theorem in $\mathfrak{T}_1$.

By 8.5 there exists a valuation v: $V_0 \to \boldsymbol{A}$ which is a model for $\mathfrak{T}$ but is not a model for α, i.e. $\alpha_A(v) \neq \mathbb{V}$. Let V_1 be the set of all propositional variables in $\mathfrak{L}_1$. Extend the mapping v: $V_0 \to \boldsymbol{A}$ to a mapping v_1: $V_1 \to \boldsymbol{A}$. We have

$$\beta_A(v_1) = \beta_A(v)$$

for every formula β in $\mathfrak{L}_0$ (see the second part of VI, 2.3). Hence it follows that v_1 is a model for $\mathfrak{T}_1$ but not for α. By 8.5, α is not a theorem in $\mathfrak{T}_1$.

§ 9. Deduction theorems. This name is given to the two following theorems.

9.1. *A formula* β *is a theorem in an intuitionistic theory* $\mathfrak{T}' = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathfrak{A} \cup [\alpha]\}$ *if and only if the formula* $(\alpha \Rightarrow \beta)$ *is a theorem in the intuitionistic theory* $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathfrak{A}\}$.

If $(\alpha \Rightarrow \beta)$ is a theorem in $\mathfrak{T}$, then it is also a theorem in $\mathfrak{T}'$. Since α is an axiom in $\mathfrak{T}'$, by *modus ponens* formula β is a theorem in $\mathfrak{T}'$.

To prove the remaining part of 9.1 we shall, as usual, denote by $\|\gamma\|$ the element of the pseudo-Boolean algebra $\boldsymbol{A} = \mathfrak{A}(\mathfrak{T})$, determined by a formula γ. Suppose that β is a theorem in $\mathfrak{T}'$. By 2.2 we have to show that $\|(\alpha \Rightarrow \beta)\| = \|\alpha\| \Rightarrow \|\beta\| = \mathbb{V}$, i.e. that $\|\alpha\| \leqslant \|\beta\|$. This is trivial when the element $g = \|\alpha\|$ is the zero element. Thus it suffices to consider the case for $g \neq \mathbb{\Lambda}$. Then the pseudo-Boolean algebra $\boldsymbol{A}_g$ (see IV, § 8) of all elements $\leqslant g$ is non-degenerate.

Let v^0 be the canonical valuation in $\boldsymbol{A} = \mathfrak{A}(\mathfrak{T})$. Since the mapping

$$h(\|\gamma\|) = \|\alpha\| \cap \|\gamma\| \tag{1}$$

is a homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{A}_g$ (see IV, 8.2), for every formula γ we have the identity

$$\gamma_{A_g}(hv^0) = h\gamma_A(v^0) \tag{2}$$

by VI, 2.4. If γ is a formula in $\mathfrak{A}$, then $\gamma_A(v^0) = \mathbb{V}$ and consequently $\gamma_{A_g}(hv^0) = h(\mathbb{V}) = g =$ the unit element of $\boldsymbol{A}_g$. If γ is the formula α, then similarly $\alpha_{A_g}(hv^0) = h(g) = g =$ the unit element of $\boldsymbol{A}_g$. Thus the valuation hv^0 in $\boldsymbol{A}_g$ is a model for the theory $\mathfrak{T}'$. Since β is a theorem in $\mathfrak{T}'$, hv^0 is also a model for β by 8.1, i.e. $\beta_{A_g}(hv^0) = g$. Hence, by (2), $h\beta_A(v^0) = g$ and consequently, by (1) and 2.3,

$$\|\alpha\| \leqslant \beta_A(v^0) = \|\beta\|, \quad \text{q. e. d.}$$

9.2. *A formula β is a theorem in an intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_i, \mathcal{A}\}$ (where $\mathcal{A}$ is a non-empty set of formulas) if and only if there exists a conjunction α of a finite number of axioms in $\mathcal{A}$ such that the implication $(\alpha \Rightarrow \beta)$ is an intuitionistic tautology.*

Suppose that β is a theorem in $\mathfrak{T}$, i.e. β belongs to $\mathfrak{C}_i(\mathcal{A})$. Then there exists a finite non-empty set $\mathcal{A}_1 \subset \mathcal{A}$ such that β is in $\mathfrak{C}_i(\mathcal{A}_1)$. Let α be a conjunction of all formulas in $\mathcal{A}_1$. By *modus ponens* and (T_6), $\mathcal{A}_1 \subset \mathfrak{C}_i([\alpha])$. Consequently β belongs to $\mathfrak{C}_i([\alpha])$, i.e. β is a theorem of the theory $\{\mathfrak{L}_0, \mathfrak{C}_i, [\alpha]\}$. By 9.1 (where $\mathcal{A}$ is empty), the implication $(\alpha \Rightarrow \beta)$ is a theorem in $\{\mathfrak{L}_0, \mathfrak{C}_i, 0\}$, i.e. it is an intuitionistic tautology.

Conversely, if there exists a conjunction α of axioms in $\mathcal{A}$ such that $(\alpha \Rightarrow \beta)$ is an intuitionistic tautology, then α is a theorem in $\mathfrak{T}$ (see VI, 10.8) and, by *modus ponens*, β is also a theorem in $\mathfrak{T}$.

§ 10. Connection between theories and filters. In this section $\|\alpha\|$ always denotes the element of $\mathfrak{A}(\mathfrak{S}_{0i})$, determined by a formula α.

For every set $\mathcal{A}$ of formulas in a propositional intuitionistic calculus $\mathfrak{S}_{0i} = \{\mathfrak{L}_0, \mathfrak{C}_i\}$ let $\nabla_{0\mathcal{A}}$ be the set of all elements $\|\alpha\|$ in $\mathfrak{A}(\mathfrak{S}_{0i})$ where α is in $\mathcal{A}$, and let $\nabla_{\mathcal{A}}$ be the set of all elements $\|\alpha\|$ in $\mathfrak{A}(\mathfrak{S}_{0i})$ where α is a theorem in the theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_i, \mathcal{A}\}$. The last definition can be formulated as follows:

(1) $\|\alpha\| \in \nabla_{\mathcal{A}}$ if and only if α is a theorem in $\mathfrak{T}$.

All theorems in this section are intuitionistic analogues of those in VII, § 9.

10.1. *$\nabla_{\mathcal{A}}$ is a filter in the pseudo-Boolean algebra $\mathfrak{A}(\mathfrak{S}_{0i})$ and $\Delta_{0\mathcal{A}}$ is a set of generators for the filter $\nabla_{\mathcal{A}}$.*

The theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_i, \mathcal{A}\}$ is consistent if and only if the filter $\nabla_{\mathcal{A}}$ is proper.

By the deduction theorem 9.2, α is a theorem in $\mathfrak{T}$ if and only if there exists a conjunction $\bigcap_{i=1}^{m} \alpha_i$ of formulas α_i in $\mathcal{A}$, such that $(\bigcap_{i=1}^{m} \alpha_i \Rightarrow \alpha)$ is an intuitionistic tautology, i.e. if $\|(\bigcap_{i=1}^{m} \alpha_i \Rightarrow \alpha)\| = V \in \mathfrak{A}(\mathfrak{S}_{0i})$. In other words, α is a theorem in $\mathfrak{T}$ if and only if there exist formulas $\alpha_1, \ldots, \alpha_m$ in $\mathcal{A}$ such that

$$\bigcap_{i=1}^{m} \|\alpha_i\| \leqslant \|\alpha\| .$$

This proves the first part of 10.1 by I, 8.1.

By V, 14.1, the theory $\mathfrak{T}$ is consistent if and only if there exists a formula which is not a theorem in $\mathfrak{T}$. By (1), this is equivalent to the statement that the filter $\nabla_{\mathcal{A}}$ is proper.

We recall that a formula α is said to be irrefutable in a theory $\mathfrak{T}$ if and only if $-\alpha$ is not any theorem in $\mathfrak{T}$, i.e. if $-\|\alpha\| \notin \nabla_{\mathcal{A}}$.

10.2. *A formula a is irrefutable in a theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathfrak{A}\}$ if and only if the theory $\{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathfrak{A} \cup [a]\}$ is consistent.*

Thus a formula $--\beta$ is not a theorem in $\mathfrak{T}$ if and only if the theory $\{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathfrak{A} \cup [-\beta]\}$ is consistent.

The first part follows from 10.1 since the filter generated by $\Delta_{0\mathfrak{A}}$ and $\|a\|$ (i.e. generated by $\nabla_{\mathfrak{A}}$ and $\|a\|$) is proper if and only if $-\|a\| \notin \nabla_{\mathfrak{A}}$ (see I, 13.9).

The second part follows from the first, where a is the formula $-\beta$.

10.3. *A theory $\mathfrak{T}' = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathfrak{A}'\}$ is a linguistically invariant extension of a theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathfrak{A}\}$ if and only if $\nabla_{\mathfrak{A}} \subset \nabla_{\mathfrak{A}'}$.*

By definition, $\mathfrak{T}'$ is a linguistically invariant extension of $\mathfrak{T}$ if and only if $\mathfrak{C}_\iota(\mathfrak{A}) \subset \mathfrak{C}_\iota(\mathfrak{A}')$. By (1), this is equivalent to the inclusion $\nabla_{\mathfrak{A}} \subset \nabla_{\mathfrak{A}'}$.

10.4. *Theories $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathfrak{A}\}$ and $\mathfrak{T}' = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathfrak{A}'\}$ are equivalent if and only if $\nabla_{\mathfrak{A}} = \nabla_{\mathfrak{A}'}$.*

This follows immediately from the definition of equivalent theories and (1) (or from 10.3).

We shall complete theorem 10.1 by the following remark:

10.5. *For every filter ∇ in $\mathfrak{A}(\mathfrak{S}_{0\iota})$ there exists an intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathfrak{A}\}$ such that $\nabla = \nabla_{\mathfrak{A}}$. Moreover, for every set ∇_0 of generators of ∇ we can choose the set $\mathfrak{A}$ of axioms for $\mathfrak{T}$ in such a way that $\nabla_0 = \Delta_{0\mathfrak{A}}$.*

For every element $e \in \nabla_0$ let a_e be a formula such that $\|a_e\| = e$, and let $\mathfrak{A}$ be the set consisting of all formulas a_e where $e \in \nabla_0$. By definition, $\nabla_{0\mathfrak{A}} = \nabla_0$. Consequently $\nabla_{\mathfrak{A}} = \nabla$ by the first part of 10.1.

Theorems 10.1, 10.4 and 10.5 state that there is a natural one-to-one correspondence between filters in $\mathfrak{A}(\mathfrak{S}_{0\iota})$ and formalized intuitionistic theories based on $\mathfrak{S}_{0\iota}$, two theories being identified if and only if they are equivalent.

To prove the next theorem we shall use the more precise notation introduced in VI, § 10, p. 245: For any formula a, the symbol $\|a\| = \|a\|_{\mathfrak{S}_{0\iota}}$ will denote the corresponding element in the algebra $\mathfrak{A}(\mathfrak{S}_{0\iota})$ of the intuitionistic propositional calculus $\mathfrak{S}_{0\iota} = \{\mathfrak{L}_0, \mathfrak{C}_\iota\}$, and the symbol $\|a\|_{\mathfrak{T}}$ will denote the corresponding element of the algebra $\mathfrak{A}(\mathfrak{T})$ of an intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathfrak{A}\}$.

If formulas $(a \Rightarrow \beta)$ and $(\beta \Rightarrow a)$ are derivable in $\mathfrak{S}_{0\iota}$, then they are also theorems in $\mathfrak{T}$. Hence it follows that the equation

$$h(\|a\|_{\mathfrak{S}_{0\iota}}) = \|a\|_{\mathfrak{T}}$$

defines a mapping from $\mathfrak{A}(\mathfrak{S}_{0\iota})$ onto $\mathfrak{A}(\mathfrak{T})$. It is easy to verify that h is a pseudo-Boolean homomorphism (this follows from 2.2, (3)). By the definition of h and $\nabla_{\mathfrak{A}}$, the element $h(\|a\|_{\mathfrak{S}_{0\iota}})$ is the unit element of $\mathfrak{A}(\mathfrak{T})$

if and only if a is a theorem in $\mathcal{T}$, i.e. if $\|a\|_{\mathfrak{S}_{0\iota}} \in V_{\mathcal{A}}$. Hence, by I, 13.4, follows

10.6. *The algebra $\mathfrak{A}(\mathcal{T})$ of an intuitionistic theory $\mathcal{T}$ based on $\mathfrak{S}_{0\iota}$ is isomorphic with the quotient algebra $\mathfrak{A}(\mathfrak{S}_{0\iota})/V_{\mathcal{A}}$.*

Theorem 10.6 gives a method of construction of $\mathfrak{A}(\mathcal{T})$ from $\mathfrak{A}(\mathfrak{S}_{0\iota})$.

10.7. *For every pseudo-Boolean algebra $\boldsymbol{A}$ there exists an intuitionistic theory $\mathcal{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$ of zero order, such that $\boldsymbol{A}$ is isomorphic with $\mathfrak{A}(\mathcal{T})$.*

Let $\boldsymbol{A}_0$ be a set of generators for $\boldsymbol{A}$, and let $\mathfrak{m}$ be the cardinal of $\boldsymbol{A}_0$. Take a formalized language $\mathfrak{L}_0$ of zero order, such that the cardinal of the set V_0 of all propositional variables in $\mathfrak{L}_0$ is equal to $\mathfrak{m}$. Let f be a mapping from V_0 onto $\boldsymbol{A}_0$. By theorems 7.1 and 7.2 there exists a homomorphism h: $\mathfrak{A}(\mathfrak{S}_{0\iota}) \to \boldsymbol{A}$ such that $h(\|a\|) = f(a)$ for every a in V_0. Since $\boldsymbol{A}_0$ generates $\boldsymbol{A}$, the homomorphism h maps $\mathfrak{A}(\mathfrak{S}_{0\iota})$ onto $\boldsymbol{A}$. By I, 13.4, the algebra $\boldsymbol{A}$ is isomorphic with $\mathfrak{A}(\mathfrak{S}_{0\iota})/V$, where V is the filter composed of all $\|a\| \in \mathfrak{A}(\mathfrak{S}_{0\iota})$ such that $h(\|a\|) = 1 \in \boldsymbol{A}$. By theorems 10.6 and 10.5 there exists a set $\mathcal{A}$ of formulas in $\mathfrak{L}_0$, such that $\mathfrak{A}(\mathfrak{S}_{0\iota})/V$ is isomorphic with $\mathfrak{A}(\mathcal{T})$ where $\mathcal{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$. Consequently $\boldsymbol{A}$ is isomorphic with $\mathfrak{A}(\mathcal{T})$.

§ 11. Maximal theories. We recall that a consistent theory $\mathcal{T}$ is said to be maximal provided every linguistically invariant consistent extension of $\mathcal{T}$ is equivalent to $\mathcal{T}$. The following theorem characterizes maximal intuitionistic theories.

11.1. *The following conditions are equivalent for every intuitionistic theory $\mathcal{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$ of zero order:*

(i) *$\mathcal{T}$ is a maximal theory;*

(ii) *$V_{\mathcal{A}}$ is a maximal filter in $\mathfrak{A}(\mathfrak{S}_{0\iota})$;*

(iii) *for every formula a in $\mathfrak{L}_0$, exactly one of the formulas a and $-a$ is a theorem in $\mathcal{T}$;*

(iv) *$\mathfrak{A}(\mathcal{T})$ is the two-element Boolean algebra.*

(i) is equivalent to (ii). This follows from the natural one-to-one correspondence between filters in $\mathfrak{A}(\mathfrak{S}_{0\iota})$ and classes of equivalent theories (see § 10, p. 401) and from theorem 10.3.

Condition (iii) means that for any element $\|a\|$ in $\mathfrak{A}(\mathfrak{S}_{0\iota})$ exactly one of the elements $\|a\|$ and $-\|a\|$ is in $V_{\mathcal{A}}$. Thus, by I, 13.10, (iii) is equivalent to (ii).

(iv) is equivalent to (ii) by 10.6 and I, 13.10.

The next theorem gives another characterization by means of models.

11.2. *For every intuitionistic theory $\mathcal{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$ the following conditions are equivalent:*

(i) $\mathfrak{T}$ *is maximal*;

(ii) $\mathfrak{T}$ *is consistent and every semantic model for* $\mathfrak{T}$ *is adequate*;

(iii) $\mathfrak{T}$ *has an adequate semantic model.*

Let $\boldsymbol{A}_0$ denote the two-element Boolean algebra.

Suppose that $\mathfrak{T}$ is maximal and v is a semantic model for $\mathfrak{T}$. If α is not a theorem in $\mathfrak{T}$, then by 11.1, (iii), the formula $-\alpha$ is a theorem, and consequently $(-\alpha)_{\boldsymbol{A}_0}(v) = \mathsf{V}$, i.e. $\alpha_{\boldsymbol{A}_0}(v) = \mathsf{\Lambda}$. This proves that v is adequate.

Thus (i) implies (ii). Since (ii) implies (iii) trivially (see 8.4, (v)), it remains to prove that (iii) implies (i). Suppose that v is an adequate semantic model for $\mathfrak{T}$ in $\boldsymbol{A}_0$. For every formula α, exactly one ot the elements $\alpha_{\boldsymbol{A}_0}(v)$, $(-\alpha)_{\boldsymbol{A}_0}(v)$ is equal to the unit element of $\boldsymbol{A}_0$. Since v is adequate, we infer that exactly one of the formulas α and $-\alpha$ is a theorem in $\mathfrak{T}$.

Thus $\mathfrak{T}$ is maximal, by 11.1.

11.3. *If v is a valuation in the two-element Boolean algebra $\boldsymbol{A}_0$ and $\mathcal{A}_0$ is the set of all formulas such that $\alpha_{\boldsymbol{A}_0}(v) = \mathsf{V}$, then every theory $\mathfrak{T}$ equivalent to $\{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}_0\}$ is maximal.*

In fact, v is then an adequate semantic model for $\mathfrak{T}$.

11.4. *Every consistent theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$ can be extended to a maximal theory $\mathfrak{T}_0 = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}_0\}$.*

Since $\mathfrak{T}$ is consistent, the filter $\mathsf{F}_{\mathcal{A}}$ in $\mathfrak{A}(\mathfrak{S}_{0\iota})$ is proper (see 10.1). By I, 8.4, the filter $\mathsf{F}_{\mathcal{A}}$ can be extended to a maximal filter F in $\mathfrak{A}(\mathfrak{S}_{0\iota})$. Let $\mathcal{A}_0$ be any set of formulas such that $\mathsf{F} = \mathsf{F}_{\mathcal{A}_0}$ (see 10.5). By 11.1, the theory $\mathfrak{T}_0 = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}_0\}$ is maximal. It is an extension of $\mathfrak{T}$ by 10.3.

Another proof of 11.4 is as follows: Let v be a semantic model for $\mathfrak{T}$ and let $\mathcal{A}_0$ be the set of all formulas α such that $\alpha_{\boldsymbol{A}_0}(v) = \mathsf{V}$. The theory $\mathfrak{T}_0 = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}_0\}$ is an extension of $\mathfrak{T}$ and is maximal, on account of 11.3.

§ 12. Prime theories. In any (classical or intuitionistic) theory, if one of the formulas α, β is a theorem, then the disjunction $(\alpha \cup \beta)$ is also a theorem (this follows easily from (T_2), (T_3) by *modus ponens*). The converse statement is not true, in general: it is possible that a disjunction $(\alpha \cup \beta)$ is a theorem in a (classical or intuitionistic) theory, but neither α nor β is a theorem. By V, § 14, p. 204, a consistent theory $\mathfrak{T}$ is said to be prime if the converse statement is true. In other words, $\mathfrak{T}$ is prime if and only if, for every disjunction $(\alpha \cup \beta)$, the disjunction $(\alpha \cup \beta)$ is a theorem in $\mathfrak{T}$ if and only if one of the formulas α, β is a theorem in $\mathfrak{T}$.

12.1. *An intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$ of zero order is prime if and only if the filter F_A in $\mathfrak{A}(\mathfrak{S}_{0\iota})$ (see § 10) is prime.*

In fact, $\mathfrak{T}$ is prime if and only if, for arbitrary formulas α, β,

$$\|\alpha\| \cup \|\beta\| \in \nabla_{\mathfrak{A}} \quad \text{if and only if either} \quad \|\alpha\| \in \nabla_{\mathfrak{A}} \text{ or } \|\beta\| \in \nabla_{\mathfrak{A}},$$

(see 2.2 and § 10, (1)), i.e. if the filter $\nabla_{\mathfrak{A}}$ is prime.

A similar theorem was proved for classical theories (VII, 10.5, (i), (ii)).

Since in Boolean algebras prime filters coincide with maximal filters, a classical theory is prime if and only if it is maximal (VII, 10.5, (i), (iii)). In pseudo-Boolean algebras the notions of prime and maximal filters do not coincide, therefore there exist prime intuitionistic theories which are not maximal. The simplest example is given by the intuitionistic propositional calculus $\mathfrak{S}_{0i}$ considered as the formalized theory $\{\mathfrak{L}_0, \mathfrak{C}_i, 0\}$ with the empty set of axioms. The simplest example of a non-prime consistent intuitionistic theory is given by the theory $\{\mathfrak{L}_0, \mathfrak{C}_i, \mathfrak{A}_0\}$ where $\mathfrak{A}_0$ is the set of all tautologies (T_{12}), i.e. of all tautologies of the form $(a \cup -a)$. In fact, it is easy to see that

$$\mathfrak{C}_i(\mathfrak{A}_0) = \mathfrak{C}(0) = \text{the set of all tautologies},$$

i.e. the intuitionistic theory $\{\mathfrak{L}_0, \mathfrak{C}_i, \mathfrak{A}_0\}$ is equivalent to the classical propositional calculus $\mathfrak{S}_0$ considered as the theory $\{\mathfrak{L}_0, \mathfrak{C}, 0\}$ with the empty set of axioms.

Since every maximal filter in a pseudo-Boolean algebra is prime, we infer from 11.1 and 12.1 that

12.2. *Every maximal intuitionistic theory is prime.*

From 12.2 and 11.4 easily follows

12.3. *Every consistent intuitionistic theory can be extended to a prime theory.*

We shall now give a characterization of prime theories. First we have to introduce the notion of characteristic valuation for a theoryl

Let $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_i, \mathfrak{A}\}$ be a consistent intuitionistic theory. By IV, 9.2, there is an isomorphism h from $\mathfrak{A}(\mathfrak{T})$ into the algebra $\mathfrak{G}(X)$ of open subsets of a topological space X. By 2.3 and VI, 2.4, for every formula a

$$h(\|a\|) = h a_{\mathfrak{A}(\mathfrak{T})}(v^0) = a_{\mathfrak{G}(X)}(h v^0) \tag{1}$$

where v^0 is the canonical valuation in $\mathfrak{A}(\mathfrak{T})$ (see § 2, (7)). By III, 5.1, X is an open subspace of a strongly compact space X_0. By the strong compactness of X_0, for any open subsets G_1, G_2 of X_0,

$$\text{if } G_1 \cup G_2 = X_0, \text{ then either } G_1 = X_0 \text{ or } G_2 = X_0. \tag{2}$$

Since X is an open subset of X_0, the mapping

$$h_0(G) = X \cap G \quad \text{for} \quad G \in \mathfrak{G}(X_0) \tag{3}$$

is a pseudo-Boolean homomorphism from $\mathfrak{G}(X_0)$ onto $\mathfrak{G}(X)$ (see IV, 8.2).

The valuation $v = \{v_a\}_{a \in V_0}$ defined as follows:

$$v_a = \begin{cases} X_0 \text{ if } h(v_a^0) = X, \text{ i.e. if } v_a^0 = \|a\| = V \in \mathfrak{A}(\mathfrak{C}), \\ h(v_a^0) \text{ if } h(v_a^0) \neq X, \end{cases} \tag{4}$$

is called the *characteristic valuation for* $\mathfrak{C}$. It follows from (3) and (4) that

$$h_0 v = h v^0. \tag{5}$$

Under the above notation the following theorem holds.

12.4. *For every formula* a,

$$\textit{either} \quad a_{\mathfrak{G}(X_0)}(v) = X_0 \quad \textit{or} \quad a_{\mathfrak{G}(X_0)}(v) = h(\|a\|). \tag{6}$$

Moreover

$$\textit{if} \quad a_{\mathfrak{G}(X_0)}(v) = X_0, \quad \textit{then} \quad h(\|a\|) = X, \quad \textit{i.e.} \quad \|a\| = V \in \mathfrak{A}(\mathfrak{C}). \tag{7}$$

It follows from VI, 2.4, from (5) and from § 2, (8) that

$$h_0\big(a_{\mathfrak{G}(X_0)}(v)\big) = a_{\mathfrak{G}(X)}(h_0 v) = a_{\mathfrak{G}(X)}(h v^0) = h(\|a\|).$$

Hence, by (3),

$$a_{\mathfrak{G}(X_0)}(v) \cap X = h(\|a\|). \tag{8}$$

Since $a_{\mathfrak{G}(X_0)}(v)$ is an open subset of X_0, and the class of all open subsets of X_0 is composed of X_0 and all open subsets of X, (8) implies (6) and (7).

12.5. *If the characteristic valuation* v *for* $\mathfrak{C}$ *is a model for a consistent intuitionistic theory* $\mathfrak{C}$, *then the theory* $\mathfrak{C}$ *is prime.*

Suppose that $(a \cup \beta)$ is a theorem in $\mathfrak{C}$. Since v is a model for $\mathfrak{C}$, we have by 8.1

$$a_{\mathfrak{G}(X_0)}(v) \cup \beta_{\mathfrak{G}(X_0)}(v) = (a \cup \beta)_{\mathfrak{G}(X_0)}(v) = X_0.$$

Hence, by (2), at least one of the open sets $a_{\mathfrak{G}(X_0)}(v)$, $\beta_{\mathfrak{G}(X_0)}(v)$ equals X_0. Suppose $a_{\mathfrak{G}(X_0)}(v) = X_0$. By (7), $h(\|a\|) = X$, i.e. $\|a\| = V \in \mathfrak{A}(\mathfrak{C})$. By 2.2, a is a theorem in $\mathfrak{C}$.

Consider the following property of formulas a in an intuitionistic theory $\mathfrak{C}$:

(p) if a is a theorem in $\mathfrak{C}$, then

$$a_{\mathfrak{G}(X_0)}(v) = X_0, \tag{9}$$

v being the characteristic valuation.

12.6. *The set* S *of all formulas* a *in an intuitionistic theory* $\mathfrak{C}$ *which have property* (p) *satisfies the following conditions*:

(a) *every propositional variable is in* S;

(b) *if* β, γ *are in* S, *then the conjunction* $(\beta \cap \gamma)$ *is in* S;

(c) *for every formula* β *the negation* $-\beta$ *is in* S;

(d) *if γ is in S, then for every formula β the implication $(\beta \Rightarrow \gamma)$ is in S.*

(a) follows directly from (4).

To prove (b), suppose that β and γ are in S and that the conjunction $(\beta \cap \gamma)$ is a theorem. Then β and γ are also theorems and, by (p), $\beta_{\mathfrak{G}(X_0)}(v) = X_0 = \gamma_{\mathfrak{G}(X_0)}(v)$. Hence

$$(\beta \cap \gamma)_{\mathfrak{G}(X_0)}(v) = \beta_{\mathfrak{G}(X_0)}(v) \cap \gamma_{\mathfrak{G}(X_0)}(v) = X_0 .$$

To prove (c), let us observe that if $-\beta$ is a theorem, then $\|-\beta\| = V$. Thus $h(\|\beta\|) = 0$. By 12.4, $\beta_{\mathfrak{G}(X_0)}(v) = 0$. Hence $(-\beta)_{\mathfrak{G}(X_0)}(v) = X_0$.

To prove (d) let us suppose that $(\beta \Rightarrow \gamma)$ is a theorem and γ is in S. For brevity, denote $(\beta \Rightarrow \gamma)$ by α. We have to prove (9). By 12.4, there are three cases 1°, 2°, 3° to be considered.

1° $$\gamma_{\mathfrak{G}(X_0)}(v) = X_0 .$$

Then (9) holds (see I, 12.2, (7)).

2° $$\beta_{\mathfrak{G}(X_0)}(v) = X_0 .$$

Then it follows from (7) and 2.2 that β is a theorem. By *modus ponens*, γ is also a theorem. Since γ is in S, we have $\gamma_{\mathfrak{G}(X_0)}(v) = X_0$. By the case 1° just proved, (9) holds.

3° $$\beta_{\mathfrak{G}(X_0)}(v) = h(\|\beta\|) \quad \text{and} \quad \gamma_{\mathfrak{G}(X_0)}(v) = h(\|\gamma\|) .$$

Since α is a theorem, we have $\|\beta\| \leqslant \|\gamma\|$. Hence $h(\|\beta\|) \leqslant h(\|\gamma\|)$. This implies (9).

12.7. *All formulas of a prime intuitionistic theory have property* (p).

We shall prove that, if the theory is prime, the set S mentioned in 14.6 also has the property:

(e) if β, γ are in S, then the disjunction $(\beta \cup \gamma)$ is also in S.

In fact, suppose that $(\beta \cup \gamma)$ is a theorem and β and γ have property (p). Since the theory is prime, one of the formulas β and γ, say β, is a theorem. Since β satisfies (p), we have $\beta_{\mathfrak{G}(X_0)}(v) = X_0$ which implies that $(\beta \cup \gamma)_{\mathfrak{G}(X_0)}(v) = X_0$.

Since the set S has properties (a), (b), (c), (d), (e), it coincides with the set of all formulas.

It follows from 12.5 and 12.7 that

12.8. *A consistent intuitionistic theory $\mathfrak{T}$ is prime if and only if the characteristic valuation is a model for $\mathfrak{T}$* [1].

---

[1] Theorems 12.4-12.9 are analogues of theorems proved by Rasiowa [4] for elementary intuitionistic theories.

The intersection F^* of all sets S satisfying conditions (a), (b), (c), (d) also satisfies these conditions. It is the least set of formulas satisfying (a)-(d). From 12.5 and 12.6 there follows immediately

12.9. *If the set $\mathcal{A}$ of axioms of a consistent intuitionistic theory $\mathcal{T} = \{\mathfrak{L}_0, \mathcal{C}_\iota, \mathcal{A}\}$ is contained in F^*, then the theory $\mathcal{T}$ is prime.*

In particular, if no axiom of the theory contains the sign of disjunction, the theory is prime.

Theorem 12.9 gives a simple method of construction of prime theories. For instance, if $\mathcal{A}$ is composed of the tautology (see (T_{30}))

$$(((a \Rightarrow b) \Rightarrow a) \Rightarrow a)$$

where a, b are propositional variables, then the theory is prime. However if $\mathcal{A}$ is composed of all the tautologies

$$(((a \Rightarrow \beta) \Rightarrow a) \Rightarrow a) \tag{10}$$

where a, β are any formulas, then the theory is not prime since it is equivalent to the classical propositional calculus.

To prove the last statement, first observe that if $\mathcal{A}$ consists of all formulas (10), then the set of all theorems in $\mathcal{T} = \{\mathfrak{L}_0, \mathcal{C}_\iota, \mathcal{A}\}$ is contained in the set of all propositional tautologies because this set is closed under *modus ponens* and contains all axioms in $\mathcal{A}$ and all logical axioms of $\mathcal{T}$. On the other hand, for all formulas a, β,

$$(\|a\| \Rightarrow \|\beta\|) \Rightarrow \|a\| \leqslant \|a\|$$

in the pseudo-Boolean algebra $\mathfrak{A}(\mathcal{T})$. Consequently, for $\|\beta\| = \Lambda$,

$$-\|a\| \Rightarrow \|a\| \leqslant \|a\|$$

and, by I, 12.2, (12),

$$--\|a\| = -\|a\| \Rightarrow \Lambda \leqslant -\|a\| \Rightarrow \|a\| \leqslant \|a\| .$$

Replacing a by $(a \cup -a)$ we obtain, by 5.3,

$$V = --\|(a \cup -a)\| \leqslant \|(a \cup -a)\| .$$

Thus the formula $(a \cup -a)$ is a theorem in $\mathcal{T}$, for every a. Since all formulas of the form (T_1)-(T_{12}) are theorems in $\mathcal{T}$, all propositional tautologies are theorems in $\mathcal{T}$.

The criterion given in 12.9 is a sufficent but not a necessary condition for a theory to be prime. For instance, if the set $\mathcal{A}$ of axioms is composed of all formulas

$$((-a \Rightarrow (\beta \cup \gamma)) \Rightarrow ((-a \Rightarrow \beta) \cup (-a \Rightarrow \gamma)))$$

where α, β, γ are any formulas, the theory is prime (1) but the criterion from 12.9 is not satisfied. Similarly, if the set $\mathcal{A}$ of axioms consists of all formulas

$$(((--\alpha \Rightarrow \alpha) \Rightarrow (\alpha \cup -\alpha)) \Rightarrow (--\alpha \cup -\alpha))$$

where α is any formula, the theory is prime (2). The proof that the two theories are prime must be given by different methods than those explained in this section.

§ 13. Connection between classical and intuitionistic theories. Let $\mathcal{A}$ be any set of formulas in a formalized language $\mathfrak{L}_0$ of zero order. We shall examine the connection between the intuitionistic theory

$$\mathcal{T} = \{\mathfrak{L}_0, C_i, \mathcal{A}\}$$

and the classical theory

$$\mathcal{T}_0 = \{\mathfrak{L}_0, C, \mathcal{A}\}$$

having the same set of axioms and the same language.

From the definition of the classical and intuitionistic consequence operations C and C_i there follows immediately

13.1. *Every theorem in* $\mathcal{T} = \{\mathfrak{L}_0, C_i, \mathcal{A}\}$ *is also a theorem in* $\mathcal{T}_0 = \{\mathfrak{L}_0, C, \mathcal{A}\}$; *in symbols*

$$C_i(\mathcal{A}) \subset C(\mathcal{A}) . \tag{1}$$

We shall now prove the following fundamental theorem.

13.2. *A formula* α *is a theorem in* $\mathcal{T}_0 = \{\mathfrak{L}_0, C, \mathcal{A}\}$ *if and only if the formula* $--\alpha$ *is a theorem in* $\mathcal{T} = \{\mathfrak{L}_0, C_i, \mathcal{A}\}$.

Suppose that α is a theorem in $\mathcal{T}_0$. By the deduction theorem VII, 8.2, there exists a conjunction β of formulas in $\mathcal{A}$ such that the implication $(\beta \Rightarrow \alpha)$ is a tautology. By 5.3 the formula $--(\beta \Rightarrow \alpha)$ is an intuitionistic tautology. Since $(--(\beta \Rightarrow \alpha) \Rightarrow (\beta \Rightarrow --\alpha))$ is also an intuitionistic tautology (see 4.1, (T_{67})), by *modus ponens* the formula $(\beta \Rightarrow --\alpha)$ is an intuitionistic tautology. By the deduction theorem 9.2, the formula $--\alpha$ is a theorem in $\mathcal{T}$.

Conversely, if a formula $--\alpha$ is a theorem in $\mathcal{T}$, then by 13.1 it is also a theorem in $\mathcal{T}_0$. Since $(--\alpha \Rightarrow \alpha)$ is a tautology, it follows from *modus ponens* that α is a theorem in $\mathcal{T}_0$.

13.3. *A formula* α *is a theorem in* $\mathcal{T}_0 = \{\mathfrak{L}_0, C, \mathcal{A}\}$ *if and only if, for every model* v *for the theory* $\mathcal{T} = \{\mathfrak{L}_0, C_i, \mathcal{A}\}$ *in any pseudo-Boolean algebra* $\boldsymbol{A}$, *the element* $a_{\boldsymbol{A}}(v)$ *is dense in* $\boldsymbol{A}$.

---

(1) Kreisel and Putnam [1].

(2) This result is due to D. S. Scott (see Kreisel and Putnam [1]).

Suppose that a is a theorem in $\mathfrak{C}_0$. By 13.2, $--a$ is a theorem in $\mathfrak{C}$. Thus $(--a)_A(v) = V$ by 8.1, i.e. $--a_A(v) = V$. This proves that $a_A(v)$ is dense in A.

To prove the remaining part of 13.3 it suffices to take as A the two-element Boolean algebra and to apply VII 7.5.

13.4. *A formula a is a theorem in $\mathfrak{C}_0 = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ if and only if the element $\|a\| \in \mathfrak{A}(\mathfrak{C})$ is dense in the algebra $\mathfrak{A}(\mathfrak{C})$ of $\mathfrak{C} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$.*

If a is a theorem in $\mathfrak{C}_0$, then $--a$ is a theorem in $\mathfrak{C}$ by 13.2, i.e. $--\|a\| = \|--a\| = V$. This proves that $\|a\|$ is a dense element of $\mathfrak{A}(\mathfrak{C})$.

Conversely, if $\|a\|$ is a dense element in $\mathfrak{A}(\mathfrak{C})$, then $\|--a\| = V$, i. e. $--a$ is a theorem in $\mathfrak{C}$. By 13.2, a is a theorem in $\mathfrak{C}_0$.

13.5. *A formula $-a$ is a theorem in $\mathfrak{C}_0 = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ if and only if it is a theorem in $\mathfrak{C} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$.*

The element $\|-a\| = -\|a\| \in \mathfrak{A}(\mathfrak{C})$ is regular by IV, 6.3. By IV, 6.6, the element $\|-a\|$ is dense in $\mathfrak{A}(\mathfrak{C})$ if and only if $\|-a\| = V$, i.e. if $-a$ is a theorem in $\mathfrak{C}$ (see 2.2). This directly implies 13.5 on account of 13.4.

13.6. *A formula $(a \Rightarrow -\beta)$ is a theorem in $\mathfrak{C}_0 = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ if and only if it is a theorem in $\mathfrak{C} = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}\}$.*

The element $\|(a \Rightarrow -\beta)\| = \|a\| \Rightarrow -\|\beta\| \in \mathfrak{A}(\mathfrak{C})$ is regular by IV, 6.3. Thus, by IV, 6.6, the element $\|(a \Rightarrow -\beta)\|$ is dense if and only if $\|(a \Rightarrow -\beta)\| = V$, i.e. $(a \Rightarrow -\beta)$ is a theorem in $\mathfrak{C}$. This implies 13.6 by 13.4.

Denote by $\mathcal{A}'$ the set of all formulas $--a$ where a is in $\mathcal{A}$.

13.7. *A formula a is a theorem in $\mathfrak{C}_0 = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ if and only if the formula $--a$ is a theorem in $\mathfrak{C}' = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}'\}$* (1).

The classical theories $\mathfrak{C}_0$ and $\mathfrak{C}_0' = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}'\}$ are equivalent by VII 9.4 since $V_{0\mathcal{A}} = V_{0\mathcal{A}'}$ and consequently $V_{\mathcal{A}} = V_{\mathcal{A}'}$ in the Boolean algebra $\mathfrak{A}(\mathfrak{S}_0)$ of the classical propositional calculus $\mathfrak{S}_0 = \{\mathfrak{L}_0, \mathfrak{C}\}$. Thus a is a theorem in $\mathfrak{C}_0$ if and only if a is a theorem in $\mathfrak{C}_0'$. By 13.2, a is a theorem in $\mathfrak{C}_0'$ if and only if $--a$ is a theorem in $\mathfrak{C}'$. This proves 13.7.

For every formula a, let a^* denote the formula defined in § 5, p. 393. Let $\mathcal{A}^*$ be the set of all formulas a^* where a is any formula in $\mathcal{A}$.

13.8. *A formula a is a theorem in $\mathfrak{C}_0 = \{\mathfrak{L}_0, \mathfrak{C}, \mathcal{A}\}$ if and only if the formula a^* is a theorem in $\mathfrak{C}^* = \{\mathfrak{L}_0, \mathfrak{C}_\iota, \mathcal{A}^*\}$* (2).

Suppose that a is a theorem in $\mathfrak{C}_0$. By the deduction theorem VII, 8.2, there exists a conjunction β of formulas in $\mathcal{A}$ such that $(\beta \Rightarrow a)$ is a tautology. By 5.9, the formula $(\beta \Rightarrow a)^*$ is an intuitionistic tautology. On the

(1) See Glivenko [1].

(2) Cf. Kleene [3], p. 495.

other hand, $(\beta \Rightarrow \alpha)^*$ is the formula $(\beta^* \Rightarrow \alpha^*)$ and β^* is a conjunction of formulas in $\mathcal{A}^*$. By the deduction theorem 9.2, the formula α^* is a theorem in $\mathcal{C}^*$.

The proof of the converse statement is similar.

To prove the next theorem, we shall use the following notation introduced in VI, § 10, p. 245: for every formula α, $\|\alpha\|_{\mathcal{C}}$ will denote the corresponding element in the pseudo-Boolean algebra $\mathfrak{A}(\mathcal{C})$, and $\|\alpha\|_{\mathcal{C}_0}$ will denote the corresponding element in the Boolean algebra $\mathfrak{A}(\mathcal{C}_0)$. If $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are theorems in $\mathcal{C} = \{\mathfrak{L}_0, \mathbf{C}_\iota, \mathcal{A}\}$, then they are also theorems in $\mathcal{C}_0 = \{\mathfrak{L}_0, \mathbf{C}, \mathcal{A}\}$ by 13.1. Hence it follows that the equation

$$h(\|\alpha\|_{\mathcal{C}}) = \|\alpha\|_{\mathcal{C}_0}$$

defines a mapping from $\mathfrak{A}(\mathcal{C})$ onto $\mathfrak{A}(\mathcal{C}_0)$. It is easy to see that h is a pseudo-Boolean homomorphism (see 2.2, (3) and the analogous identities VII 1.1, (3)). By the definition of h, $h(\|\alpha\|_{\mathcal{C}})$ is the unit element of $\mathfrak{A}(\mathcal{C}_0)$ if and only if α is a theorem in $\mathcal{C}_0$, i.e. if and only if $\|\alpha\|_{\mathcal{C}}$ is a dense element in $\mathfrak{A}(\mathcal{C})$ (see 13.4). Hence because of I, 13.4, follows

13.9. *The algebra $\mathfrak{A}(\mathcal{C}_0)$ of the classical theory $\mathcal{C}_0 = \{\mathfrak{L}_0, \mathbf{C}, \mathcal{A}\}$ is isomorphic to $\mathfrak{A}(\mathcal{C})/V$ where V is the filter of all dense elements in the algebra $\mathfrak{A}(\mathcal{C})$ of the intuitionistic theory $\mathcal{C} = \{\mathfrak{L}_0, \mathbf{C}_\iota, \mathcal{A}\}$.*

From 13.9 and IV, 6.7, follows

13.10. *The Boolean algebra $\mathfrak{A}(\mathcal{C}_0)$ is isomorphic with the Boolean algebra $\mathfrak{R}(\mathfrak{A}(\mathcal{C}))$ of all regular elements in $\mathfrak{A}(\mathcal{C})$.*

Denote by $\mathcal{A}_0$ the set of all formulas $(\alpha \cup -\alpha)$ where α is any formula in $\mathfrak{L}_0$. It follows immediately from the definition of the classical and intuitionistic consequence operations $\mathbf{C}$ and $\mathbf{C}_\iota$ that

13.11. *A formula α is a theorem in the classical theory $\{\mathfrak{L}_0, \mathbf{C}, \mathcal{A}\}$ if and only if it is a theorem in the intuitionistic theory $\{\mathfrak{L}_0, \mathbf{C}_\iota, \mathcal{A} \cup \mathcal{A}_0\}$.*

We conclude this section by two remarks.

13.12. *If the theory $\mathcal{C} = \{\mathfrak{L}_0, \mathbf{C}_\iota, \mathcal{A}\}$ is maximal, then the theory $\mathcal{C}_0 = \{\mathfrak{L}_0, \mathbf{C}, \mathcal{A}\}$ is maximal, and $\mathbf{C}_\iota(\mathcal{A}) = \mathbf{C}(\mathcal{A})$.*

The theory $\mathcal{C}_0$ is maximal, if and only if, for every formula α, exactly one of formulas $-\alpha$, $--\alpha$ is a theorem in $\mathcal{C}$.

First note that $\mathcal{C}$ is consistent if and only if $\mathcal{C}_0$ is consistent by 8.4, (i), (vi).

The first part of 13.12 follows immediately from (1) and 13.11.

The second part follows immediately from 13.2, 13.5 and VII, 10.1.

CHAPTER X

INTUITIONISTIC ELEMENTARY FORMALIZED THEORIES

§ 1. Preliminaries. In this Chapter we shall deal with a fixed formalized language $\mathfrak{L} = \{A, T, F\}$ of the first order. The alphabet A, the set T of all terms and the set F of all formulas in $\mathfrak{L}$ were described in V, § 3. The letter V will always denote the set of all free individual variables $x, y, z, \ldots$ in $\mathfrak{L}$. In this Chapter by a formula we shall mean a formula in $\mathfrak{L}$, unless the contrary is explicitly stated.

To facilitate our considerations we shall always assume that

the language $\mathfrak{L}$ is enumerable,

i.e. the set of all signs in $\mathfrak{L}$ is enumerable (1). Consequently the set T of all terms and the set F of all formulas in $\mathfrak{L}$ are enumerable. All other formalized languages considered in this Chapter will be enumerable, too.

We shall now examine the deductive system

$$\mathfrak{S}_i = \{\mathfrak{L}, \mathfrak{C}_i\}$$

called the *intuitionistic predicate calculus* based on the language $\mathfrak{L}$, and the elementary *intuitionistic theories* based on $\mathfrak{L}$

$$\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_i, \mathfrak{A}\}$$

where $\mathfrak{A}$ is any set of formulas. According to IX, § 1, the symbol $\mathfrak{C}_i$ always denotes the intuitionistic consequence operation. Sometimes we shall use the classical consequence operation. It will always be denoted by $\mathfrak{C}$ according to V, § 11, VII and VIII. Sometimes we shall also deal with the classical predicate calculus based on $\mathfrak{L}$. It will be denoted by $\mathfrak{S}$.

The following definitions are intuitionistic analogues of definitions from VIII, § 1, for the classical predicate calculus.

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(1) Many theorems in Chapter X can be proved without this hypothesis which seems to be essential only wherever we make use of theorems IV, 9.3 (the last part) and IV, 10.1, i.e. where we pass from pseudo-Boolean algebras to topological spaces.

A finite sequence $a_1, \ldots, a_n$ of formulas is said to be a *formal proof of a formula* a in the intuitionistic predicate calculus $\mathbb{S}_i$ *from a set* S of formulas (or: an *intuitionistic formal proof of* a *from* S) provided a_n is the formula a, and for every $j \leqslant n$ either a_j is one of the logical axioms (T_1)-(T_{11}) from IX, § 1, or it is a formula in S, or it is an immediate consequence of some formulas a_{j_1}, a_{j_2} with $j_1, j_2 < j$ by means of the rules (r_1)-(r_6) from V, § 8. A formula a is said to be *derivable in* $\mathbb{S}_i$ *from* S (or: *intuitionistically derivable from* S) if there exists a formal proof of a from S in $\mathbb{S}_i$.

By this definition and by the definition of C_i from IX, § 1, the set $C_i(S)$ of all *intuitionistic consequences of a set* S of formulas is the set of all formulas derivable from S in $\mathbb{S}_i$.

It follows immediately from the definition of the intuitionistic consequence operation C_i and the classical consequence operation C that

$$C_i(S) \subset C(S) \tag{1}$$

for every set S of formulas in $\mathfrak{L}$.

The set $C_i(\mathcal{A})$ plays a special role for the theory $\mathfrak{T} = \{\mathfrak{L}, C_i, \mathcal{A}\}$: formulas in $C_i(\mathcal{A})$ are called *theorems* of the theory $\mathfrak{T}$.

We recall that when the set $\mathcal{A}$ is empty, the intuitionistic theory $\{\mathfrak{L}, C_i, 0\}$ is identified with the intuitionistic predicate calculus $\mathbb{S}_i = \{\mathfrak{L}, C_i\}$ [1]. Theorems in $\{\mathfrak{L}, C_i, 0\}$ are called *intuitionistically derivable formulas* or, more precisely, *formulas derivable in* $\mathbb{S}_i$. By definition, a formula a is intuitionistically derivable if and only if there exists a formal proof for a in $\mathbb{S}_i$ from the empty set of formulas.

Unless the contrary is explicitly stated, by *theory* we shall understand in this Chapter any intuitionistic formalized theory of the first order $\mathfrak{T} = \{\mathfrak{L}, C_i, \mathcal{A}\}$.

We recall (see VI, § 10, p. 245) that the algebra $\mathfrak{A}(\mathfrak{T})$ of a theory $\mathfrak{T}$ is the algebra $F/\approx$ where $\approx$ is the congruence in the algebra of formulas $\{F, \cup, \cap, \Rightarrow, -\}$ defined as follows:

(2) $a \approx \beta$ if and only if the formulas $(a \Rightarrow \beta)$ and $(\beta \Rightarrow a)$ are theorems in $\mathfrak{T}$.

In particular, $\mathfrak{A}(\mathbb{S}_i)$ is the algebra $F/\approx$ where $\approx$ is the congruence in $\{F, \cup, \cap, \Rightarrow, -\}$ defined as follows:

(2′) $a \approx \beta$ if and only if the formulas $(a \Rightarrow \beta)$ and $(\beta \Rightarrow a)$ are derivable in $\mathbb{S}_i$.

From VI, 10.4, 10.5, 10.7, 10.10, 11.2 follows theorem 1.1.

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[1] Heyting [1]. This formalization of the intuitionistic predicate calculus differs from that of Heyting, but is equivalent.

1.1 [1]. *The algebra* $\{\mathfrak{A}(\mathcal{C}), \cup, \cap, \Rightarrow, -\}$ *of an intuitionistic theory* $\mathcal{C}$ *is a pseudo-Boolean algebra. For any formulas* α, β

$$\begin{aligned} \|\alpha\| \cup \|\beta\| &= \|(\alpha \cup \beta)\|, \quad \|\alpha\| \cap \|\beta\| = \|(\alpha \cap \beta)\|, \\ \|\alpha\| \Rightarrow \|\beta\| &= \|(\alpha \Rightarrow \beta)\|, \quad -\|\alpha\| = \|-\alpha\|. \end{aligned} \tag{3}$$

Moreover,

$$\|\alpha\| \leqslant \|\beta\| \quad \textit{if and only if } (\alpha \Rightarrow \beta) \textit{ is a theorem in } \mathcal{C}. \tag{4}$$

For every formula α,

$$\|\alpha\| = V \quad \textit{if and only if } \alpha \textit{ is a theorem in } \mathcal{C}, \tag{5}$$

$$\|\alpha\| \neq \Lambda \quad \textit{if and only if } \alpha \textit{ is irrefutable in } \mathcal{C}. \tag{6}$$

Thus the pseudo-Boolean algebra $\mathfrak{A}(\mathcal{C})$ *is non-degenerate if and only if the theory* $\mathcal{C}$ *is consistent.*

For every formula $\beta(x)$,

$$\begin{aligned} \Big\|\bigcup_\xi \beta(\xi)\Big\| &= \bigcup_{\tau \in T} \|\beta(\tau)\|, \\ \Big\|\bigcap_\xi \beta(\xi)\Big\| &= \bigcap_{\tau \in T} \|\beta(\tau)\|. \end{aligned} \tag{Q}$$

According to VI, § 11, we shall always conceive of $\mathfrak{A}(\mathcal{C})$ as the generalized algebra

$$\{\mathfrak{A}(\mathcal{C}), \cup, \cap, \Rightarrow, -, \bigcup|\mathfrak{Q}, \bigcap|\mathfrak{Q}\} \tag{7}$$

where $\bigcup$ and $\bigcap$ are respectively the infinite join and meet. As the common domain $\mathfrak{Q}$ of $\bigcup$ and $\bigcap$ in $\mathfrak{A}(\mathcal{C})$ we assume only the class of all indexed sets

$$\{\|\beta(\tau)\|\}_{\tau \in T} \tag{8}$$

where $\beta(x)$ is any formula in $\mathfrak{k}$. In other words, $\bigcup$ and $\bigcap$ are restricted in $\mathfrak{A}(\mathcal{C})$ to the infinite joins and meets (Q) only, i.e. to the infinite joins and meets corresponding to quantifiers. The generalized algebra (7) is called the *quantifier algebra of the theory* $\mathcal{C}$, or briefly the $\mathfrak{Q}$-*algebra of* $\mathcal{C}$.

According to VI, § 11, by a Q-homomorphism from $\mathfrak{A}(\mathcal{C})$ into any pseudo-Boolean algebra we shall always mean any pseudo-Boolean homomorphism preserving all the infinite joins and meets (Q).

Every set $\mathcal{A}$ of formulas in $\mathfrak{k}$ determines two theories: the intuitionistic theory $\{\mathfrak{k}, \mathcal{C}_i, \mathcal{A}\}$ and the classical theory $\{\mathfrak{k}, \mathcal{C}, \mathcal{A}\}$. It follows from (1) that $\mathcal{C}_i(\mathcal{A}) \subset \mathcal{C}(\mathcal{A})$, i.e. every theorem in $\{\mathfrak{k}, \mathcal{C}_i, \mathcal{A}\}$ is also a theorem in $\{\mathfrak{k}, \mathcal{C}, \mathcal{A}\}$. The converse statement is not true, in general, i.e. the two theories are not equivalent.

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[1] Rasiowa [4]. Cf. also Henkin [2] and Rasiowa [2].

On the other hand, every classical theory $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ can be interpreted as an intuitionistic theory. More precisely, every classical theory $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ is equivalent to an intuitionistic theory $\{\mathfrak{L}, \mathfrak{C}_i, \mathcal{A}'\}$, i.e. there exists a set $\mathcal{A}'$ of formulas in $\mathfrak{L}$ such that $\mathfrak{C}(\mathcal{A}) = \mathfrak{C}_i(\mathcal{A}')$. In fact, it follows directly from the definitions of the consequence operations $\mathfrak{C}$ and $\mathfrak{C}$ that it suffices to assume that $\mathcal{A}'$ is the union

$$\mathcal{A}' = \mathcal{A} \cup \mathcal{A}_0$$

where $\mathcal{A}_0$ is the set of all formulas (T_{12}), i.e. of all formulas $(a \cup -a)$, a being any formula in $\mathfrak{L}$.

§ 2. Models. We remind the reader that by a realization of a formalized language $\mathfrak{L} = \{A, T, F\}$ of the first order in a set $J \neq 0$ and in a complete algebra $\{A, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ we understand (see VI, § 6) any mapping R defined on the set of all functors in $\mathfrak{L}$ and on the set of all predicates in $\mathfrak{L}$ such that:

(a) R assigns to every m-argument functor φ in $\mathfrak{L}$ an m-argument function φ_R in J, i.e. a mapping $\varphi_R\colon J^m \to J$,

(b) R assigns to every m-argument predicate ϱ in $\mathfrak{L}$ an m-argument function defined in J with values in $\boldsymbol{A}$, i. e. a mapping $\varrho_R\colon J^m \to \boldsymbol{A}$.

As we explained in VI, § 6, given any realization R of $\mathfrak{L}$ in J and $\boldsymbol{A}$, every formula a in $\mathfrak{L}$ uniquely determines a mapping

$$a_R\colon J^V \to \boldsymbol{A} \tag{1}$$

where V is the set of all free individual variables in $\mathfrak{L}$. The value of a_R at a point $v = \{v_x\}_{x \in V}$ of the Cartesian product J^V is denoted by $a_R(v)$.

If, for a given mapping R defined on the set of all functors and the set of all predicates in $\mathfrak{L}$ and satisfying conditions (a), (b) where $\boldsymbol{A}$ is an incomplete generalized algebra, all infinite operations occurring in the definitions of $a_R(v)$ for every a in $\mathfrak{L}$ and for every $v \in J^V$ are feasible in $\boldsymbol{A}$, then R is also called a realization of $\mathfrak{L}$ in J and $\boldsymbol{A}$ (see VI, § 7).

In this Chapter we shall always assume that

$\{\boldsymbol{A}, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ *is a non-degenerate pseudo-Boolean algebra*

and $\bigcup$, $\bigcap$ are respectively the infinite join and meet [1].

Observe that, on account of IV, 9.1, for every pseudo-Boolean algebra $\boldsymbol{A}$ there exists a complete pseudo-Boolean algebra $\boldsymbol{A}^*$ and an isomorphism of $\boldsymbol{A}$ into $\boldsymbol{A}^*$ which preserves all infinite joins and meets in $\boldsymbol{A}$. In other

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[1] The idea to interpret any formula a of an intuitionistic predicate calculus as a mapping a_R, where R is a realization in a non-empty set J and in a complete pseudo-Boolean algebra is due to Mostowski [4]. See also Rasiowa [2] and Henkin [2].

words, A can be extended to a complete pseudo-Boolean algebra A^* with preservation of all infinite joins and meets. Therefore any realization R in $J \neq 0$ and in a incomplete pseudo-Boolean algebra A can be considered a realization in J and in the complete pseudo-Boolean algebra A^* which is a complete extension of A preserving all infinite joins and meets in A. However, if only elements in A are values of $a_R(v)$ and only infinite operations in A are used to calculate $a_R(v)$, then there is no reason to introduce any extension A^* of A and it is much more natural to consider R as a realization in A.

Let $\mathfrak{L}$ be a fixed formalized language and let R be a fixed realization of $\mathfrak{L}$ in a set $J \neq 0$ and in a pseudo-Boolean algebra A. Any element $v = \{v_x\}_{x \in V}$ of J^V, i.e. any mapping $v\colon V \to J$ is said to be a valuation of $\mathfrak{L}$ in J. Sometimes we shall also say that v is a *valuation of* $\mathfrak{L}$ *in* R, since R determines the set J.

A valuation $v\colon V \to J$ is said to *satisfy a formula* a *in the realization* R provided that

$$a_R(v) = 1 . \tag{2}$$

A formula a is said to be *satisfiable in* R if there exists a valuation v in R which satisfies a, i.e. for which (2) is satisfied. A formula a is said to be *valid in* R, if every valuation in R satisfies a, i.e. if (2) holds for every valuation v in R. In this case we shall say that R is a *model for* a. Observe that for a closed formula a the notions of satisfiability and validity coincide since a_R is a fixed element of A.

A set $\mathcal{A}$ of formulas in $\mathfrak{L}$ is said to be *valid in* R provided every formula a in $\mathcal{A}$ is valid in R. In this case we shall also say that R is a *model for* $\mathcal{A}$, or that R is a *model for the intuitionistic theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_i, \mathcal{A}\}$ [1]. By definition, every realization R is a model for $\mathfrak{S}_i = \{\mathfrak{L}, \mathfrak{C}_i, 0\}$.

A realization (or: model) R of $\mathfrak{L}$ in a set $J \neq 0$ and a pseudo-Boolean algebra A is said to be *enumerable* if the set J is enumerable.

Modifying the definition from IX, § 2, p. 383, we shall say that a formula a in $\mathfrak{L}$ is *intuitionistically valid*, or: is an *intuitionistic predicate tautology* if a is valid in every realization R in any pseudo-Boolean algebra [2].

Let $\mathfrak{L}_0$ be a formalized language of the zero order with a set V_0 of propositional variables. We recall (see VI, § 2) that if s is a substitution from $\mathfrak{L}_0$ into $\mathfrak{L}$, then for every formula β in $\mathfrak{L}_0$ the result $s\beta$ of the substitution s in β is a formula in $\mathfrak{L}$.

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[1] The above notion of satisfiability, validity and model is due to Rasiowa [2], [4]; see also Rasiowa and Sikorski [3].

[2] For another definition see Beth [3]; see also Kreisel [1], [2], [3] and Kleene [2].

2.1. *If β is an intuitionistic propositional tautology and s is a substitution from $\mathfrak{L}_0$ into $\mathfrak{L}$, then $s\beta$ is an intuitionistic predicate tautology.*

Suppose that R is a realization of $\mathfrak{L}$ in a set $J \neq 0$ and a pseudo-Boolean algebra $\boldsymbol{A}$. If v: $V \to J$ is a valuation of $\mathfrak{L}$ in J, then the equation

$$v_0(a) = (sa)_R(v) \quad \text{for} \quad a \in V_0 \tag{3}$$

defines a valuation v_0: $V_0 \to \boldsymbol{A}$. By VI, 6.9, for every formula β in $\mathfrak{L}_0$

$$(s\beta)_R(v) = \beta_{\boldsymbol{A}}(v_0)\,. \tag{4}$$

Thus if β is an intuitionistic propositional tautology, then $\beta_{\boldsymbol{A}}(v_0) = \mathrel{|}\!-$. By (4), $(s\beta)_R(v) = \mathrel{|}\!-$ for every valuation v, i.e. $s\beta$ is valid in every realization R. This proves 2.1.

2.2. *If a realization R is a model for an intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_i, \mathcal{A}\}$, then every theorem a of $\mathfrak{T}$ is valid in R.*

We have to show that the set $F(R)$ of all formulas valid in R contains the set $\mathfrak{C}_i(\mathcal{A})$ of all theorems.

Since every logical axiom of the intuitionistic theory $\mathfrak{T}$ is the result of a substitution s in an intuitionistic prepositional tautology, by 2.1 all logical axioms for $\mathfrak{T}$ belong to $F(R)$. Since R is a model for $\mathfrak{T}$, $F(R)$ contains the set $\mathcal{A}$. Thus in order to prove the inclusion $\mathfrak{C}_i(\mathcal{A}) \subset F(R)$ it suffices to show that $F(R)$ is closed under the rules of inference V, § 8, (r_1)-(r_6). The proof is analogous to the proof of the analogous statement for classical theories (see the proof of VIII, 2.2) and therefore it is omitted.

2.3. *Every formula derivable in the intuitionistic predicate calculus is an intuitionistic predicate tautology.*

This follows directly from 2.2 (the case in which $\mathcal{A}$ is the empty set). The statement converse to 2.3 is also true but it will be proved later (see § 4).

2.4. *If an intuitionistic theory $\mathfrak{T}$ has a model R in a non-degenerate pseudo-Boolean algebra, then $\mathfrak{T}$ is consistent.*

The proof is the same as that of the analogous statement VIII, 2.4. Suppose that a is a theorem in $\mathfrak{T}$. By 2.2, $a_R(v) = \mathrel{|}\!-$ for any valuation in R. Thus $(-a)_R = \wedge$. By 2.2, the formula $-a$ is not a theorem in $\mathfrak{T}$.

2.5. *The intuitionistic predicate calculus $\mathfrak{S}_i = \{\mathfrak{L}, \mathfrak{C}_i\}$ is consistent.*

This follows directly from 2.4 since every realization is a model for the theory $\mathfrak{S}_i = \{\mathfrak{L}, \mathfrak{C}_i, 0\}$.

Let $\boldsymbol{B}$ be a non-degenerate complete pseudo-Boolean algebra and let h be a homomorphism from a non-degenerate pseudo-Boolean algebra $\boldsymbol{A}$ into $\boldsymbol{B}$ preserving all infinite joins and meets in $\boldsymbol{A}$. If R is a realization

of the language $\mathfrak{L}$ in $J \neq 0$ and in $\boldsymbol{A}$, then the following equalities define another realization (denoted by hR) of $\mathfrak{L}$ in J and $\boldsymbol{B}$:

$$\begin{aligned} \varphi_{hR} &= \varphi_R \quad \text{for every functor } \varphi \text{ in } \mathfrak{L}, \\ \varrho_{hR} &= h\varrho_R \quad \text{for every predicate } \varrho \text{ in } \mathfrak{L}. \end{aligned} \tag{5}$$

Under the hypotheses admitted above the following theorem holds.

2.6. *If R is a model for a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$, then hR is also a model for $\mathfrak{T}$. Moreover for any formula α*

$$a_{hR}(v) = h\big(a_R(v)\big) \quad \textit{for any valuation } v \textit{ in } R. \tag{6}$$

Identity (6) follows from VI, 6.6. If α is in $\mathcal{A}$, then $a_R(v) = \mathbb{V}$. By (6) $a_{hR}(v) = \mathbb{V}$ for any valuation v. Thus hR is a model.

Let a formalized language $\mathfrak{L}'$ be an extension of a language $\mathfrak{L}$ (see V, § 14) and let a realization R' of $\mathfrak{L}'$ in a set $J \neq 0$ and in a pseudo-Boolean algebra $\boldsymbol{A}$ be an extension of a realization R of $\mathfrak{L}$ in J and $\boldsymbol{A}$. Under the above hypotheses we have

2.7. *R is a model for a theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ if and only if R' is a model for the linguistic extension $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}_\iota, \mathcal{A}\}$ of $\mathfrak{T}$. More exactly, for any formula α in $\mathfrak{L}$ the following conditions hold:*

(i) *α is satisfiable in R if and only if α is satisfiable in R',*

(ii) *α is valid in R if and only if α is valid in R'.*

(i) and (ii) follow from VI, 6.4. Hence, by the definition of model, R is a model for $\mathfrak{T}$ if and only if R' is a model for $\mathfrak{T}'$ since $\mathcal{A}$ is a set of formulas in $\mathfrak{L}$.

2.8. *Let R be a model for an intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ in a set $J \neq 0$ and in a complete pseudo-Boolean algebra $\boldsymbol{A}$. Let f be a mapping from J onto a set J', and let R' be a realization of $\mathfrak{L}$ in J' and $\boldsymbol{A}$ such that*

$$\varphi_{R'}\big(f(j_1), \ldots, f(j_m)\big) = f\big(\varphi_R(j_1, \ldots, j_m)\big)$$

for every m-argument functor φ in $\mathfrak{L}$ ($m = 0, 1, 2, \ldots$) and

$$\varrho_{R'}\big(f(j_1), \ldots, f(j_m)\big) = \varrho_R(j_1, \ldots, j_m)$$

for every m-argument predicate ϱ in $\mathfrak{L}$ ($m = 1, 2, \ldots$). Then R' is a model for $\mathfrak{T}$ in J' and $\boldsymbol{A}$. Moreover, for any formula α in $\mathfrak{L}$,

α is valid in R if and only if α is valid in R'.

This is an immediate result of VI, 6.8.

By a *topological realization of* $\mathfrak{L}$ we shall understand any realization of $\mathfrak{L}$ in a set $J \neq 0$ and the pseudo-Boolean algebra $\mathfrak{G}(X)$ of all open

subsets of a topological space X. Any model of an intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$ in a set J and in the pseudo-Boolean algebra $\mathfrak{G}(X)$ is said to be a *topological model* for $\mathfrak{T}$.

According to VI, § 6, by a *semantic realization* of $\mathfrak{L}$ we shall understand any realization in a set $J \neq 0$ and in the two-element Boolean algebra $\boldsymbol{A}_0$. Similarly, every model for an intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$ in a set $J \neq 0$ and in the two-element Boolean algebra $\boldsymbol{A}_0$ is said to be a *semantic model for* $\mathfrak{T}$.

§ 3. Canonical models. Consistency and the existence of models [1]. Let $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$ be a consistent intuitionistic theory and let $\mathfrak{A}(\mathfrak{T})$ be the Q-algebra of $\mathfrak{T}$. By 1.1, $\mathfrak{A}(\mathfrak{T})$ has at least two elements. Let h be a Q-homomorphism (see VI § 11) from $\mathfrak{A}(\mathfrak{T})$ into a non-degenerate pseudo-Boolean algebra $\boldsymbol{A}$. Let R^0 be the mapping defined on the set of all functors in $\mathfrak{L}$ and on the set of all predicates in $\mathfrak{L}$ such that

(a) R^0 restricted to the set of all functors in $\mathfrak{L}$ is the canonical realization of terms in $\mathfrak{L}$, i.e.

(1) $\varphi_{R^0} = \varphi$ for any m-argument functor φ in $\mathfrak{L}$ $(m = 0, 1, \ldots)$.

More exactly (see VI, § 7, (1')) the mapping φ_{R^0}: $T^m \to T$ is defined as follows:

$$\varphi_{R^0}(\tau_1, \ldots, \tau_m) = \varphi(\tau_1 \ldots \tau_m)$$

where $\tau_1, \ldots, \tau_m$ are any terms in $\mathfrak{L}$.

(b) For every m-argument predicate ϱ in $\mathfrak{L}$ $(m = 1, 2, \ldots)$ and for arbitrary terms $\tau_1, \ldots, \tau_m$ in $\mathfrak{L}$

$$\varrho_{R^0}(\tau_1, \ldots, \tau_m) = h(\|\varrho(\tau_1 \ldots \tau_m)\|) \,. \tag{2}$$

By VI, 7.3, and 11.3 the mapping R^0 is a realization of $\mathfrak{L}$ in the set T of all terms in $\mathfrak{L}$ and in the algebra $\boldsymbol{A}$. According to VI, § 7, the mapping R^0 will be called the *canonical realization* for $\mathfrak{L}$ *determined by the* Q-*homomorphism* h.

We recall that any valuation v: $V \to T$ of $\mathfrak{L}$ in the set T of all terms in $\mathfrak{L}$ can be conceived of as a substitution in $\mathfrak{L}$. Therefore, for any term τ in $\mathfrak{L}$ and for any formula α in $\mathfrak{L}$, $v\tau$ and $v'\alpha$ will denote respectively the result of the substitution v in τ and in α. In particular for the identity valuation $\mathfrak{i} = \{x\}_{x \in V}$ (see VI, § 7) we have

$$\mathfrak{i}\tau = \tau \quad \text{and} \quad \mathfrak{i}'\alpha = \alpha \,, \tag{3}$$

for any term τ and for any formula α in $\mathfrak{L}$.

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[1] The main results of this section were given by Rasiowa [2], [4], Rasiowa and Sikorski [3], Sikorski [7].

The following theorem is a particular case of VI, 7.3 on account of VI, 11.3.

3.1. *If R^0 is the canonical realization determined by a* Q*-homomorphism h from $\mathfrak{A}(\mathfrak{T})$ into a pseudo-Boolean algebra $\boldsymbol{A}$, then for every valuation $v\colon V \to T$ and for every formula a in $\mathfrak{L}$*

$$a_{R^0}(v) = h(\|v'a\|)\,. \tag{4}$$

In particular for the identity valuation i

$$a_{R^0}(\mathrm{i}) = h(\|a\|)\,. \tag{5}$$

A model R for a theory $\mathfrak{T}$ is said to be *adequate* provided that, for any formula a in $\mathfrak{T}$, a is a theorem of $\mathfrak{T}$ if and only if a is valid in R.

3.2. *If $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ is a consistent intuitionistic theory, then any canonical realization R^0 determined by a* Q*-homomorphism h from $\mathfrak{A}(\mathfrak{T})$ into a non-degenerate pseudo-Boolean algebra $\boldsymbol{A}$ is a model for $\mathfrak{T}$ in the set T of all terms in $\mathfrak{L}$ and in $\boldsymbol{A}$.*

If h is a Q*-isomorphism (in particular if $\boldsymbol{A} = \mathfrak{A}(\mathfrak{T})$ and h is the identity mapping), then R^0 is an adequate model for $\mathfrak{T}$. Moreover, for any formula a in $\mathfrak{L}$,*

$$a \text{ is a theorem of } \mathfrak{T} \text{ if and only if } a_{R^0}(\mathrm{i}) = V\,, \tag{6}$$

where i *is the identity valuation.*

It follows from 3.1, 2.2 and 1.1 that for any formula a in $\mathcal{A}$

$$a_{R^0}(v) = h(\|v'a\|) = V$$

for every valuation $v\colon V \to T$ since the substitution $v'a$ is a theorem in $\mathfrak{T}$. Thus any mathematical axiom for $\mathfrak{T}$ is valid in R^0, i.e. R^0 is a model for $\mathfrak{T}$. If h is an Q-isomorphism from $\mathfrak{A}(\mathfrak{T})$ into a pseudo-Boolean algebra $\boldsymbol{A}$, then $h(\|a\|) = V$ if and only if $\|a\| = V$. Hence, by 1.1 and (5) we obtain (6). In consequence R^0 is an adequate model for $\mathfrak{T}$.

3.3. *Every consistent intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ has an adequate enumerable topological model in the set T of all terms and in the pseudo-Boolean algebra $\mathfrak{G}(X_0)$ of all open subsets of a set X_0 of irrational numbers.*

Since $\mathfrak{T}$ is consistent, the Q-algebra $\mathfrak{A}(\mathfrak{T})$ is non-degenerate. The set (Q) of all infinite joins and meets corresponding to quantifiers is enumerable in $\mathfrak{A}(\mathfrak{T})$ because the set of all formulas is enumerable. By IV, 10.1, there exists a set X_0 of irrational numbers and a Q-isomorphism h from $\mathfrak{A}(\mathfrak{T})$ into $\mathfrak{G}(X_0)$. By the second part of 3.2, the canonical realization for $\mathfrak{T}$, determined by h, is an adequate model for $\mathfrak{T}$ in T and $\mathfrak{G}(X_0)$.

3.4. *If a formula a is irrefutable in an intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_t, \mathfrak{A}\}$, then there exists an enumerable topological model R in the pseudo-Boolean algebra $\mathfrak{G}(X)$ of all open subsets of a set X of irrational numbers, such that a is satisfiable in R.*

Let h and X_0 have the same meanings as in the proof of 3.3, let $X = h(\|a\|) \in \mathfrak{G}(X_0)$ and let

$$h'(Y) = Y \cap X \quad \text{for all} \quad Y \in \mathfrak{G}(X_0).$$

Since a is irrefutable, $\|a\| \neq \Lambda$ and consequently $X \neq 0$. The superposition $h_0 = h'h$ of the Q-isomorphism $h: \mathfrak{A}(\mathfrak{T}) \to \mathfrak{G}(X_0)$ and of the homomorphism $h': \mathfrak{G}(X_0) \to \mathfrak{G}(X)$ preserving all infinite operations (see IV, 8.2) is a Q-homomorphism. The canonical realization R^0 determined by the Q-homomorphism h_0 has the required properties. In fact, it is a model for $\mathfrak{T}$ by 3.2. Moreover

$$a_{R^0}(\mathfrak{i}) = h_0(a_{R^0}(\mathfrak{i})) = X = V_{\mathfrak{G}(X)}$$

for the identity valuation $\mathfrak{i}$ (see 3.1, (5)). Thus a is satisfiable in R.

3.5. *If a formula a is not a theorem in an intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_t, \mathfrak{A}\}$, then there exists an enumerable topological model R in the pseudo-Boolean algebra $\mathfrak{G}(X_0)$ of all open subsets of a set X_0 of irrational numbers, such that a is not valid in R.*

Theorem 3.5 follows immediately from 3.3. As X_0 we can take the set X_0 mentioned in 3.3. Thus in theorem 3.5 we can add that the space X_0 does not depend on the formula a.

Also in theorem 3.4 we can add the remark that the space X_0 mentioned in 3.4 can be chosen in such a way that it is common to all irrefutable formulas a. This does not follow directly from the above proof of 3.4 but can be obtained additionally as follows. Let S be the set of all irrefutable formulas in $\mathfrak{T}$. By 3.4, for every a in S there exists a set X_a of irrational numbers such that a is satisfiable in $\mathfrak{G}(X_a)$. Let $X = \boldsymbol{P}_{a \in S} X_a$. Since the projection π_a from X onto X_a is an interior mapping (see III, § 9, p. 113), the equation

$$h_a(A) = \pi^{-1}(A) \quad \text{for} \quad A \in \mathfrak{G}(X)$$

defines a pseudo-Boolean isomorphism h_a from $\mathfrak{G}(X_a)$ into $\mathfrak{G}(X)$ (see IV, 2.2). If R is a realization in $\mathfrak{G}(X_a)$ and v is a valuation such that $a_R(v) = X_a$, then $a_{h_a R}(v) = X$ by 2.6. Thus every irrefutable formula is satisfiable in $\mathfrak{G}(X)$. Denote by Y the space of all irrational numbers. Since $X_a \subset Y$, we have $X \subset Y^S$. Since the set S is enumerable, the space Y^S is homeomorphic to Y (see III, 9.1). Therefore the set X can be interpreted as a subset of the space of all irrational numbers.

The following theorem states some relationships which hold between the consistency of a theory and the existence of various kinds of models.

3.6. *The following conditions are equivalent for any intuitionistic theory* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$

(i) $\mathcal{T}$ *is consistent,*

(ii) *there exists a model for* $\mathcal{T}$,

(iii) *there exists a model for* $\mathcal{T}$ *in a complete pseudo-Boolean algebra,*

(iv) *there exists a topological model for* $\mathcal{T}$,

(v) *there exists an adequate enumerable topological model for* $\mathcal{T}$ *in the pseudo-Boolean algebra* $\mathfrak{G}(X_0)$ *of all open subsets of a set* X_0 *of irrational numbers.*

This follows directly from 3.3 and 2.4.

We shall now formulate conditions necessary and sufficient for any formula to be a theorem of an enumerable theory.

3.7. *For any formula* α *in a consistent intuitionistic theory* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$, *the following conditions are equivalent:*

(i) α *is a theorem of* $\mathcal{T}$,

(ii) α *is valid in every model for* $\mathcal{T}$,

(iii) α *is valid in every model for* $\mathcal{T}$ *in any complete pseudo-Boolean algebra,*

(iv) α *is valid in every topological model for* $\mathcal{T}$,

(v) α *is valid in every model for* $\mathcal{T}$ *in an enumerable set* J_0 *and in every pseudo-Boolean algebra* $\mathfrak{G}(X_0)$ *where* X_0 *is a set of irrational numbers,*

(vi) *there is an infinite set* J *such that* α *is valid in every model for* $\mathcal{T}$ *in* J *and in every pseudo-Boolean algebra* $\mathfrak{G}(X_0)$ *where* X_0 *is a set of irrational numbers,*

(vii) $a_{R^0}(\mathfrak{i}) = V$ *for the canonical model* R^0 *for* $\mathcal{T}$ *in* $\mathfrak{A}(\mathcal{T})$ *and for the identity valuation* $\mathfrak{i}$.

(i) implies (ii) by 2.2. Obviously (ii) implies (iii) and (vii), (iii) implies (iv), (iv) implies (v) and (v) implies (vi). (v) implies (i) by 3.5. (vii) implies (i) by the second part of 3.2 (the case where $\boldsymbol{A} = \mathfrak{A}(\mathcal{T})$).

(vi) implies (v). In fact, suppose that (v) does not hold, i.e. there exists a model R for $\mathcal{T}$ in an enumerable set J_0 and in the pseudo-Boolean algebra $\mathfrak{G}(X_0)$ (where X_0 is a set of irrational numbers) such that α is not valid in R. Since J is infinite and J_0 is enumerable, there exists a mapping f from J onto J_0. Since f is a mapping onto J_0, there exists

a realization R' in J and $\mathfrak{G}(X_0)$ such that

$$f(\varphi_{R'}(j_1, \ldots, j_m)) = \varphi_R(f(j_1), \ldots, f(j_m)),$$

$$\varrho_{R'}(j_1, \ldots, j_m) = \varrho_R(f(j_1), \ldots, f(j_m))$$

for every m-argument functor $\varphi (m = 0, 1, 2, \ldots)$, for every m-argument predicate $\varrho (m = 1, 2, \ldots)$ and for all $j_1, \ldots, j_m \in J$. By 2.8, R' is a model for $\mathfrak{T}$ in J and $\mathfrak{G}(X_0)$, and a is not valid in R', i.e. (vi) does not hold.

Let a formalized language $\mathfrak{L}'$ be an extension of a language $\mathfrak{L}$. Under the above hypothesis the following theorem holds.

3.8. *A formula a in $\mathfrak{L}$ is a theorem of an intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_i, \mathfrak{A}\}$ if and only if it is a theorem of the theory $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}_i, \mathfrak{A}\}$, i.e. every linguistic extension $\mathfrak{T}'$ of $\mathfrak{T}$ is inessential (see* V, § 14). *In consequence $\mathfrak{T}$ is consistent if and only if $\mathfrak{T}'$ is consistent.*

If a is a theorem of $\mathfrak{T}$, then clearly it is a theorem of $\mathfrak{T}'$. Conversely, if a is not a theorem of $\mathfrak{T}$, then by 3.5 a is not valid in a model R for $\mathfrak{T}$ in a set $J \neq 0$ and in a complete pseudo-Boolean algebra $\boldsymbol{A}$. Let R' be an extension of the realization R (in J and $\boldsymbol{A}$) to a realization of $\mathfrak{L}'$. By 2.7 R' is a model for $\mathfrak{T}'$ and a is not valid in R'. By 2.2, a is not a theorem of $\mathfrak{T}$.

The second part of 3.8 results from the first.

§ 4. The completeness of intuitionistic predicate calculi. According to the convention in § 1, p. 412, and in V, § 10, p. 187, the intuitionistic predicate calculus $\mathfrak{S}_i = \{\mathfrak{L}, \mathfrak{C}_i\}$ is identified with the theory $\{\mathfrak{L}, \mathfrak{C}_i, 0\}$ having the empty set of mathematical axioms. Therefore we can apply to $\mathfrak{S}_i$ all the results obtained in previous sections.

Since $\mathfrak{S}_i$ has the empty set of mathematical axioms, every realization R of $\mathfrak{L}$ in a set $J \neq 0$ and in a pseudo-Boolean algebra $\boldsymbol{A}$ is a model for $\mathfrak{S}_i$. Thus the notion of model for $\mathfrak{S}_i$ coincides with the notion of realization. By the terminology adopted in § 1, a formula in $\mathfrak{L}$ is a theorem of $\{\mathfrak{L}, \mathfrak{C}_i, 0\}$ if and only if it is derivable in $\mathfrak{S}_i$. Thus the following theorem is a particular case of theorem 3.7.

4.1. *For every formula a in the intuitionistic predicate calculus $\mathfrak{S}_i = \{\mathfrak{L}, \mathfrak{C}_i\}$ the following conditions are equivalent:*

(i) *a is derivable in $\mathfrak{S}_i$;*

(ii) *a is valid in every realization of $\mathfrak{L}$;*

(iii) *a is valid in every realization of $\mathfrak{L}$ in any complete pseudo-Boolean algebra;*

(iv) *a is valid in every topological realization of $\mathfrak{L}$;*

(v) *α is valid in every realization of $\mathfrak{L}$ in an enumerable set J_0 and in every pseudo-Boolean algebra $\mathfrak{G}(X_0)$ where X_0 is a set of irrational numbers*;

(vi) *there is an infinite set J such that α is valid in every realization of $\mathfrak{L}$ in J and in every pseudo-Boolean algebra $\mathfrak{G}(X_0)$ where X_0 is a set of irrational numbers*;

(vii) $a_{R^0}(\mathrm{i}) = V$ *for the canonical realization R^0 in $\mathfrak{A}(\mathfrak{S}_\iota)$, and the identity valuation* i [1].

Theorem 4.1, which is an intuitionistic analogue of theorem VIII, 6.1, is called the *completeness theorem for the intuitionistic predicate calculus* $\mathfrak{S}_\iota$. Theorem 4.1 states that the phrases "intuitionistic predicate tautology" and "formula derivable in $\mathfrak{S}_\iota$" have the same meaning. In practice we shall use the former more frequently.

The following theorem follows immediately from 3.3.

4.2. *There exists a metric space X_0 such that, for every formula α, α is derivable in $\mathfrak{S}_\iota$ if and only if α is valid in every realization in an enumerable set J and the pseudo-Boolean algebra $\mathfrak{G}(X_0)$* [2].

It follows from the proof of 3.3 that as X_0 we can take any set such that there exists a Q-isomorphism from $\mathfrak{A}(\mathfrak{S}_\iota)$ into $\mathfrak{G}(X_0)$. We can take as X_0 a set of irrational numbers. It is not known whether we can take as X_0 in 4.2 the set of all irrational numbers or the set of all real numbers.

It follows from 4.1 that if a formula α is an intuitionistic predicate tautology, then there exists a formal proof of α in $\mathfrak{S}_\iota$. Theorem 4.1 does not yield any method of construction of the formal proof. In the case of the classical predicate calculus such a method was described in VIII, § 7. A similar method also exists for the intuitionistic predicate calculus but it is omitted here.

§ 5. The algebra of the intuitionistic predicate calculus. Let E be the set of all elementary formulas in an intuitionistic predicate calculus $\mathfrak{S}_\iota = \{\mathfrak{L}, \mathfrak{C}_\iota\} = \{\mathfrak{L}, \mathfrak{C}_\iota, 0\}$.

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[1] The implication (i)—(iii) was proved by Mostowski [4]. The converse statement was proved by Rasiowa [2]. Independently Henkin [2] proved the equivalence of (i) and (ii). Both authors used the equivalence of (i) and (vii). The equivalence of (i) and (iv) was stated by Rasiowa and Sikorski [3]. The equivalence of (i) and (v) was proved by Sikorski [7].

It follows from the results of Beth [3] (see also Beth [4]) — according to his remark announced at the Colloquium in Amsterdam (1957) — that the intuitionistic predicate calculus is complete even when the realizations are restricted to realizations in topological spaces which are closed subsets of Cantor's discontinuum; for a proof see Dyson and Kreisel [1].

[2] Sikorski [7]. See also Rasiowa and Sikorski [3].

5.1. *The* Q*-algebra* $\mathfrak{A}(\mathfrak{S}_i)$ *is a generalized free algebra for the class of all complete pseudo-Boolean algebras, the set of all elements* $\|\alpha\|$ *with* α *in* E *being the set of free generators.*

The proof is similar to that of the analogous theorem VIII, 24.1, on the algebra of a two-valued predicate calculus.

Let f be a mapping defined on the set of all $\|\alpha\| \in \mathfrak{A}(\mathfrak{S}_i)$ where α is in E, with values in a complete pseudo-Boolean algebra $\boldsymbol{A}$. The mapping f induces a mapping f_0: $\overline{F} \to \boldsymbol{A}$ defined as follows:

$$f_0(|\alpha|) = f(\|\alpha\|) \quad \text{for} \quad \alpha \text{ in } E .$$

We recall that $\overline{F}$ denotes the Q-algebra of the language $\mathfrak{L}$ and $|\alpha|$ denotes the element of $\overline{F}$ determined by a formula α (see VI, § 4). Since $\overline{F}$ is a free generalized algebra (see VI, 4.1), the mapping f_0 can be extended to a homomorphism g from $\{\overline{F}, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ into $\{\boldsymbol{A}, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$.

Let R be the realization of $\mathfrak{L}$ in the set T of all terms and in the complete pseudo-Boolean algebra $\boldsymbol{A}$, such that R is the canonical realization of terms and, for every m-argument predicate ϱ $(m = 1, 2, \ldots)$ and for any terms $\tau_1, \ldots, \tau_m$

$$\varrho_R(\tau_1, \ldots, \tau_m) = g(|\varrho(\tau_1 \ldots \tau_m)|) \in \boldsymbol{A} .$$

By VI 7.2, for every formula γ

$$\gamma_R(\mathfrak{i}) = g(|\gamma|) , \tag{1}$$

where $\mathfrak{i}$ is the identity valuation.

Suppose that α and β are formulas such that $\|\alpha\| = \|\beta\|$, i.e. both $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are predicate intuitionistic tautologies (see 4.1). Then

$$(\alpha \Rightarrow \beta)_R(\mathfrak{i}) = V = (\beta \Rightarrow \alpha)_R(\mathfrak{i}) .$$

Hence, on account of (1)

$$g(|\alpha|) \Rightarrow g(|\beta|) = (\alpha \Rightarrow \beta)_R(\mathfrak{i}) = V$$

and similarly

$$g(|\beta|) \Rightarrow g(|\alpha|) = (\beta \Rightarrow \alpha)_R(\mathfrak{i}) = V .$$

This proves that

$$\|\alpha\| = \|\beta\| \quad \text{implies} \quad g(|\alpha|) = g(|\beta|) .$$

Consequently the formula

$$h(\|\alpha\|) = g(|\alpha|) \quad \text{for} \quad \alpha \text{ in } F$$

defines a mapping h: $\mathfrak{A}(\mathfrak{S}_i) \to \boldsymbol{A}$. It is easy to verify that the mapping h is a Q-homomorphism and that it is an extension of f.

We have proved that every mapping f defined on the set of all $\|\alpha\|$ with α in E, with values in a complete pseudo-Boolean algebra A, can be extended to a homomorphism $h\colon \mathfrak{A}(\mathfrak{S}_t) \to A$. Thus $A(\mathfrak{S}_t)$ is a generalized free algebra for the class of all complete pseudo-Boolean algebras, and the set of all $\|\alpha\|$, where α is in E, is a set of free generators.

§ 6. Examples of intuitionistic tautologies. The simplest examples of intuitionistic tautologies are yielded by theorem 2.1 stating that all formulas in $\mathfrak{L}$ which are substitutions of intuitionistic propositional tautologies are intuitionistic predicate tautologies. In particular, all formulas of the form (T_0), (T_{13})-(T_{17}), (T_{19}), (T_{21})-(T_{23}), (T_{25})-(T_{27}), (T_{65})-(T_{68}) (see IX, 4.1) where α, β, γ are any formulas in $\mathfrak{L}$ are intuitionistic predicate tautologies.

We shall complete theorem 6.1 by the following two theorems.

6.1. *If a substitution s is a one-to-one mapping from the set V_0 of all propositional variables of a language $\mathfrak{L}_0$ of zero order into the set of all elementary formulas of a language $\mathfrak{L}$ of the first order, then for every formula δ in $\mathfrak{L}_0$, δ is an intuitionistic propositional tautology if and only if $s\delta$ is an intuitionistic tautology.*

By 6.1 it suffices to prove that if δ is not an intuitionistic propositional tautology, then $s\delta$ is not an intuitionistic predicate tautology.

By IX, 3.2, there exists a valuation $v = \{v_a\}_{a \in V_0} \in A^{V_0}$ (where A is a complete pseudo-Boolean algebra and V_0 is the set of all propositional variables in $\mathfrak{L}_0$) such that

$$\delta_A(v) \neq V .$$

We define a realization R of $\mathfrak{L}$ in the set T of all terms in $\mathfrak{L}$ as follows. R is a canonical realization of terms, i.e.

$$\varphi_R(\tau_1, \ldots, \tau_m) = \varphi(\tau_1 \ldots \tau_m)$$

for every m-argument functor $\varphi (m = 0, 1, 2, \ldots)$. For every m-argument predicate $\varrho\ (m = 1, 2, \ldots)$

$$\varrho_R(\tau_1, \ldots, \tau_m) = \begin{cases} v_a & \text{if } sa \text{ is the elementary formula } \varrho(\tau_1, \ldots, \tau_m), \\ V & \text{otherwise.} \end{cases}$$

Let $\mathfrak{i}\colon V \to T$ be the identity valuation in $\mathfrak{L}$. We have

$$sa_R(\mathfrak{i}) = v_a$$

and consequently

$$(s\delta)_R(\mathfrak{i}) = \delta_A(s\mathfrak{i}) = \delta_A(v) \neq V$$

on account of VI, 6.9, since

$$s\mathfrak{i} = \{sa_R(\mathfrak{i})\}_{a \in V_0} = \{v_a\}_{a \in V_0} = v .$$

Since $s\delta_R$ is not identically equal to $\vee$, $s\delta$ is not an intuitionistic predicate tautology.

Observe that in the above theorem it suffices to assume that s is one-to-one on the set of all propositional variables which appear in δ.

Every open formula α in a formalized language $\mathfrak{L}$ of the first order is of the form $s\delta$ where s is a one-to-one mapping from a language $\mathfrak{L}_0$ of zero order (with an enumerable set V_0 of propositional variables) into the set of all elementary formulas in $\mathfrak{L}$. Hence, by 2.1 and 6.1, follows

6.2. *An open formula in a formalized language $\mathfrak{L}$ of the first order is an intuitionistic predicate tautology if and only if it is a result of a substitution performed on an intuitionistic propositional tautology.*

The following theorem is an intuitionistic analogue of VIII, 6.4.

6.3. *All formulas of any of the following forms are derivable in* $\mathfrak{S}_i = \{\mathfrak{L}, \mathfrak{C}_i\}$:

(T_{31}) $(\bigcap_\xi \alpha(\xi) \Rightarrow \alpha(\tau))$, *where τ is an arbitrary term,*

(T_{32}) $(\alpha(\tau) \Rightarrow \bigcup_\xi \alpha(\xi))$, *where τ is an arbitrary term,*

(T_{33}) $(\bigcap_\xi \alpha(\xi) \Rightarrow \bigcup_\xi \alpha(\xi))$,

(T_{35}) $(\bigcup_\xi -\alpha(\xi) \Rightarrow -\bigcap_\xi \alpha(\xi))$,

(T_{36}) $(-\bigcup_\xi \alpha(\xi) \Rightarrow \bigcap_\xi -\alpha(\xi))$,

(T_{37}) $(\bigcap_\xi -\alpha(\xi) \Rightarrow -\bigcup_\xi \alpha(\xi))$,

(T_{38}) $(\bigcap_\xi (\alpha(\xi) \Rightarrow \beta) \Rightarrow (\bigcup_\xi \alpha(\xi) \Rightarrow \beta))$,

(T_{39}) $((\bigcup_\xi \alpha(\xi) \Rightarrow \beta) \Rightarrow (\bigcap_\xi (\alpha(\xi) \Rightarrow \beta))$,

(T_{40}) $(\bigcap_\xi (\alpha \Rightarrow \beta(\xi)) \Rightarrow (\alpha \Rightarrow \bigcap_\xi \beta(\xi)))$,

(T_{41}) $((\alpha \Rightarrow \bigcap_\xi \beta(\xi)) \Rightarrow \bigcap_\xi (\alpha \Rightarrow \beta(\xi)))$,

(T_{42}) $(\bigcup_\xi (\alpha(\xi) \Rightarrow \beta) \Rightarrow (\bigcap_\xi \alpha(\xi) \Rightarrow \beta))$,

(T_{44}) $(\bigcup_\xi (\alpha \Rightarrow \beta(\xi)) \Rightarrow (\alpha \Rightarrow \bigcup_\xi \alpha(\xi)))$,

(T_{46}) $(\bigcap_\xi (\alpha \cap \beta(\xi)) \Rightarrow (\alpha \cap \bigcap_\xi \beta(\xi)))$,

(T_{47}) $((\alpha \cap \bigcap_\xi \beta(\xi)) \Rightarrow \bigcap_\xi (\alpha \cap \beta(\xi)))$,

(T_{49}) $((\alpha \cup \bigcap_\xi \beta(\xi)) \Rightarrow \bigcap_\xi (\alpha \cup \beta(\xi)))$,

(T_{50}) $(\bigcup_\xi (\alpha \cap \beta(\xi)) \Rightarrow (\alpha \cap \bigcup_\xi \beta(\xi)))$,

(T_{51}) $((\alpha \cap \bigcup_\xi \beta(\xi)) \Rightarrow \bigcup_\xi (\alpha \cap \beta(\xi)))$,

(T_{52}) $(\bigcup_\xi (\alpha \cup \beta(\xi)) \Rightarrow (\alpha \cup \bigcup_\xi \beta(\xi)))$,

(T_{53}) $((\alpha \cup \bigcup_\xi \beta(\xi)) \Rightarrow (\bigcup_\xi (\alpha \cup \beta(\xi)))$,

(T_{54}) $(\bigcap_\xi (\alpha(\xi) \Rightarrow \beta(\xi)) \Rightarrow (\bigcap_\xi \alpha(\xi) \Rightarrow \bigcap_\xi \beta(\xi)))$,

(T_{55}) $(\bigcap_\xi (\alpha(\xi) \Rightarrow \beta(\xi)) \Rightarrow (\bigcup_\xi \alpha(\xi) \Rightarrow \bigcup_\xi \beta(\xi)))$,

(T_{56}) $(\bigcap_\xi (\alpha(\xi) \cap \beta(\xi)) \Rightarrow (\bigcap_\xi \alpha(\xi) \cap \bigcap_\xi \beta(\xi)))$,

(T_{57}) $((\bigcap_\xi \alpha(\xi) \cap \bigcap_\xi \beta(\xi)) \Rightarrow \bigcap_\xi (\alpha(\xi) \cap \beta(\xi)))$,

(T_{58}) $(\bigcup_\xi (\alpha(\xi) \cup \beta(\xi)) \Rightarrow (\bigcup_\xi \alpha(\xi) \cup \bigcup_\xi \beta(\xi)))$,

(T_{59}) $((\bigcup_\xi \alpha(\xi) \cup \bigcup_\xi \beta(\xi)) \Rightarrow \bigcup_\xi (\alpha(\xi) \cup \beta(\xi)))$,

(T_{60}) $((\bigcap_\xi \alpha(\xi) \cup \bigcap_\xi \beta(\xi)) \Rightarrow \bigcap_\xi (\alpha(\xi) \cup \beta(\xi)))$,

(T_{61}) $(\bigcup_\xi (\alpha(\xi) \cap \beta(\xi)) \Rightarrow (\bigcup_\xi \alpha(\xi) \cap \bigcup_\xi \beta(\xi)))$,

(T_{62}) $(\bigcup_\xi \bigcap_\eta \alpha(\xi, \eta) \Rightarrow \bigcap_\eta \bigcup_\xi \alpha(\xi, \eta))$,

(T_{63}) $(\bigcap_\xi \bigcap_\eta \alpha(\xi, \eta) \Rightarrow \bigcap_\eta \bigcap_\xi \alpha(\xi, \eta))$,

(T_{64}) $(\bigcup_\xi \bigcup_\eta \alpha(\xi, \eta) \Rightarrow \bigcup_\eta \bigcup_\xi \alpha(\xi, \eta))$.

We shall indicate only the number of statements in I, § 7, and IV, § 7 from which the intuitionistic validity of the formulas in question follows:

(T_{31}) I, 7.1, (3);	(T_{32}) I, 7.1, (2);	(T_{33}) I, 7.1, (6);
(T_{35}) IV, 7.3, (9);	(T_{36}) IV, 7.3, (10);	(T_{37}) IV, 7.3, (10);
(T_{38}) IV, 7.2, (7);	(T_{39}) IV, 7.2, (7);	(T_{40}) IV, 7.2, (8);
(T_{41}) IV, 7.2, (8);	(T_{42}) IV, 7.2, (3);	(T_{44}) IV, 7.2, (4);
(T_{46}) I, 7.1, (10);	(T_{47}) I, 7.1, (10);	(T_{49}) IV, 7.2, (2);
(T_{50}) IV, 7.1, (1);	(T_{51}) IV, 7.1, (1);	(T_{52}) I, 7.1, (9);
(T_{53}) I, 7.1, (9);	(T_{54}) IV, 7.2, (5);	(T_{55}) IV, 7.2, (6);
(T_{56}) I, 7.1, (15);	(T_{57}) I, 7.1, (15);	(T_{58}) I, 7.1, (14);
(T_{59}) I, 7.1, (14);	(T_{60}) I, 7.1, (16);	(T_{61}) I, 7.1, (17);
(T_{62}) I, 7.1, (13);	(T_{63}) I, 7.1, (12);	(T_{64}) I, 7.1, (11).

The list of intuitionistic predicate tautologies quoted in 6.3 does not contain all classical tautologies mentioned in the analogous theorem VIII, 6.4. Namely, it does not contain the tautologies (T_{34}), (T_{43}), (T_{45}) and (T_{48}). In fact, if π and ϱ are one-argument predicates, then the following formulas

(1) $$(-\bigcap_\xi \pi(\xi) \Rightarrow \bigcup_\xi -\pi(\xi)),$$

(2) $$((\bigcap_\xi \pi(\xi) \Rightarrow \varrho(x)) \Rightarrow \bigcup_\xi (\pi(\xi) \Rightarrow \varrho(x)),$$

(3) $$((\pi(x) \Rightarrow \bigcup_\xi \varrho(\xi)) \Rightarrow \bigcup_\xi (\pi(x) \Rightarrow \varrho(\xi)),$$

(4) $$(\bigcap_\xi (\pi(x) \cup \varrho(\xi)) \Rightarrow (\pi(x) \cup \bigcap_\xi \varrho(\xi)),$$

are of the form (T_{34}), (T_{43}), (T_{45}), (T_{48}) respectively but are not intuitionistic tautologies [1]. To prove this it suffices to show that they are

[1] The first algebraic proof that (4) is not an intuitionistic tautology was given by Mostowski [4].

not valid in a realization R in a set $J \neq 0$ and in the algebra $\mathfrak{G}(X)$ of open subsets of a topological space X. Take as J the set of all positive integers j and take as X the space of all real numbers w. Let $\{w_j\}$ be a sequence composed of all rational numbers.

If

$$\pi_R(j) = \text{the set of all } w \neq w_j \ (w \in X),$$

then formula (1) is not valid in R.

If $\pi_R(j)$ is defined as above and $\varrho_R(j) = 0$ for all j, then formula (2) is not valid in R.

If

$$\pi_R(j) = \text{the set of all } w \neq 0,$$

$$\varrho_R(j) = \text{the set of all } w \text{ such that } |w| > 1/j$$

then formula (3) is not valid in R.

If

$$\pi_R(j) = \text{the set of all } w \neq 0,$$

$$\varrho_R(j) = \text{the set of all } w \text{ such that } |w| < 1/j,$$

then formula (4) is not valid in R.

Note moreover that neither of the tautologies

$$\bigcap_\xi \big(\varrho(\xi) \cup -\varrho(\xi)\big), \tag{5}$$

$$--\bigcap_\xi \big(\varrho(\xi) \cup -\varrho(\xi)\big) \tag{6}$$

is an intuitionistic tautology [1]. In fact, let α denote formula (5). If

$$\varrho_R(j) = \text{the set of all real numbers } w \neq w_j, \tag{7}$$

then

$$\alpha_R \equiv 0, \tag{8}$$

and consequently

$$(-\alpha)_R \equiv X, \tag{8}$$

$$(--\alpha)_R \equiv 0. \tag{9}$$

The formula $--\alpha$ coincides with (6).

§ 7. The connection between tautologies and intuitionistic tautologies. This section is an analogue to IX, § 5, where the same problem was investigated for propositional calculi. However, there is no complete analogy between the case of propositional calculi and predicate calculi. Instead of the equivalence of conditions (i) and (ii) in IX, 5.1, we have only the following results.

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[1] Heyting [2], Kleene [2] with Nelson [1].

7.1. *Let α be a formula in a language $\mathfrak{L}$ of the first order. If for every realization R in a set $J \neq 0$ and any non-degenerate pseudo-Boolean algebra $\boldsymbol{A}$ and for every valuation v, the element $\alpha_R(v)$ is dense in $\boldsymbol{A}$, then α is a predicate tautology.*

To prove this it suffices to take as $\boldsymbol{A}$ the two-element Boolean algebra. Since the unit element is the only dense element in $\boldsymbol{A}$, we have $\alpha_R(v) = V$ for every realization R in $\boldsymbol{A}$ and for every valuation v, which proves that α is a predicate tautology.

The converse statement is not true, in general. There are predicate tautologies α such that $\alpha_R(v)$ is not dense for a realization R in a pseudo-Boolean algebra $\boldsymbol{A}$ and a valuation v (see e.g. § 6, (5), and (7), (8)).

We shall now prove a theorem which is a partial converse of 7.1. For this purpose we shall introduce an auxiliary consequence operation $\mathfrak{C}^*$ in $\mathfrak{L}$. Namely, for every set S of formulas in $\mathfrak{L}$, $\mathfrak{C}^*(S)$ will denote the least set (of formulas in $\mathfrak{L}$) containing all formulas in S and all logical axioms for the classical predicate calculus (i.e. all formulas in $\mathfrak{L}$ of the form (T_1)-(T_{12})—see V, § 11) and closed under the rules of inference (r_1), (r_2), (r_5), (r_6). In other words, the definition of $\mathfrak{C}^*$ is obtained from the definition of the classical consequence operation $\mathfrak{C}$ by cancelation of the rules of introduction of existential and universal quantifiers. By definition,

$$\mathfrak{C}^*(S) \subset \mathfrak{C}(S) .$$

7.2. *If a formula α in $\mathfrak{L}$ belongs to $\mathfrak{C}^*(0)$, then for every realization R of $\mathfrak{L}$ in any set $J \neq 0$ and in any non-degenerate pseudo-Boolean algebra $\boldsymbol{A}$, the element $\alpha_R(v)$ is dense in $\boldsymbol{A}$ for every valuation v.*

Let D be the set of all formulas α such that $\alpha_R(v)$ is always dense. It follows from VI, 6.9, and from IX, 5.1, that D contains all formulas which are obtained by a substitution in a propositional tautology. In particular D contains all the logical axioms of the classical predicate calculus. It follows from IV, 5.6, and the identity $(\beta \Rightarrow \gamma)_R(v) = \beta_R(v) \Rightarrow \gamma_R(v)$ that D is closed under *modus ponens* (r_1). D is also closed under the rule (r_2) of substitution for free individual variables on account of VI, 6.7, (10). Since

$$(\bigcup_\xi \beta(\xi) \Rightarrow \gamma)_R(v) \leqslant (\beta(x) \Rightarrow \gamma)_R(v) \text{ and } (\beta \Rightarrow \bigcap_\xi \gamma(\xi))_R(v) \leqslant (\beta \Rightarrow \gamma(x))_R(v),$$

by I, 12.2, (11) and (12), the set D is also closed under the rules (r_5) and (r_6) of elimination of existential and universal quantifiers by IV, 5.5. This proves that $\mathfrak{C}^*(0) \subset D$.

7.3. *For any formula α in $\mathfrak{L}$, if α is in $\mathfrak{C}^*(0)$, then $--\alpha$ is an intuitionistic predicate tautology* [1].

[1] Cf. Kleene [3], p. 492.

Suppose that α is in $\mathbf{C}^*(0)$. Then, by 7.2, $-\alpha_R(v) = \wedge$ for every realization R of $\mathfrak{L}$ in any set $J \neq 0$ and in any pseudo-Boolean algebra $\boldsymbol{A}$. Since $-\wedge = \vee$, we obtain $--\alpha_R(v) = \vee$. This proves that $--\alpha$ is an intuitionistic predicate tautology.

7.4. *For any formula α in $\mathfrak{L}$, if $-\alpha$ is in $\mathbf{C}^*(0)$, then $-\alpha$ is an intuitionistic predicate tautology.*

Suppose that $-\alpha$ is in $\mathbf{C}^*(0)$. By 7.2, $-\alpha_R(v)$ is dense in every realization R in any set $J \neq 0$ and in every pseudo-Boolean algebra $\boldsymbol{A}$. By IV, 6.1, and IV, 6.6, $-\alpha_R(v) = \vee$, which proves that $-\alpha$ is an intuitionistic predicate tautology.

7.5. *For any formulas α, β in $\mathfrak{L}$, if $(\alpha \Rightarrow -\beta)$ is in $\mathbf{C}^*(0)$, then $(\alpha \Rightarrow -\beta)$ is an intuitionistic tautology.*

If $(\alpha \Rightarrow -\beta)$ is in $\mathbf{C}^*(0)$, then by 7.2 for any realization R in every set $J \neq 0$ and in any pseudo-Boolean algebra $\boldsymbol{A}$ the element $(\alpha \Rightarrow -\beta)_R(v) = \alpha_R(v) \Rightarrow -\beta_R(v)$ is dense in $\boldsymbol{A}$. By IV, 6.3, and 6.6 $(\alpha \Rightarrow -\beta)_R(v) = \vee$, which proves that $(\alpha \Rightarrow -\beta)$ is an intuitionistic predicate tautology.

§ 8. Theorem on intuitionistically derivable disjunctions and existential formulas. In a classical predicate calculus it can happen that a disjunction $(\alpha \cup \beta)$ is a predicate tautology but neither α nor β is a predicate tautology. As an example we quote here the disjunction $(\varrho(x) \cup -\varrho(x))$ where ϱ is a one-argument predicate. In the case of intuitionistic predicate calculi, analogous to intuitionistic propositional calculi (see IX, 6.1) such a situation never holds.

8.1. *A disjunction $(\beta \cup \gamma)$ is an intuitionistic predicate tautology if and only if either β or γ is an intuitionistic predicate tautology* [1].

If at least one of the formulas β, γ is an intuitionistic predicate tautology, i.e. is derivable in an intuitionistic predicate calculus $\mathfrak{S}_i = \{\mathfrak{L}, \mathbf{C}_i\}$ (see 5.1) then by VI, 10.8, the disjunction $(\beta \cup \gamma)$ is also derivable, i.e. is an intuitionistic predicate tautology (this part of 8.1 also holds, of course, for classical predicate calculi).

Suppose that $(\beta \cup \gamma)$ is an intuitionistic predicate tautology, i.e. is derivable in an intuitionistic predicate calculus $\mathfrak{S}_i = \{\mathfrak{L}, \mathbf{C}_i\}$.

By IV, 10.1, there exists an isomorphism h from $\mathfrak{A}(\mathfrak{S}_i)$ into the algebra $\mathfrak{G}(X)$ of all open subsets of a topological space X. By III, 5.1, X is an open subset of a strongly compact space X_0. By IV, 8.2, the mapping

$$h_0(A) = A \cap X \quad \text{for} \quad A \in \mathfrak{G}(X_0) \tag{1}$$

[1] Rasiowa and Sikorski [5]. This theorem can be also deduced from Gentzen [1]. For other proofs of 8.1, see Kleene [6] and references in Kleene [6].

is a pseudo-Boolean homomorphism from $\mathfrak{G}(X_0)$ onto $\mathfrak{G}(X)$ and h_0 preserves all infinite joins and meets. Moreover

$$h_0(A) = A \quad \text{for} \quad A \in \mathfrak{G}(X) . \tag{2}$$

Let R^0 be the canonical realization of $\mathfrak{L}$ determined by the homomorphism h. By definition, R^0 is a realization in the set T of all terms and in the pseudo-Boolean algebra $\mathfrak{G}(X)$. By 3.2, (6) and 4.1, for every formula α

(3) α is an intuitionistic predicate tautology if and only if

$$\alpha_{R^0}(\mathfrak{i}) = X ,$$

$\mathfrak{i}$ being the identity valuation.

Since $\mathfrak{G}(X)$ is a subalgebra of $\mathfrak{G}(X_0)$, R^0 can also be considered a realization in T and in the pseudo-Boolean algebra $\mathfrak{G}(X_0)$. Denote the last realization by R. The realizations R and R^0 are the same functions defined on the set of all functors and predicates in $\mathfrak{L}$, but for any formula α in $\mathfrak{L}$ they determine different mappings α_R and α_{R^0} because they are interpreted as realizations in different pseudo-Boolean algebras.

Using the notation from theorem VI, 6.6, we have

$$R^0 = h_0 R$$

by (2). Hence it follows, by VI, 6.6, that

$$\alpha_{R^0}(\mathfrak{i}) = h_0\big(\alpha_R(\mathfrak{i})\big) = X \cap \alpha_R(\mathfrak{i}) \tag{4}$$

for every formula α and for the identity valuation $\mathfrak{i}$. Since $(\beta \cup \gamma)$ is an intuitionistic predicate tautology, we have $(\beta \cup \gamma)_R(\mathfrak{i}) = X_0$, i.e.

$$\beta_R(\mathfrak{i}) \cup \gamma_R(\mathfrak{i}) = X_0 .$$

Since X_0 is strongly compact, one of the open sets on the left-hand side of this equation is equal to X_0. Suppose, e.g., that $\beta_R(\mathfrak{i}) = X_0$. By (4), $\beta_{R^0}(\mathfrak{i}) = X$. Hence, by (3), β is an intuitionistic predicate tautology.

In a classical predicate calculus it can happen that an existential formula, i.e. a formula of the form $\bigcup_\xi \alpha(\xi)$, is a predicate tautology, i.e. is derivable, but no term τ exists such that the formula $\alpha(\tau)$ is a predicate tautology, i.e. is derivable. This does not hold for an intuitionistic predicate calculus.

8.2. *An existential formula $\bigcup_\xi \beta(\xi)$ in a language $\mathfrak{L}$ of the first order is an intuitionistic predicate tautology if and only if there exists a term τ in $\mathfrak{L}$ such that $\beta(\tau)$ is an intuitionistic predicate tautology* [1].

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[1] Rasiowa and Sikorski [5]. This theorem can be also deduced from Gentzen [1]. For other proofs, see Kleene [6] and references in Kleene [6].

Suppose that $\beta(\tau)$ in an intuitionistic predicate tautology, i.e. is derivable in $\mathfrak{S}_i = \{\mathfrak{L}, \mathfrak{C}_i\}$. By 6.4 ($T_{32}$) and *modus ponens* ($r_1$), the formula $\bigcup_\xi \beta(\xi)$ is also derivable in $\mathfrak{S}_i$, i.e. it is an intuitionistic predicate tautology.

Let X, X_0, h, h_0, R^0, R and i have the same meaning as in the proof of 6.1. Let $\beta(x_0)$ be a formula such that $\bigcup_\xi \beta(\xi)$ is an intuitionistic predicate tautology. By 4.1 and VI, 7.1, (3),

$$\bigcup_{\tau \epsilon T} \beta(\tau)_R(\mathrm{i}) = \big(\bigcup_\xi \beta(\xi)\big)_R(\mathrm{i}) = X_0 . \tag{5}$$

Since the space X_0 is strongly compact, at least one of the summands on the left-hand side of (5) is equal to X_0. In other words, there exists a term τ such that

$$\beta(\tau)_R(\mathrm{i}) = X_0 .$$

Thus by (4), $\beta(\tau)_{R^0}(\mathrm{i}) = X \cap \beta(\tau)_R(\mathrm{i}) = X$. By (3), $\beta(\tau)$ is an intuitionistic tautology.

§ 9. The deduction theorems. This name is given to the following two theorems.

9.1. *Let α be a closed formula in $\mathfrak{L}$. A formula β is a theorem is an intuitionistic theory $\mathfrak{T}' = \{\mathfrak{L}, \mathfrak{C}_i, \mathfrak{A} \cup [\alpha]\}$ if and only if the formula $(\alpha \Rightarrow \beta)$ is a theorem in the intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_i, \mathfrak{A}\}$.*

If $(\alpha \Rightarrow \beta)$ is a theorem in $\mathfrak{T}$ then it is also a theorem in $\mathfrak{T}'$. Since α is an axiom in $\mathfrak{T}'$, by *modus ponens* the formula β is a theorem in $\mathfrak{T}'$.

If $(\alpha \Rightarrow \beta)$ is not a theorem in $\mathfrak{T}$, then by 3.5 there exists a model R for $\mathfrak{T}$ in a set $J \neq 0$ and in a complete pseudo-Boolean algebra $\boldsymbol{A}$ such that $(\alpha \Rightarrow \beta)$ is not valid in R, i.e. there is a valuation v such that

$$\alpha_R(v) \Rightarrow \beta_R(v) = (\alpha \Rightarrow \beta)_R(v) \neq V . \tag{1}$$

Since α is a closed formula, α_R is a constant function. Say g is an element in $\boldsymbol{A}$ such that α_R is identically equal to g. By (1), $g \neq \Lambda$ (see I, 12.2, (4)), i.e. the pseudo-Boolean algebra $\boldsymbol{A}_g$ described in IV, § 8, is non-degenerate. It follows from (1) that g is not a subelement of $\beta_R(v)$, i.e.

$$g \cap \beta_R(v) \neq g . \tag{2}$$

By IV, 8.2, the mapping

$$h(a) = g \cap a \qquad (a \in A)$$

is a homomorphism from $\boldsymbol{A}$ onto $\boldsymbol{A}_g$ preserving all infinite joins and meets in $\boldsymbol{A}$. By 2.6, the realization hR of $\mathfrak{L}$ in J and the pseudo-Boolean algebra $\boldsymbol{A}_g$ is a model for $\mathfrak{T}$. Moreover, it is a model for α since α_{hR} is identically equal to $h(g)$ (see 2.6, (6)), i.e. to g which is the unit element

in A_g. Thus hR is a model for $\mathfrak{C}'$. The formula β is not valid in hR since $\beta_{hR}(v) = h(\beta_R(v)) = g \cap \beta_R(v) \neq g$ by (2) and 2.6 (6). Thus β is not a theorem in $\mathfrak{C}'$ on account of 2.2.

9.2. *A formula β is a theorem in an intuitionistic theory $\mathfrak{C} = \{\mathfrak{L}, C_\iota, \mathcal{A}\}$ with a non-empty set $\mathcal{A}$ of axioms if and only if there exists a conjunction α of closures of a finite numbers of axioms in $\mathcal{A}$ such that the implication $(\alpha \Rightarrow \beta)$ is an intuitionistic predicate tautology.*

Suppose that β is a theorem in $\mathfrak{C}$, i.e. β belongs to $C_\iota(\mathcal{A})$. Then there exists a finite non-empty set $\mathcal{A}_1 \subset \mathcal{A}$ such that β is in $C_\iota(\mathcal{A}_1)$ (see V, 10.1). Let α be a conjunction of closures of all formulas in $\mathcal{A}_1$. By *modus ponens*, (T_6) and VI, 11.5, $\mathcal{A}_1 \subset C_\iota([\alpha])$. Consequently β belongs to $C_\iota([\alpha])$, i.e. β is a theorem in the theory $\{\mathfrak{L}, C_\iota, [\alpha]\}$. By 9.1 (where $\mathcal{A}$ is the empty set), the implication $(\alpha \Rightarrow \beta)$ is a theorem in $\{\mathfrak{L}, C_\iota, 0\}$, i.e. it is an intuitionistic tautology.

Conversely, if there exists a conjunction α of closures of axioms in $\mathcal{A}$ such that $(\alpha \Rightarrow \beta)$ is an intuitionistic tautology, then α is a theorem in $\mathfrak{C}$ (see VI, 11.5, and VI, 10.8) and, by *modus ponens*, β is also a theorem in $\mathfrak{C}$.

§ 10. Connection between theories and filters. In this section (which is an analogue to VII, § 9, VIII, § 11, and IX, § 10) the symbol $\|\alpha\|$ will always denote the element of the pseudo-Boolean algebra $\mathfrak{A}(\mathfrak{S}_\iota)$, determined by a formula α. We recall that $\mathfrak{S}_\iota = \{\mathfrak{L}, C_\iota\}$ is the intuitionistic predicate calculus based on $\mathfrak{L}$.

A filter ∇ in $\mathfrak{A}(\mathfrak{S}_\iota)$ is said to be an $\bigcap$-filter if it has the following property:

if $\|\alpha(x)\|$ is in ∇, then the element $\|\bigcap_\xi \alpha(\xi)\|$ (i.e. the meet $\bigcap_{\tau \in T}\|\alpha(\tau)\|$) is also in ∇.

In other words, a filter ∇ in $\mathfrak{A}(\mathfrak{S}_\iota)$ is said to be a $\bigcap$-filter if it has the following property:

if $\|\alpha\| \in \nabla$, and $\bar{\alpha}$ is a closure of the formula α, then also $\|\bar{\alpha}\| \in \nabla$.

The symbol $\overline{\mathfrak{A}}(\mathfrak{S}_\iota)$ will denote the set of all elements $\|\alpha\| \in \mathfrak{A}(\mathfrak{S}_\iota)$ where α is a closed formula. It is easy to verify that $\overline{\mathfrak{A}}(\mathfrak{S}_\iota)$ is closed under the pseudo-Boolean operations $\cup, \cap, \Rightarrow, -$ in $\mathfrak{A}(\mathfrak{S}_\iota)$, i.e. $\overline{\mathfrak{A}}(\mathfrak{S}_\iota)$ is a subalgebra of the pseudo-Boolean algebra $\mathfrak{A}(\mathfrak{S}_\iota)$.

For any set $\mathcal{A}$ of formulas in $\mathfrak{L}$, the symbol $\nabla_{0\mathcal{A}}$ will denote the set of all $\|\alpha\| \in \mathfrak{A}(\mathfrak{S}_\iota)$ where α is in $\mathcal{A}$, and the symbol $\overline{\nabla}_{0\mathcal{A}}$ will denote the set of all $\|\bar{\alpha}\| \in \overline{\mathfrak{A}}(\mathfrak{S}_\iota)$ where $\bar{\alpha}$ is a closure of a formula α in $\mathcal{A}$ (all closures of a formula α determine the same element of $\mathfrak{A}(\mathfrak{S}_\iota)$ by VI, 11.7). The symbol $\nabla_{\mathcal{A}}$ will denote the set of all $\|\alpha\| \in \mathfrak{A}(\mathfrak{S}_\iota)$ where α is a theorem in the intuitionistic theory $\mathfrak{C} = \{\mathfrak{L}, C_\iota, \mathcal{A}\}$, and $\overline{\nabla}_{\mathcal{A}}$ will denote the set

of all $\|\alpha\|$ where α is a closed formula and a theorem in $\mathfrak{T}$. The definition of $\nabla_{\mathfrak{A}}$ can be formulated as follows:

(1) $\|\alpha\| \in \nabla_{\mathfrak{A}}$ if and only if α is a theorem in $\mathfrak{T}$.

The set $\overline{\nabla}_{\mathfrak{A}}$ is the intersection:

(2) $$\overline{\nabla}_{\mathfrak{A}} = \nabla_{\mathfrak{A}} \cap \overline{\mathfrak{A}}(\mathfrak{S}_\iota) .$$

A set S of elements in $\mathfrak{A}(\mathfrak{S}_\iota)$ is said to $\bigcap$-*generate* an $\bigcap$-filter ∇ in $\mathfrak{A}(\mathfrak{S}_\iota)$ (or: to be a set of $\bigcap$-*generators* for ∇) if ∇ is the smallest $\bigcap$-filter containing S.

10.1. *The set $\nabla_{\mathfrak{A}}$ is a $\bigcap$-filter in $\mathfrak{A}(\mathfrak{S}_\iota)$. The set $\nabla_{0\mathfrak{A}}$ $\bigcap$-generates the $\bigcap$-filter $\nabla_{\mathfrak{A}}$. The set $\overline{\nabla}_{0\mathfrak{A}}$ generates the filter $\nabla_{\mathfrak{A}}$.*

The set $\overline{\nabla}_{\mathfrak{A}}$ is a filter in $\overline{\mathfrak{A}}(\mathfrak{S}_\iota)$. The set $\overline{\nabla}_{0\mathfrak{A}}$ generates the filter $\overline{\nabla}_{\mathfrak{A}}$.

By the deduction theorem 9.2, α is a theorem in $\mathfrak{T}$ if and only if there exists a conjunction $\bigcap_{i=1}^{m} \bar{\alpha}_i$ of closures of α_i in $\mathfrak{A}$, such that the implication $(\bigcap_{i=1}^{m} \bar{\alpha}_i \Rightarrow \alpha)$ is a predicate tautology, i.e. if

$$\|(\bigcap_{i=1}^{m} \bar{\alpha}_i \Rightarrow \alpha)\| = V \in \mathfrak{A}(\mathfrak{S}_\iota) .$$

In other words, α is a theorem in $\mathfrak{T}$ if and only if there exist formulas $\alpha_1, \ldots, \alpha_m$ in $\mathfrak{A}$ such that

$$\bigcap_{i=1}^{m} \|\bar{\alpha}_i\| \leqslant \|\alpha\| ,$$

where $\bar{\alpha}_i$ is a closure of α_i $(i = 1, \ldots, m)$. This proves the first part of 10.1 by (1) and I, 8.1.

The second part follows from the first part and (2).

We have deduced the first part of 10.1 from the deduction theorem 9.2. It is easy to show that, conversely, 9.2 results immediately from the first part of 10.1. Thus the first part of 10.1 is only another algebraic formulation of the deduction theorem 9.2. It also explains the algebraic sense of the set of axioms and the set of all theorems in an intuitionistic theory.

10.2. *An elementary intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$ is consistent if and only if the filter $\nabla_{\mathfrak{A}}$ is proper*, i.e. *if the filter $\overline{\nabla}_{\mathfrak{A}}$ is proper.*

A closed formula α is irrefutable in the intuitionistic theory $\mathfrak{T}$ if and only if the intuitionistic theory $\{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A} \cup [\alpha]\}$ is consistent.

Thus a closed formula $--\beta$ is not a theorem in $\mathfrak{T}$ if and only if the theory $\{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A} \cup [-\beta]\}$ is consistent.

The first part of 10.2 follows from the definition of $\nabla_{\mathfrak{A}}$ or $\overline{\nabla}_{\mathfrak{A}}$ (see V, 14.1).

The second part of 10.2 follows from the part just proved. For the filter generated by $\overline{\nabla}_{0\mathcal{A}}$ and $\|\alpha\|$ (i.e. generated by $\overline{\nabla}_{\mathcal{A}}$ and $\|\alpha\|$) is proper if and only if $-\|\alpha\| \notin \nabla_{\mathcal{A}}$ (see I 13.7).

The third part of 10.2 follows from the second where α is the formula $-\beta$.

Without the hypothesis that the formula α in question is closed, the second part of 10.2 can be formulated as follows: a formula $\alpha(x_1, \ldots, x_n)$ in which $x_1, \ldots, x_n$ are all free individual variables occurring in this formula, is irrefutable in $\mathfrak{T}$ if and only if the intuitionistic theory $\{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A} \cup [\bigcup_{\xi_1} \ldots \bigcup_{\xi_n} \alpha(\xi_1, \ldots, \xi_n)]\}$ is consistent. This statement follows from the second part of 10.2 and from the fact that $\alpha(x_1, \ldots, x_n)$ is irrefutable in $\mathfrak{T}$ if and only if $\bigcup_{\xi_1} \ldots \bigcup_{\xi_n} \alpha(\xi_1, \ldots, \xi_n)$ is irrefutable in $\mathfrak{T}$. The last remark follows from the definition of a formula irrefutable in $\mathfrak{T}$ and from (T_{38}), (T_{37}) (see 6.3).

Replacing $\alpha(x_1, \ldots, x_n)$ by $-\beta(x_1, \ldots, x_n)$ in the above statement we obtain the following generalization of the third part of 10.2: a formula $--\beta(x_1, \ldots, x_n)$ (where $x_1, \ldots, x_n$ are all free individual variables occurring in this formula) is not a theorem of $\mathfrak{T}$ if and only if the intuitionistic theory $\{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A} \cup [\bigcup_{\xi_1} \ldots \bigcup_{\xi_n} -\beta(\xi_1, \ldots, \xi_n)]\}$ is consistent.

10.3. *An elementary intuitionistic theory* $\mathfrak{T}' = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}'\}$ *is a linguistically invariant extension of an elementary intuitionistic theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *if and only if* $\nabla_{\mathcal{A}} \subset \nabla_{\mathcal{A}'}$, i.e. *if* $\overline{\nabla}_{\mathcal{A}} \subset \overline{\nabla}_{\mathcal{A}'}$.

By definition, $\mathfrak{T}'$ is a linguistically invariant extension of $\mathfrak{T}$ if and only if $\mathfrak{C}_\iota(\mathcal{A}) \subset \mathfrak{C}_\iota(\mathcal{A}')$. By (1), this is equivalent to the inclusion $\nabla_{\mathcal{A}} \subset \nabla_{\mathcal{A}'}$. By (2) and 10.1, the last inclusion is equivalent to $\overline{\nabla}_{\mathcal{A}} \subset \overline{\nabla}_{\mathcal{A}'}$.

10.4. *Intuitionistic theories* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *and* $\mathfrak{T}' = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}'\}$ *are equivalent if and only if* $\nabla_{\mathcal{A}} = \nabla_{\mathcal{A}'}$, *i.e. if* $\overline{\nabla}_{\mathcal{A}} = \overline{\nabla}_{\mathcal{A}'}$.

This follows immediately from 10.3.

We shall complete theorem 10.1 by the following remark.

10.5. *For every filter* $\overline{\nabla}$ *in* $\overline{\mathfrak{A}}(\mathfrak{S}_\iota)$ *there exists an elementary intuitionistic theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *such that* $\overline{\nabla} = \overline{\nabla}_{\mathcal{A}}$. *Moreover, for every set* $\overline{\nabla}_0$ *of generators of* $\overline{\nabla}$ *we can choose a set* $\mathcal{A}$ *of axioms for* $\mathfrak{T}$ *in such a way that* $\overline{\nabla}_0 = \overline{\nabla}_{0\mathcal{A}}$.

For every $\bigcap$*-filter* ∇ *in* $\mathfrak{A}(\mathfrak{S}_\iota)$ *there exists an elementary intuitionistic theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *such that* $\nabla = \nabla_{\mathcal{A}}$. *Moreover, for every set* ∇_0 *of* $\bigcap$*-generators of* ∇ *we can choose a set* $\mathcal{A}$ *of axioms for* $\mathfrak{T}$ *in such a way that* $\nabla_0 = \nabla_{0\mathcal{A}}$.

For every element $e \in \nabla_0$, let α_e be a formula such that $\|\alpha_e\| = e$, and let $\mathcal{A}$ be the set of all α_e, $e \in \nabla_0$. The theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ has the required property.

Theorems 10.1, 10.4 and 10.5 state that there exists a natural one-to-one correspondence between $\bigcap$-filters in $\mathfrak{A}(\mathcal{S}_\iota)$, filters in $\overline{\mathfrak{A}}(\mathcal{S}_\iota)$, and formalized elementary intuitionistic theories based on $\mathcal{S}_\iota$, two theories being identified if and only if they are equivalent.

To prove the next theorem we shall use the more precise notation introduced in VI, § 10, p. 245: for every formula α, the symbol $\|\alpha\|_{\mathcal{S}_\iota}$ will denote the corresponding element in the algebra $\mathfrak{A}(\mathcal{S}_\iota)$ of the intuitionistic predicate calculus $\mathcal{S}_\iota = \{\mathfrak{L}, \mathfrak{C}_\iota\}$ and the symbol $\|\alpha\|_{\mathcal{T}}$ will denote the corresponding element of the algebra $\mathfrak{A}(\mathcal{T})$ of an intuitionistic theory $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$.

If formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are derivable in $\mathcal{S}_\iota$, then they are also theorems in $\mathcal{T}$. Hence it follows that the equation

$$h(\|\alpha\|_{\mathcal{S}_\iota}) = \|\alpha\|_{\mathcal{T}}$$

defines a mapping from $\mathfrak{A}(\mathcal{S}_\iota)$ onto $\mathfrak{A}(\mathcal{T})$. It is easy to verify that h is a pseudo-Boolean homomorphism and even a Q-homomorphism (this follows from 2.1, (2), and (Q)). By the definitions of h and $\nabla_{\mathcal{A}}$, the element $h(\|\alpha\|_{\mathcal{S}_\iota})$ is the unit element of $\mathfrak{A}(\mathcal{T})$ if and only if α is a theorem in $\mathcal{T}$, i.e. if $\|\alpha\|_{\mathcal{S}_\iota} \in \nabla_{\mathcal{A}}$. Hence by I, 13.4, follows

10.6. *The algebra $\mathfrak{A}(\mathcal{T})$ of an intuitionistic theory $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ is isomorphic to the algebra $\mathfrak{A}(\mathcal{S}_\iota)/\nabla_{\mathcal{A}}$.*

Theorem 10.6 yields a method of construction of $\mathfrak{A}(\mathcal{T})$ from $\mathfrak{A}(\mathcal{S}_\iota)$.

§ 11. Maximal theories. We recall that according to V, § 14, a consistent theory $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ is said to be maximal if every linguistically invariant consistent extension of $\mathcal{T}$ is equivalent to $\mathcal{T}$.

In the first characterization of maximal intuitionistic theories we shall use the notation $\nabla_{\mathcal{A}}$, $\overline{\nabla}_{\mathcal{A}}$ from § 10. We recall that $\mathfrak{A}(\mathcal{S}_\iota)$ is the pseudo-Boolean algebra of the intuitionistic predicate calculus $\mathcal{S}_\iota = \{\mathfrak{L}, \mathfrak{C}_\iota, 0\}$. A $\bigcap$-filter ∇ in $\mathfrak{A}(\mathcal{S}_\iota)$ is said to be a *maximal $\bigcap$-filter* provided that it is proper and is not a proper subset of a proper $\bigcap$-filter.

11.1. *The following conditions are equivalent for every intuitionistic theory $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ of the first order:*

(i) *$\mathcal{T}$ is maximal,*

(ii) *$\nabla_{\mathcal{A}}$ is a maximal $\bigcap$-filter in $\mathfrak{A}(\mathcal{S}_\iota)$,*

(iii) *$\overline{\nabla}_{\mathcal{A}}$ is a maximal filter in $\overline{\mathfrak{A}}(\mathcal{S}_\iota)$.*

Theorem 11.1 follows from the one-to-one natural correspondence between $\bigcap$-filters in $\mathfrak{A}(\mathcal{S}_\iota)$, filters in $\overline{\mathfrak{A}}(\mathcal{S}_\iota)$ and classes of equivalent theories, and from the fact that a theory $\{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}'\}$ is an extension of a theory $\{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ if and only if $\nabla_{\mathcal{A}} \subset \nabla_{\mathcal{A}'}$, i.e. if and only if $\overline{\nabla}_{\mathcal{A}} \subset \overline{\nabla}_{\mathcal{A}'}$.

11.2. *An intuitionistic theory* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *is maximal if and only if for every closed formula* α, *exactly one of the formulas* α *and* $-\alpha$ *is a theorem of* $\mathcal{T}$. *In other words,* $\mathcal{T}$ *is maximal if and only if in the* Q*-algebra* $\mathfrak{A}(\mathcal{T})$, *the class of all elements* $\|\alpha\| \in \mathfrak{A}(\mathcal{T})$ *where* α *is a closed formula is a two element subalgebra of* $\mathfrak{A}(\mathcal{T})$, *i.e. the subalgebra formed only of* $\wedge, \vee \in \mathfrak{A}(\mathcal{T})$, $\wedge \neq \vee$.

The condition that for every closed formula α exactly one of the formulas α and $-\alpha$ is a theorem of $\mathcal{T}$ means that for any element $\|\alpha\|$ in $\overline{\mathfrak{A}}(\mathfrak{S}_\iota)$ exactly one of the elements $\|\alpha\|$ and $-\|\alpha\|$ is in $\overline{\nabla}_{\mathcal{A}}$. Thus, by I, 13.10, (ii) is equivalent to the condition that $\overline{\nabla}_{\mathcal{A}}$ is a maximal filter in $\overline{\mathfrak{A}}(\mathfrak{S}_\iota)$ and consequently, by 11.1, to the condition that $\mathcal{T}$ is maximal. The second part of 11.2 follows from the first.

11.3. *If* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *is a maximal intuitionistic theory, then every semantic model for* $\mathcal{T}$ *is adequate.*

Suppose that R is a semantic model for a maximal theory $\mathcal{T}$. Let $\mathcal{A}'$ be the set of all formulas which are valid in R. By definition, the theory $\mathcal{T}' = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}'\}$ is an extension of $\mathcal{T}$. $\mathcal{T}'$ is consistent since R is a model for $\mathcal{T}'$. Since $\mathcal{T}$ is maximal, $\mathcal{T}'$ is equivalent to $\mathcal{T}$. By the definition of $\mathcal{T}'$, R is an adequate model for $\mathcal{T}'$. Thus R is also an adequate model for the equivalent theory $\mathcal{T}$.

Note that not every maximal theory has a semantic model (see 11.7).

11.4. *If an intuitionistic theory* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *has an adequate semantic model, then* $\mathcal{T}$ *is a maximal theory.*

Suppose a semantic realization R of the language $\mathfrak{L}$ is an adequate model for $\mathcal{T}$. For every closed formula α exactly one of elements α_R and $-\alpha_R$ is equal to the unit element $\vee$. Since R is adequate, we infer that exactly one of the formulas α and $-\alpha$ is a theorem in $\mathcal{T}$. Thus by 11.2 $\mathcal{T}$ is maximal.

11.5. *If* R *is a semantic realization of a formalized language* $\mathfrak{L}$ *and* $\mathcal{A}_0$ *is the set of all formulas which are valid in* R, *then every theory* $\{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *equivalent to* $\{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}_0\}$ *is maximal.*

In fact, R is then an adequate semantic model for $\{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$.

11.6. *Every consistent intuitionistic theory* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *can be extended to a maximal theory* $\mathcal{T}_0 = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}_0\}$.

Since $\mathcal{T}$ is consistent, by 10.2 the filter $\overline{\nabla}_{\mathcal{A}}$ is proper. By I, 8.4, the filter $\overline{\nabla}_{\mathcal{A}}$ can be extended to a maximal filter ∇ in $\overline{\mathfrak{A}}(\mathfrak{S}_\iota)$. Let $\mathcal{A}_0$ be any set of formulas such that $\nabla = \nabla_{\mathcal{A}_0}$ (see 10.5). By 11.1 the theory $\mathcal{T}_0 = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}_0\}$ is maximal. It is an extension of $\mathcal{T}$ by 10.3.

11.7. *There exists a maximal intuitionistic theory* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *which has no semantic model.*

Let $\mathcal{T}_0 = \{\mathfrak{L}, \mathfrak{C}_\iota, [-\alpha]\}$ where α is the formula

$$\bigcap_\xi (\varrho(\xi) \cup -\varrho(\xi)),$$

ϱ being a one-argument predicate. The theory $\mathcal{T}_0$ has a model, viz. the topological realization R described in § 6, (7), p. 428. Thus $\mathcal{T}_0$ is a consistent theory. Let $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ be a maximal extension of $\mathcal{T}_0$ (see 11.6). The theories $\mathcal{T}_0$ and $\mathcal{T}$ have no semantic model. In fact, α is a predicate tautology. Therefore in every semantic realization R, $\alpha_R(v) = V$ and consequently $(-\alpha)_R(v) = \Lambda$ which proves that R is not a model for $\mathcal{T}_0$ or for $\mathcal{T}$.

The example of theories $\mathcal{T}_0$ or $\mathcal{T}$ also show

11.8. *There exists an intuitionistic consistent theory* $\{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *such that the corresponding classical theory* $\{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ *is inconsistent.*

Also note that:

11.9. *An intuitionistic theory* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *has a semantic model if and only if the corresponding classical theory* $\mathcal{T}_0 = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ *is consistent.*

Let $\mathcal{A}_0$ be the set of all formulas (T_{12}) in $\mathfrak{L}$, i.e. the set of all formulas $(\alpha \cup -\alpha)$ where α is a formula in $\mathfrak{L}$. It is easy to see that every semantic realization is a model for $\mathcal{A}_0$. Thus if a realization R is a semantic model for $\mathcal{T}$, it is also a semantic model for the theory $\{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A} \cup \mathcal{A}_0\}$. Since this theory is equivalent to $\mathcal{T}_0$ (see the remark at the end of § 1), R is a semantic model for $\mathcal{T}_0$. By VIII, 2.4, $\mathcal{T}_0$ is consistent.

Conversely, if $\mathcal{T}_0$ is consistent, then $\mathcal{T}_0$ has a semantic model R (see e.g. VIII, 5.3). The realization R is also a model for $\mathcal{T}$.

§ 12. Prime theories. According to V § 14, a consistent intuitionistic theory $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}, \mathcal{A}\}$ is said to be *prime* provided, for any closed formulas β, γ

(1) if $(\beta \cup \gamma)$ is a theorem in $\mathcal{T}$, then either β or γ is a theorem in $\mathcal{T}$.

If (1) holds for all (closed or not) formulas, then $\mathcal{T}$ is said to be a *strongly prime* theory. Clearly every strongly prime theory is prime.

It follows immediately from the definition that, using the notation from § 10, we have

12.1. *An intuitionistic theory* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *is prime if and only if the filter* $\overline{V}_{\mathcal{A}}$ *is prime.*

12.2. *An intuitionistic theory* $\mathcal{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathcal{A}\}$ *is strongly prime if and only if the* $\bigcap$*-filter* $V_{\mathcal{A}}$ *is prime.*

Generalizing a definition from VIII, § 12, p. 323, we shall say that an intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_i, \mathcal{A}\}$ is an $\bigcup$-*theory* provided for every existential formula $\bigcup_\xi \beta(\xi)$,

(2) if $\bigcup_\xi \beta(\xi)$ is a theorem in $\mathfrak{T}$, then there exists a term τ such that $\beta(\tau)$ is a theorem in $\mathfrak{T}$.

A filter $\mathfrak{F}$ in the algebra $\mathfrak{A}(\mathfrak{S}_i)$ of the intuitionistic predicate calculus $\mathfrak{S}_i = \{\mathfrak{L}, \mathfrak{C}_i\}$ is said to be a $\bigcup$-*filter* provided that, for every existential formula $\bigcup_\xi \beta(\xi)$

(3) if $\|\bigcup_\xi \beta(\xi)\| \in \mathfrak{F}$, then there exists a term τ such that $\|\beta(\tau)\| \in \mathfrak{F}$.

From the definition follows immediately:

12.3. *An intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_i, \mathcal{A}\}$ is a $\bigcup$-theory if and only if the filter $\mathfrak{F}_{\mathcal{A}}$ is a $\bigcup$-filter.*

Theorem 8.1 asserts that the intuitionistic predicate calculus $\mathfrak{S}_i$ is a strongly prime theory. Theorem 8.2 asserts that $\mathfrak{S}_i$ is also a $\bigcup$-theory.

Theorem 12.1 is an intuitionistic analogue of a part of theorem VIII 12.7. The remaining part of VIII, 12.7, fails for intuitionistic theories: $\mathfrak{S}_i$ is an example of a prime theory which is not maximal.

Theorem 12.2 is an intuitionistic analogue of a part of theorem VIII, 12.8. The remaining part of VIII, 12.8, fails for intuitionistic theories. The example of $\mathfrak{S}_i$ shows that, in contrast to the case of classical theories, there are interesting strongly prime intuitionistic theories. Other interesting examples will be yielded by theorem 13.6.

Theorem 12.3 is an intuitionistic analogue of theorem VIII, 12.9.

We proved in VIII, § 12 (p. 323) that every classical strongly prime theory is a $\bigcup$-theory. This statement does not hold for intuitionistic theories. An example will be quoted in § 13, p. 444. In VIII, p. 323, we noted that there are classical $\bigcup$-theories which are not prime or strongly prime. Interpreting those theories as intuitionistic theories (see § 1, p. 414) we assert that there are intuitionistic $\bigcup$-theories which are not prime or strongly prime. Similarly we infer that there are prime theories which are not strongly prime (see VIII, § 12, p. 322).

Since every maximal filter in a pseudo-Boolean algebra is prime (see I, 12.1 and I, 9.1), there follows from 11.1 and 12.1:

12.4. *Every maximal intuitionistic theory is prime.*

Hence, by 11.6, we have

12.5. *Every consistent intuitionistic theory can be extended to a prime theory.*

§ 13. Constructive theories. An intuitionistic theory is said to be *constructive* provided it is simultaneously a strongly prime theory and a $\bigcup$-theory.

For instance, it follows from theorems 8.1 and 8.2 that the intuitionistic predicate calculus (interpreted as a theory with the empty set of axioms) is a constructive theory.

From 12.2 and 12.3 follows immediately:

13.1. *An intuitionistic theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_i, \mathcal{A}\}$ *is constructive if and only if the* $\bigcap$*-filter* $V_{\mathcal{A}}$ *in the algebra* $\mathfrak{A}(\mathfrak{S}_i)$ *of the intuitionistic predicate calculus* $\mathfrak{S}_i = \{\mathfrak{L}, \mathfrak{C}_i\}$ *is prime and is a* $\bigcup$*-filter.*

We shall now give a characterization of constructive theories by means of a kind of models.

Let $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_i, \mathcal{A}\}$ be a consistent intuitionistic theory. Let h be a Q-isomorphism from $\mathfrak{A}(\mathfrak{T})$ into the algebra $\mathfrak{G}(X)$ of all open subsets of a topological space X (see 1.1 and IV, 9.2), and let R^0 be the canonical realization for $\mathfrak{T}$ determined by h (see § 3). By definition, R^0 is a realization in the set T of all terms and in the algebra $\mathfrak{G}(X)$. By 3.2, R^0 is an adequate model for $\mathfrak{T}$. Moreover, by 3.2, (6), for every formula α,

(1) $\qquad \alpha$ is a theorem in $\mathfrak{T}$ if and only if $\alpha_{R^0}(\mathfrak{i}) = X$,

$\mathfrak{i}$ being the identity valuation.

By III, 5.1, X is an open subset of a strongly compact space X_0. By IV, 8.2, the mapping

(2) $\qquad h_0(A) = X \cap A \quad$ for $\quad A \in \mathfrak{G}(X_0)$,

where $\mathfrak{G}(X_0)$ is the pseudo-Boolean algebra of all open subsets of X_0, is a pseudo-Boolean homomorphism from $\mathfrak{G}(X_0)$ onto $\mathfrak{G}(X)$ and h_0 preserves all infinite joins and meets. Let R be the realization of $\mathfrak{L}$ in the set T of all terms and in $\mathfrak{G}(X_0)$, defined as follows:

(3) $\qquad \varphi_R = \varphi_{R^0}$ for every m-argument functor $\quad (m = 0, 1, 2, \ldots)$;

(4) $$\varrho_R(\tau_1, \ldots, \tau_m) = \begin{cases} X_0 & \text{if} \quad \varrho_{R^0}(\tau_1, \ldots, \tau_m) = X, \\ \varrho_{R^0}(\tau_1, \ldots, \tau_m) & \text{in the opposite case} \end{cases}$$

for every m-argument predicate ϱ $(m = 1, 2, \ldots)$ and for all terms $\tau_1, \ldots, \tau_m$.

Thus R is also a canonical realization of terms. R considered as a realization of predicates is a slight modification of R^0: X is everywhere replaced by X_0 in the definition of ϱ_R.

The realization R defined by (3) and (4) will be called the *characteristic realization* for $\mathfrak{T}$. By definition,

$$R^0 = h_0 R$$

using the notation from VI, § 6, p. 232. Hence it follows, by VI, 6.6, that

(5) $$\alpha_{R^0}(\mathfrak{i}) = h_0\big(\alpha_R(\mathfrak{i})\big) = X \cap \alpha_R(\mathfrak{i})$$

for every formula α and for the identity valuation i. Since the class of all open subsets of X_0 is composed of X_0 and open subsets of X (see III, 5.1), for every formula α

(6) $$\text{either} \quad \alpha_R(\mathrm{i}) = X_0 \quad \text{or} \quad \alpha_R(\mathrm{i}) = \alpha_{R^0}(\mathrm{i}) ;$$

(7) $$\text{if} \quad \alpha_{R^0}(\mathrm{i}) \neq X, \quad \text{then} \quad \alpha_R(\mathrm{i}) = \alpha_{R^0}(\mathrm{i}) ;$$

(8) $$\text{if} \quad \alpha_R(\mathrm{i}) = X_0, \quad \text{then} \quad \alpha_{R^0}(\mathrm{i}) = X .$$

13.2. *If the characteristic realization R for $\mathfrak{T}$ is a model for $\mathfrak{T}$, then the theory $\mathfrak{T}$ is constructive.*

The proof is a repetition of the proofs of 8.1 and 8.2.

Suppose that R is a model for $\mathfrak{T}$. If $(\beta \cup \gamma)$ is a theorem in $\mathfrak{T}$, then

$$\beta_R(\mathrm{i}) \cup \gamma_R(\mathrm{i}) = (\beta \cup \gamma)_R(\mathrm{i}) = X_0 .$$

Since X_0 is strongly compact, at least one of the open sets $\beta_R(\mathrm{i})$, $\gamma_R(\mathrm{i})$ is equal to X_0. Assume, e.g., that $\beta_R(\mathrm{i}) = X_0$. By (8), $\beta_{R^0}(\mathrm{i}) = X$, i.e. β is a theorem in $\mathfrak{T}$ on account of (1).

Suppose now that $\bigcup_\xi \beta(\xi)$ is a theorem in $\mathfrak{T}$. By VI, 7.1, (3),

$$\bigcup_{\tau \in T} \beta(\tau)_R(\mathrm{i}) = \big(\bigcup_\xi \beta(\xi)\big)_R(\mathrm{i}) = X_0 .$$

Since X_0 is strongly compact, at least one of the open sets $\beta(\tau)_R(\mathrm{i})$ is equal to X_0. For this term τ, we have $\beta(\tau)_{R^0}(\mathrm{i}) = X$ on account of (8), i.e. $\beta(\tau)$ is a theorem in $\mathfrak{T}$ by (1).

Consider the following property of a formula α in an intuitionistic consistent theory $\mathfrak{T}$.

(p) if α is a theorem in $\mathfrak{T}$, then

(9) $$\alpha_R(\mathrm{i}) = X_0 .$$

13.3. *The set S of all formulas α in an intuitionistic theory $\mathfrak{T}$ which have the property* (p) *satisfies the following conditions:*

(a) *every elementary formula is in S,*

(b) *if β, γ are in S, then the conjunction $(\beta \cap \gamma)$ is in S,*

(c) *for every formula β the negation $-\beta$ is in S,*

(d) *if γ is in S, then for every formula β the implication $(\beta \Rightarrow \gamma)$ is in S,*

(e) *if for every term τ the formula $\beta(\tau)$ is in S, then the formula $\bigcup_\xi \beta(\xi)$ is in S.*

(a) follows directly from (3) and (4).

To prove (b) suppose that β and γ are in S and that the conjunction $(\beta \cap \gamma)$ is a theorem. Then β and γ are also theorems in $\mathfrak{T}$ and, by (p), $\beta_R(\mathrm{i}) = X_0 = \gamma_R(\mathrm{i})$. Hence $(\beta \cap \gamma)_R(\mathrm{i}) = X_0$.

To prove (c), let us observe that if $-\beta$ is a theorem, then $-\beta_{R^0}(\mathfrak{i}) = X$ by (1). Thus $\beta_{R^0}(\mathfrak{i}) = 0$. Hence, by (7), $\beta_R(\mathfrak{i}) = 0$. Hence $(-\beta)_R(\mathfrak{i}) = X_0$.

To prove (d), let us suppose that $(\beta \Rightarrow \gamma)$ is a theorem and γ is in S. For brevity, denote $(\beta \Rightarrow \gamma)$ by α. We have to prove (9). By (6) there are three cases (i), (ii), (iii) to be considered.

(i) $$\gamma_R(\mathfrak{i}) = X_0 .$$

Then (9) holds (see I, 12.2, (7)).

(ii) $$\beta_R(\mathfrak{i}) = X_0 .$$

Then it follows from (8) and (1) that β is a theorem. By *modus ponens*, γ is also a theorem. Since γ is in S, we have $\gamma_R(\mathfrak{i}) = X_0$. By the case (i) just proved, (9) holds.

(ii) $$\beta_R(\mathfrak{i}) = \beta_{R^0}(\mathfrak{i}), \quad \gamma_R(\mathfrak{i}) = \gamma_{R^0}(\mathfrak{i}) .$$

Since α is a theorem, $\beta_{R^0}(\mathfrak{i}) \Rightarrow \gamma_{R^0}(\mathfrak{i}) = (\beta \Rightarrow \gamma)_{R^0}(\mathfrak{i}) = X$, i. e. $\beta_{R^0}(\mathfrak{i}) \leqslant \gamma_{R^0}(\mathfrak{i})$. This implies (9).

To prove (e), suppose that, for a formula $\beta(x_0)$ the formula $\beta(\tau)$ is in S for every term τ and the formula $\bigcap_\xi \beta(\xi)$ is a theorem of $\mathfrak{C}$. Then for every τ the formula $\beta(\tau)$ is a theorem of $\mathfrak{C}$ by *modus ponens* and (T_{81}) (see p. 426). In consequence

$$\beta(\tau)_R(\mathfrak{i}) = X_0 \quad \text{for every term } \tau .$$

Hence, by VI, 7.1, (3') we obtain

$$\big(\bigcap_\xi \beta(\xi)\big)_R(\mathfrak{i}) = \bigcap_{\tau \in T} \beta(\tau)_R(\mathfrak{i}) = X_0 .$$

13.4. *All formulas of a constructive intuitionistic theory* $\mathfrak{C}$ *have property* (p).

First we shall prove that if $\mathfrak{C}$ is constructive, then the set S mentioned in 13.3 also has the following properties:

(f) if β, γ are in S, then the disjunction $(\beta \cup \gamma)$ is in S,

(g) if for every term τ the formula $\beta(\tau)$ is in S, then the formula $\bigcup_\xi \beta(\xi)$ is in S.

In fact, suppose that $(\beta \cup \gamma)$ is a theorem and β, γ have property (p). Since $\mathfrak{C}$ is constructive, one of the formulas β, γ is a theorem; say β is a theorem. Since β satisfies (p), we have $\beta_R(\mathfrak{i}) = X_0$ which implies that $(\beta \cup \gamma)_R(\mathfrak{i}) = X_0$.

Now, let us suppose that a formula $\bigcup_\xi \alpha(\xi)$ is a theorem, and that for every term τ the formula $\beta(\tau)$ is in S. Since $\mathfrak{C}$ is constructive, there exists a term τ such that $\beta(\tau)$ is a theorem, and consequently

$$\beta(\tau)_R(\mathfrak{i}) = X_0 .$$

Since $\left(\bigcup_\xi \beta(\xi)\right)_R(\mathfrak{i}) = \bigcup_{\tau \in T} \beta(\tau)_R(\mathfrak{i})$ by VI 7.1 (3), we infer that $\left(\bigcup_\xi \beta(\xi)\right)_R(\mathfrak{i}) = X_0$.

Since the set S has properties (a), (b), (c), (d), (e), (f), (g) it coincides with the set of all formulas.

In fact, it follows from (b)-(g) that the set of all elements $|\alpha|$ where α is in S is a subalgebra of the Q-algebra $\{\overline{F}, \cup, \cap, \Rightarrow, -, \bigcup, \bigcap\}$ of the language $\mathfrak{L}$ (see VI, § 4). By (a), this set contains all elements $|\alpha|$ where α is an elementary formula, i.e. it contains a set of generators for the Q-algebra $\overline{F}$. Thus this set coincides with the whole algebra $\overline{F}$. If α is in S and β is congruent with α, then β is also in S. Hence it follows that all formulas are in S.

13.5. *A consistent intuitionistic theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$ *is constructive if and only if its characteristic realization* R *is a model for* $\mathfrak{T}$ [1].

By 13.2 it suffices to show that if $\mathfrak{T}$ is constructive then all axioms in $\mathfrak{A}$ are valid in R.

Let $v\colon V \to T$ be any valuation. Interpreting v as a substitution, we have, by VI, 6.7

$$\alpha_R(v) = v'\alpha_R(\mathfrak{i})$$

where $v'\alpha$ is the result of the substitution v in α.

If α is in $\mathfrak{A}$, then the formula $v'\alpha$ is a theorem in $\mathfrak{T}$. Since the formula $v'\alpha$ has the property (p) by 13.4, we have $v'\alpha_R(\mathfrak{i}) = X_0$. Consequently

$$\alpha_R(v) = X_0$$

for every valuation v, which proves that every axiom α in $\mathfrak{A}$ is valid in R.

The intersection F^* of all sets S satisfying conditions (a), (b), (c), (d), (e) also satisfies these conditions. It is the least set (of formulas) satisfying (a)-(e).

13.6. *If the set* $\mathfrak{A}$ *of axioms of a consistent intuitionistic theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$ *is contained in* F^*, *then* $\mathfrak{T}$ *is constructive.*

In particular, if no axiom of the theory $\mathfrak{T}$ *contains the disjunction sign or the existential quantifier, then the theory* $\mathfrak{T}$ *is constructive.*

First observe that if a formula α is in F^* and $v\colon V \to T$ is any valuation, then the substitution $v'\alpha$ is also in F^* and, consequently, has the property (p), by 13.3.

Suppose $\mathfrak{A} \subset F^*$. If α is in $\mathfrak{A}$, then the formula $v'\alpha$ is a theorem in $\mathfrak{T}$. Consequently

$$\alpha_R(v) = v'\alpha_R(\mathfrak{i}) = X_0$$

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[1] The notion of constructive theories and theorems 13.2-13.6 were given by Rasiowa [4], [5].

since $v'a$ has the property (p). The last identity proves that the characteristic realization R is a model for $\mathfrak{T}$. By 13.2, the theory $\mathfrak{T}$ is constructive.

The second part of 13.6 follows from the first since the set F^* contains all formulas without the disjunction sign and the existential quantifier.

Theorem 13.6 yields a simple method to create constructive theories. The criteria given in 13.6 are sufficient but not necessary conditions for a theory to be constructive. For instance, the theory $\mathfrak{T} = \{\mathfrak{L}, C_\iota, \mathcal{A}\}$ where $\mathcal{A}$ is the set of all intuitionistic tautologies does not satisfy those criteria but it is constructive since it is equivalent to the intuitionistic predicate calculus $\mathfrak{S}_\iota = \{\mathfrak{L}, C_\iota, 0\}$.

In V, § 13, we defined the classical theories $\mathfrak{T} = \{\mathfrak{L}, C, \mathcal{A}\}$ of ordering, of quasi-ordering, of linear ordering, of lattices, of distributive lattices, and of Boolean algebras. Replacing in their definition the classical consequence operation C by the intuitionistic consequence operation C_ι, the language $\mathfrak{L}$ and the sets $\mathcal{A}$ of axioms being the same, we obtain an intuitionistic theory of ordering, of quasi-ordering, etc. It follows immediately that all these intuitionistic theories are constructive. The same remark is true for the theory of groups based on axioms V, § 13, (C_1), (C_2), (C_3). The intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, C_\iota, \mathcal{A}\}$ where $\mathfrak{L}$ is the language of formalized arithmetic from V, § 13, D) and $\mathcal{A}$ is the set of all axioms for arithmetic, the induction schema being restricted to formulas in F^*, is also a constructive theory.

We are now able to give an example (announced at the end of § 12, p. 439) of a strongly prime theory $\mathfrak{T} = \{\mathfrak{L}, C_\iota, \mathcal{A}\}$ which is not constructive, i.e. which is not a $\bigcup$-theory. Let $\mathfrak{L}$ be a formalized language of the first order such that $\mathfrak{L}$ contains only one one-argument predicate ϱ and no functors. The theory $\mathfrak{T} = \{\mathfrak{L}, C_\iota, [\bigcup_\xi \varrho(\xi)]\}$ has the required properties.

In fact, the only terms in $\mathfrak{T}$ are free individual variables. However, no formula $\varrho(x)$ is a theorem in $\mathfrak{T}$ since it is easy to define a semantic model for $\mathfrak{T}$ in which $\varrho(x)$ is not valid. On the other hand, the theory $\mathfrak{T}$ is strongly prime. To prove it, let us introduce the auxiliary theory $\mathfrak{T}' = \{\mathfrak{L}', C_\iota, [\varrho(c)]\}$ where $\mathfrak{L}'$ is the language obtained from $\mathfrak{L}$ by adding an individual constant c. It is easy to contract a semantic model for $\mathfrak{T}'$. Thus $\mathfrak{T}'$ is consistent. Since $\bigcup_\xi \varrho(\xi)$ is a theorem in $\mathfrak{T}'$, the theory $\mathfrak{T}'$ is an extension of $\mathfrak{T}$. The theory $\mathfrak{T}'$ is an inessential extension of $\mathfrak{T}$. For suppose that a formula a in $\mathfrak{L}$ is not a theorem in $\mathfrak{T}$. By the deduction theorem 9.2, the formula $(\bigcup_\xi \varrho(\xi) \Rightarrow a)$ is not an intuitionistic tautology, i.e. the inequality

$$\bigcup_{x \in V} \|\varrho(x)\| \leqslant \|a\|$$

does not hold in the pseudo-Boolean algebra $\mathfrak{A}(\mathcal{S}_t)$ of the intuitionistic predicate calculus $\mathcal{S}_t = \{\mathfrak{L}, \mathcal{C}_t\}$ (V denotes, as usual, the set of all free individual variables in $\mathfrak{L}$). Thus there exists a free individual variable x_0 such that the inequality

$$\|\varrho(x_0)\| \leqslant \|a\| \tag{10}$$

is not true. Let R^0 be the canonical realization of $\mathfrak{L}$ in the algebra $\mathfrak{A}(\mathcal{S}_t)$ (see § 3 or VI, § 7). We extend R^0 to a realization R' of $\mathfrak{L}'$ by assuming that $c_{R'} = x_0$. Consequently $\varrho(c)_{R'} = \|\varrho(x_0)\|$ and

$$(\varrho(c) \Rightarrow a)_{R'}(\mathfrak{i}) = \varrho(c)_{R'}(\mathfrak{i}) \Rightarrow a_{R'}(\mathfrak{i}) = \|\varrho(x_0)\| \Rightarrow a_{R^0}(\mathfrak{i}) = \|\varrho(x_0)\| \Rightarrow \|a\| \neq V$$

by 3.1, (5) (where h is the identity mapping) and (10). This proves that the implication $(\varrho(c) \Rightarrow a)$ is not an intuitionistic tautology, i.e. by the deduction theorem 9.2, a is not a theorem in $\mathcal{C}'$. This proves the inessentiality of the extension $\mathcal{C}'$ of $\mathcal{C}$. Suppose now that a formula $(\beta \cup \gamma)$ is a theorem in $\mathcal{C}$. Then it is also a theorem in $\mathcal{C}'$. Since $\mathcal{C}'$ is constructive by 13.6, one of the formulas β, γ, say β, is a theorem in $\mathcal{C}'$. Since $\mathcal{C}'$ is an inessential extension of $\mathcal{C}$, β is a theorem in $\mathcal{C}$. Thus $\mathcal{C}$ is strongly prime.

§ 14. Cancellation of initial quantifiers in formulas of a $\bigcup$-theory. Let

$$x_0, x_1, x_2, \ldots \tag{1}$$

be a fixed sequence of different free individual variables in a language $\mathfrak{L}$ of the first order.

If the first sign of a formula a in $\mathfrak{L}$ is a quantifier, then $Z(a)$ will denote the set of formulas in $\mathfrak{L}$ defined as follows.

1) If a is of the form $\bigcap_\xi \beta(\xi)$, then $Z(a)$ contains only the formula $\beta(x_i)$ where x_i is the first free individual variable in the sequence (1) which does not appear in a.

2) If a is of the form $\bigcup_\xi \beta(\xi)$, then $Z(a)$ consists of all formulas $\beta(\tau)$ where τ is any term such that all free individual variables occurring in τ appear in a, provided such terms exist; if such terms do not exist, then

2') $Z(a)$ consists of all formulas $\beta(\tau)$ where τ is any term such that x_0 is the only free individual variable occurring in τ.

14.1. *A formula a whose first sign is a quantifier is a theorem in a $\bigcup$-theory $\mathcal{C} = \{\mathfrak{L}, \mathcal{C}_t, \mathcal{A}\}$ if and only if at least one formula in $Z(a)$ is a theorem in $\mathcal{C}$.*

If the universal quantifier is the first sign of a, then 14.1 follows from VI, 11.4 and (T_{31}), see p. 426 (the hypothesis that $\mathcal{C}$ is a $\bigcup$-theory is then superflous).

Suppose that α is of the form $\bigcup_\xi \beta(\xi)$. If one of the formulas in $Z(\tau)$ is a theorem in $\mathfrak{T}$, i.e. if $\beta(\tau)$ is a theorem for a term τ of the type described in 2) or 2'), then $\bigcup_\xi \beta(\xi)$ is also a theorem in $\mathfrak{T}$ by ***modus ponens*** and the intuitionistic tautology (T_{32}). Conversely, if $\bigcup_\xi \beta(\xi)$ is a theorem, then there exists a term τ' such that $\beta(\tau')$ is a theorem. Let τ be the term obtained from τ' by replacing all free individual variables which do not occur in α by a fixed individual constant or by a free individual variable occurring in α, or—in the case 2')—by x_0. Then $\beta(\tau)$ is a formula obtained from $\beta(\tau')$ by the substitution just mentioned. Since $\beta(\tau')$ is a theorem, so is $\beta(\tau)$ by the rule (r_2) of substitution for individual variables. By definition $\beta(\tau)$ is in $Z(\alpha)$.

Consider now a formula in $\mathfrak{L}$ of the following form:

$$\Xi^1_{\xi_1} \dots \Xi^k_{\xi_k} \beta_0(\xi_1, \dots, \xi_k) \tag{2}$$

where $\Xi^1, \dots, \Xi^k$ are existential or universal quantifiers. We define sets $Z_0, \dots, Z_k$ of formulas by induction as follows: Z_0 contains only the formula (2); for $i < k$, Z_{i+1} is the union of all sets $Z(\alpha)$ where α is any formula in Z_i.

14.2. (1) *The formula* (2) *is a theorem in a* $\bigcup$*-theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$ *if and only if at least one of the formulas in* Z_k *is a theorem in* $\mathfrak{T}$.

If β_0 *is an open formula, then all formulas in* Z_k *are open.*

If $\mathfrak{L}$ *contains only a finite number of individual constants and no* m*-argument functor with* $m > 0$, *then the set* Z_k *is finite.*

It follows easily from 14.1, by induction with respect to i, that (2) is a theorem in $\mathfrak{T}$ if and only if Z_i contains a theorem in $\mathfrak{T}$. For $i = k$ we get the first part of 14.2.

The remaining parts of 14.2 follow directly from the definition of Z_i.

§ 15. Theories with the sign of equality. Suppose that $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$ is a formalized theory with the sign $\mathfrak{e}$ of equality. We recall that, according to V, § 12, p. 191, the last statement implies automatically that all axioms for equality (see V, § 12, (e_1)-(e_5)) are included to the set $\mathfrak{A}$ of mathematical axioms of $\mathfrak{T}$.

Let R be a model for $\mathfrak{T}$ in a set $J \neq 0$ and a pseudo-Boolean algebra $\boldsymbol{A}$. R is said to be a *semantic realization of the sign* $\mathfrak{e}$ *of equality* provided, for every $j_1, j_2 \in J$,

$$\mathfrak{e}_R(j_1, j_2) = V \text{ or } \Lambda\,. \tag{1}$$

R is said to be an *ordinary realization of* $\mathfrak{e}$ provided

$$\mathfrak{e}_R(j_1, j_2) = V \quad \text{if and only if} \quad j_1 = j_2\,. \tag{2}$$

(1) Rasiowa and Sikorski [5], Rasiowa [4], [5].

Thus R is said to be an *ordinary semantic realization of* $\mathfrak{e}$ provided

$$\mathfrak{e}_R(j_1, j_2) = \begin{cases} V & \text{if} \quad j_1 = j_2 , \\ \Lambda & \text{if} \quad j_1 \neq j_2 . \end{cases} \tag{3}$$

15.1. *If a intuitionistic theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_t, \mathfrak{A}\}$ *with the sign* $\mathfrak{e}$ *of equality has a model* R *in a pseudo-Boolean algebra* $\boldsymbol{A}$ *and a set* $J \neq \mathbf{0}$, *then* $\mathfrak{T}$ *has an ordinary model* R' *in* $\boldsymbol{A}$ *and the set* $J/\!\sim$ *where* $\sim$ *is defined as follows: for any* $j_1, j_2 \in J$

$$j_1 \sim j_2 \quad \textit{if and only if} \quad \mathfrak{e}_R(j_1, j_2) = V . \tag{4}$$

It follows from the axioms V, § 12, $(\mathfrak{e}_1)$-$(\mathfrak{e}_3)$ that the relation $\sim$ defined by (4) is an equivalence relation. Denote by $|j|$ the element of $J/\!\sim$ which is determined by $j \in J$. It follows from axioms V, § 12, $(\mathfrak{e}_4)$-$(\mathfrak{e}_5)$ that the equations

$$\varphi_{R'}(|j_1|, \ldots, |j_m|) = |\varphi_R(j_1, \ldots, j_m)| , \tag{5}$$

$$\varrho_{R'}(|j_1|, \ldots, |j_m|) = \varrho_R(j_1, \ldots, j_m) , \tag{6}$$

where φ is any m-argument functor ($m = 0, 1, 2, \ldots$) and ϱ is any m-argument predicate ($m = 1, 2, \ldots$), define a realization R' of $\mathfrak{L}$. Since the function $f(j) = |j|$ maps J onto $J/\!\sim$, and (5) and (6) hold, it follows from 2.8 that R' is a model for $\mathfrak{T}$. By (6) and (4), $\mathfrak{e}_{R'}(|j_1|, |j_2|) = V$ if and only if $\mathfrak{e}_R(j_1, j_2) = V$, i.e. if $|j_1| = |j_2|$. Thus R' is an ordinary realization of $\mathfrak{e}$.

By 3.6 and 15.1, every consistent intuitionistic theory with the sign $\mathfrak{e}$ of equality has a model (and even an adequate topological model) which is an ordinary realization of $\mathfrak{e}$. However, a consistent intuitionistic theory with the sign $\mathfrak{e}$ of equality does not always have a model which is a semantic realization of $\mathfrak{e}$.

For instance, let $\mathfrak{L}$ be a language of the first order such that $\mathfrak{L}$ contains no functors and only one predicate, viz. the sign $\mathfrak{e}$ of equality. Let $\mathfrak{A}$ contain all the axioms of equality V, § 12, $(\mathfrak{e}_1)$-$(\mathfrak{e}_3)$ and the formula α

$$-\bigcap_\xi \bigcap_\eta (\mathfrak{e}(\xi\, \eta) \cup -\mathfrak{e}(\xi\, \eta)) .$$

The theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_t, \mathfrak{A}\}$ is consistent since it has a model R in the set J of all rational numbers and the pseudo-Boolean algebra $\mathfrak{G}(X)$, X being the space of all real numbers. Namely, the equation

$$\mathfrak{e}_R(j_1, j_2) = \mathbf{I}(X - \mathbf{I}\langle j_1, j_2\rangle) ,$$

where $\langle j_1, j_2\rangle$ is the interval with the end points $j_1, j_2 \in J$, defines the required model. On the other hand, $\mathfrak{T}$ has no model which is a semantic realization for $\mathfrak{e}$ because in every realization R satisfying (1) we have $\alpha_R \equiv \Lambda$.

Suppose now that $\mathfrak{T} = \{\mathfrak{L}, C_i, \mathcal{A}\}$ is an intuitionistic theory without the sign of equality. Let $\mathfrak{L}'$ be the formalized language obtained from $\mathfrak{L}$ by addition of a two-argument predicate $\mathfrak{e}$. Let $\mathcal{A}'$ be the set consisting of all formulas in $\mathcal{A}$ and of all formulas in $\mathfrak{L}'$ which are axioms for equality V, § 12, (e_1)-(e_5). By definition, $\mathfrak{T}' = \{\mathfrak{L}', C_i, \mathcal{A}'\}$ is an intuitionistic theory with the sign $\mathfrak{e}$ of equality. With the above notation the following theorem holds.

15.2. *Every model R of $\mathfrak{T}$ can be extended to a model of $\mathfrak{T}'$ which is an ordinary semantic realization of $\mathfrak{e}$. Thus $\mathfrak{T}'$ is an inessential extension of $\mathfrak{T}$. Consequently $\mathfrak{T}'$ is consistent if and only $\mathfrak{T}$ is consistent.*

If $\mathfrak{T}$ is constructive, so is $\mathfrak{T}'$.

If R is a realization of $\mathfrak{L}$ in a set $J \neq 0$ and a pseudo-Boolean algebra $\boldsymbol{A}$, let R' be its extension obtained by assuming

$$\mathfrak{e}_{R'}(j_1, j_2) = \begin{cases} V & \text{if} \quad j_1 = j_2\,, \\ \Lambda & \text{if} \quad j_1 \neq j_2 \end{cases}$$

for all $j_1, j_2 \in J$.

It is easy to see that R' is a model for all axioms V, § 12, (e_1)-(e_5). Moreover, if a is an arbitrary formula in $\mathfrak{L}$, then

$$a_R(v) = a_{R'}(v) \quad \text{for every valuation} \quad v\colon V \to J \tag{1}$$

since the realization R' is an extension of the realization R.

It follows from these remarks that R is a model for $\mathfrak{T}$ if and only if R' is a model for $\mathfrak{T}'$. If a formula a in $\mathfrak{L}$ is a theorem of $\mathfrak{T}$, then of course it is a theorem of $\mathfrak{T}$. Conversely, if a formula a in $\mathfrak{L}$ is not a theorem of $\mathfrak{T}$, then there exists a model R for $\mathfrak{T}$ such that a is not valid in R (see 3.5). Thus R' is a model for $\mathfrak{T}'$ and a is not valid in R'. Consequently (by 2.2) a is not a theorem of $\mathfrak{T}'$. Thus $\mathfrak{T}'$ is an inessential extension of $\mathfrak{T}$.

The second part of 15.2 follows from the first part applied to the characteristic realization of $\mathfrak{T}$ (see theorem 13.5).

§ 16. Open intuitionistic theories. We recall that a theory $\mathfrak{T} = \{\mathfrak{L}, C_i, \mathcal{A}\}$ is said to be open if all formulas in $\mathcal{A}$ are open, i.e. if no mathematical axiom of the theory contains any quantifiers.

If $\mathfrak{L} = \{A, T, F\}$ is a formalized language of the first order, then $\mathfrak{L}^0 = \{A^0, T, F^0\}$ will denote the language of open formulas in $\mathfrak{L}$, according to V, § 14 and VI, § 9 (p. 205 and p. 243). We recall that A^0 is the alphabet obtained from the alphabet A of $\mathfrak{L}$ by omitting quantifiers and bound variables, and the set F^0 of all formulas in $\mathfrak{L}^0$ is the set of all open formulas in $\mathfrak{L}$.

The symbol C_i^0 will always denote the intuitionistic consequence operation in $\mathfrak{L}^0$, determined (see V, § 10, (S), p. 182) by *modus ponens* (r_1)

and the rule (r_2) of substitution for free individual variables and by the set $\mathcal{A}_I^0$ of logical axioms, composed of all formulas of the form IX, § 1, (T_1)-(T_{11}) where α, β, γ are any formulas in F^0. According to the definition, for any set S of formulas in $\mathfrak{L}^0$, the set $C_i^0(S)$ is formed of all formulas in $\mathfrak{L}^0$ which are derivable from S by means of (r_1), (r_2) and the logical axioms in $\mathcal{A}_I^0$ (see V, § 9). By V, 10.2, $C_i^0(S)$ is the least set (of formulas in $\mathfrak{L}^0$) containing S and $\mathcal{A}_I^0$ and closed under the rules of inference (r_1) and (r_2). By definition, $C_i^0(S)$ consists of open formulas in $\mathfrak{L}$ only. Moreover,

$$C_i^0(S) \subset C_i(S) . \tag{1}$$

Any set $\mathcal{A}$ of open formulas in $\mathfrak{L}$ uniquely determines two intuitionistic theories, viz. the open theory $\mathfrak{T} = \{\mathfrak{L}, C_i, \mathcal{A}\}$ of the first order and the theory $\mathfrak{T}^0 = \{\mathfrak{L}^0, C_i^0, \mathcal{A}\}$ which has an auxiliary character. Formulas in $C_i^0(\mathcal{A})$ are called theorems in $\mathfrak{T}^0$.

According to VI, § 10, the algebra $\mathfrak{A}(\mathfrak{T}^0)$ of the theory $\mathfrak{T}^0$ is the quotient algebra $F^0/\sim$ where $\sim$ is the congruence relation in the algebra $\{F^0, \cup, \cap, \Rightarrow, -\}$ of open formulas, defined as follows:

(1) $\alpha \sim \beta$ if and only if both the formulas $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are theorems in $\mathfrak{T}^0$, i.e. belong to $C_i^0(\mathcal{A})$.

The following theorem follows immediately from VI, 10.4, 10.5 and 10.7:

16.1. *The algebra $\mathfrak{A}(\mathfrak{T}^0)$ of $\mathfrak{T}^0 = \{\mathfrak{L}^0, C_i^0, \mathcal{A}\}$ is a pseudo-Boolean algebra. Moreover, for any formulas α, β in $\mathfrak{L}^0$,*

$$\begin{gathered} |\alpha| \cup |\beta| = |(\alpha \cup \beta)|, \quad |\alpha| \cap |\beta| = |(\alpha \cap \beta)|, \\ |\alpha| \Rightarrow |\beta| = |(\alpha \Rightarrow \beta)|, \quad -|\alpha| = |-\alpha| . \end{gathered} \tag{3}$$

The relation

$$|\alpha| \leqslant |\beta| \tag{4}$$

holds if and only if $(\alpha \Rightarrow \beta)$ is a theorem in $\mathfrak{T}^0$. The element $|\alpha|$ is the unit of $\mathfrak{A}(\mathfrak{T}^0)$ if and only if α is a theorem in $\mathfrak{T}^0$. The pseudo-Boolean algebra $\mathfrak{A}(\mathfrak{T}^0)$ is non-degenerate if and only if the theory $\mathfrak{T}^0$ is consistent, i.e. $C_i^0(\mathcal{A}) \neq F^0$.

Suppose that the theory $\mathfrak{T}^0$ is consistent. Let h be a pseudo-Boolean homomorphism of $\mathfrak{A}(\mathfrak{T}^0)$ into a complete pseudo-Boolean algebra $\boldsymbol{A}$. Let R^0 be the realization of the language $\mathfrak{L}$ in the set T of all terms and the Boolean algebra $\boldsymbol{A}$, defined as follows:

(a) R^0 is the canonical realization of terms, i.e. (see VI, § 7, (1))

$$\varphi_{R^0}(\tau_1, \ldots, \tau_m) = \varphi(\tau_1 \ldots \tau_m) \tag{5}$$

for every m-argument functor φ in $\mathfrak{L}$ $(m = 0, 1, 2 \ldots)$;

(b) for every m-argument predicate ϱ in $\mathfrak{L}$ ($m = 1, 2, \ldots$),

$$\varrho_{R^0}(\tau_1, \ldots, \tau_m) = h(|\varrho(\tau_1 \ldots \tau_m)|)\,, \tag{6}$$

$\tau_1, \ldots, \tau_m$ being arbitrary terms.

We recall that any valuation v: $V \to T$ is also a substitution in $\mathfrak{L}$. For every term τ and for every formula a, the symbols $v\tau$ and $v'a$ denote the result of the substitution v in τ and a respectively. In particular, for the identity valuation i, we have $\mathrm{i}\tau = \tau$ and $\mathrm{i}'a = a$. Under the above hypotheses and notation the following theorem holds.

16.2. *For every term τ and for every open formula a in $\mathfrak{L}$,*

$$\tau_{R^0}(v) = v\tau\,, \tag{7}$$

$$a_{R^0}(v) = h(|v'a|) \tag{8}$$

for any valuation v: $V \to T$. In particular, for the identity valuation i:

$$a_{R^0}(\mathrm{i}) = h(|a|)\,. \tag{9}$$

Identity (7) was proved in VI, 7.1.

By VI, 9.3, for every fixed v both sides of (8), considered as functions of a, are homomorphisms from the algebra F^0 of open formulas into $\boldsymbol{A}$ (see VI, § 9). It follows from (6) and (7) that both homomorphisms coincide on the set of all elementary formulas which generate the algebra F^0. Thus they are equal, i.e. (8) holds.

(9) follows immediately from (8).

From 16.2, (8) immediately follows.

16.3. *R^0 is a model for the open intuitionistic theory $\mathfrak{C}$ in the set T of all terms and the pseudo-Boolean algebra $\boldsymbol{A}$.*

Copying the terminology adopted in VI, § 9 for canonical realizations, we shall call R^0 the *canonical model determined by the homomorphism h.*

16.4. *An open formula is a theorem in an open intuitionistic theory $\mathfrak{C} = \{\mathfrak{L}, \mathrm{C}_\iota, \mathcal{A}\}$ if and only if it is a theorem in the theory $\mathfrak{C}^0 = \{\mathfrak{L}^0, \mathrm{C}^0_\iota, \mathcal{A}\}$* [1].

In symbols,

$$\mathrm{C}^0_\iota(\mathcal{A}) = \mathrm{C}_\iota(\mathcal{A}) \cap F^0\,, \tag{10}$$

where F^0 is the set of all open formulas in $\mathfrak{C}$.

By (1), it suffices to prove that if an open formula a is not a theorem in $\mathfrak{C}^0$, it is not a theorem in $\mathfrak{C}$. By IV, 9.1 or 9.2, there exists an isomorphism h from $\mathfrak{A}(\mathfrak{C}^0)$ into a complete pseudo-Boolean algebra $\boldsymbol{A}$. If a is not a theorem in $\mathfrak{C}^0$, then $|a|$ is not the unit element of $\mathfrak{A}(\mathfrak{C}^0)$ and consequently $h(|a|)$ is not the unit element of $\boldsymbol{A}$. It follows from (8)

[1] Rasiowa [6], [7].

that the formula a is not valid in the canonical model R^0 for $\mathfrak{T}$, determined by h. Thus a is not a theorem in $\mathfrak{T}$, by 2.2.

Theorem 16.4 is called the *theorem on elimination of quantifiers in formal proofs*. To explain the name, let us recall that a formula a is a theorem in $\mathfrak{T}$ if there exists a sequence $a_1, \ldots, a_n$ which is a formal proof of a in $\mathfrak{T}$. Theorem 16.4 asserts that if $\mathfrak{T}$ is open and an open formula a has a formal proof in $\mathfrak{T}$, then a has a formal proof $a_1, \ldots, a_n$ formed of formulas without quantifiers.

Let $\mathfrak{A}(\mathfrak{T})$ be the algebra of $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$. We recall that elements of $\mathfrak{A}(\mathfrak{T})$ are equivalence classes $\|a\|$ of the relation $\approx$ defined as follows:

(11) $a \approx \beta$ if and only if both formulas $(a \Rightarrow \beta)$ and $(\beta \Rightarrow a)$ are theorems in $\mathfrak{T}$.

It follows immediately from definitions (2) and (11) and from theorem 16.4 that, for any open formulas a, β,

(12) $a \sim \beta$ if and only if $a \approx \beta$.

Let $\mathfrak{A}^0(\mathfrak{T})$ denote the subalgebra of $\mathfrak{A}(\mathfrak{T})$ composed of all elements $\|a\|$ where a is any open formula.

16.5. *The mapping*

$$h_0(|a|) = \|a\| \quad (a \text{ open})$$

is an isomorphism from the pseudo-Boolean algebra $\mathfrak{A}(\mathfrak{T}^0)$ *onto the pseudo-Boolean algebra* $\mathfrak{A}^0(\mathfrak{T})$.

Theorem 16.5 follows immediately from (12).

16.6. *Every consistent open intuitionistic theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$ *has a semantic model in the set of all terms.*

Let V be a maximal filter in the pseudo-Boolean algebra $\mathfrak{A}(\mathfrak{T}^0)$ and let h be the natural homomorphism from $\mathfrak{A}(\mathfrak{T}^0)$ onto $\boldsymbol{A} = \mathfrak{A}(\mathfrak{T}^0)/V$. The canonical model R^0 determined by the homomorphism h has the required property. In fact, it is a model in the set T of all terms and in $\boldsymbol{A}$. By I, 13.10 and II, § 4, $\boldsymbol{A}$ is a two-element Boolean algebra, i.e. R^0 is a semantic model.

According to a general remark in § 1, p. 414, the classical predicate calculus $\{\mathfrak{L}, \mathfrak{C}, 0\}$ can be interpreted as the intuitionistic theory $\{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}_0\}$ where $\mathfrak{A}_0$ is the set of all formulas of the form $(a \cup -a)$, a being any formula in $\mathfrak{L}$.

16.7. *The classical predicate calculus considered as an intuitionistic theory is not equivalent to any open intuitionistic theory.*

Suppose that an open intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$ is equivalent to $\{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}_0\}$. Formulas in $\mathfrak{A}$ have to be open classical tautologies.

Let X be the space of all rational numbers and let J be the set of all positive integers.

Suppose the language $\mathfrak{L}$ contains a one-argument predicate π. Let R be the following realization of $\mathfrak{L}$ in J and $\mathfrak{G}(X)$:

(13) $$\varphi_R(j_1, \ldots, j_m) = 1 \quad \text{for all} \quad j_1, \ldots, j_m \in J$$

and for every m-argument functor φ $(m = 0, 1, 2, \ldots)$;

(14) $$\pi_R(j) = \text{the set of all } x \in X \text{ such that } -1-\frac{\sqrt{2}}{j} < x < 1+\frac{\sqrt{2}}{j}, \quad \text{for all } j \in J;$$

(15) $$\varrho_R(j_1, \ldots, j_m) = X \quad \text{for all} \quad j_1, \ldots, j_m \in J$$

for all m-argument predicates ϱ different from π $(m = 1, 2, \ldots)$.

By definition, the values of ϱ_R are always both open and closed subsets of X, for every predicate ϱ. Consequently (see the last part of IV, 1.4), for every open formula a and for every valuation v, the set $a_R(v)$ is both open and closed in X. If a is an open classical tautology, then $a_R(v) = X$ for every valuation v, since $a_R(v)$ is a closed and dense subset of X (see 7.2). This proves that R is a model for $\mathfrak{T}$.

Let β be the formula $\bigcap_\xi \pi(\xi)$. By (13) and (14),

$$\beta_R = \text{the set of all } x \in X \text{ such that } -1 < x < 1 .$$

Consequently

$$-\beta_R = \text{the set of all } x \in X \text{ such that either } x < -1 \text{ or } x > 1 .$$

Hence it follows that

$$(\beta \cup -\beta)_R = X - [-1, 1] \neq X .$$

Thus the tautology $(\beta \cup -\beta)$ is not valid in the model R. Consequently $(\beta \cup -\beta)$ is not a theorem in $\mathfrak{T}$. This contradicts the hypothesis that $\mathfrak{T}$ is equivalent to $\{\mathfrak{L}, \mathfrak{C}_i, \mathfrak{A}_0\}$.

If $\mathfrak{L}$ does not contain any one-argument predicate, then π should be an arbitrary predicate. Definition (14) and some details in the proof then require obvious modifications.

§ 17. The deduction theorem for open intuitionistic theories. This name is given to the following theorem.

17.1. *An open formula a is a theorem in an open intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_i, \mathfrak{A}\}$ if and only if there exist substitutions $s_1, \ldots, s_n$ and axioms $a_1, \ldots, a_n$ in $\mathfrak{A}$ such that the implication*

(1) $$\left(\bigcap_{i=1}^n s_i a_i \Rightarrow a\right)$$

is an intuitionistic predicate tautology.

If a_i are axioms in $\mathcal{A}$, then the formulas $s_i a_i$ are theorems in $\mathfrak{C}$, by the rule (r_2) of substitution for free individual variables. Consequently the conjunction on the left-hand side of implication (1) is also a theorem in $\mathfrak{C}$ (see VI, 10.8). If (1) is an intuitionistic tautology, then (1) is a theorem in $\mathfrak{C}$ and, by *modus ponens*, a is also a theorem in $\mathfrak{C}$.

On the other hand, the set S_0 of all open formulas a such that, for some substitutions $s_1, \ldots, s_n$ and some axioms $a_1, \ldots, a_n$ in $\mathcal{A}$, implication (1) is an intuitionistic tautology, contains all mathematical axioms a in $\mathcal{A}$ (take $n = 1$, a_1 equal to a, and $s_1 =$ the identity substitution). S_0 also contains any logical axioms (T_1)-(T_{11}) (take arbitrary $s_1, \ldots, s_n$ and $a_1, \ldots, a_n$ in $\mathcal{A}$). The set S_0 is closed under the rule (r_2) of substitution for free individual variables, for if s is any substitution and (1) is an intuitionistic tautology, then

$$(\bigcap_{i=1}^{n} s s_i a_i \Rightarrow s a)$$

is also an intuitionistic tautology as the result of the substitution s in the tautology (1). The set S_0 is also closed with respect to *modus ponens* (r_1). In fact, suppose that a and $(a \Rightarrow \beta)$ are in S_0, i.e. there exist substitutions $s_1, \ldots, s_n, \ldots, s_m$ and axioms $a_1, \ldots, a_n, \ldots, a_m$ in $\mathcal{A}$ such that (1) is an intuitionistic tautology and

$$(\bigcap_{i=n+1}^{m} s_i a_i \Rightarrow (a \Rightarrow \beta))$$

is an intuitionistic tautology. Let γ be the conjunction $\bigcap_{i=1}^{m} s_i a_i$. By 2.1 and IX, 4.1, (T_{13}) and *modus ponens*, the formulas

$$(\gamma \Rightarrow a) \text{ and } (\gamma \Rightarrow (a \Rightarrow \beta))$$

are intuitionistic tautologies. By the intuitionistic tautology (T_{68}) (see IX, 4.1) and *modus ponens*, the formula $(\gamma \Rightarrow \beta)$ is an intuitionistic tautology, which proves that β is in S_0.

Since S_0 contains $\mathcal{A}$ and all logical axioms (T_1)-(T_{11}) and S_0 is closed under (r_1) and (r_2), we infer that $C_i^0(\mathcal{A}) \subset S_0$. By 16.4, $C_i^0(\mathcal{A})$ is the set of all open formulas which are theorems in $\mathfrak{C}$. Thus every open theorem in $\mathfrak{C}$ belongs to S_0, which completes the proof of 17.1.

§ 18. A theorem on the extension of topological realizations. The theorem in question is an intuitionistic substitute for theorem VIII, 8.1 on the extension of semantic realizations of a language $\mathfrak{L}$ to realizations of a rich extension $\mathfrak{L}'$ of $\mathfrak{L}$.

We recall that by a rich extension of a language $\mathfrak{L}$ of the first order we understand the language $\mathfrak{L}'$ constructed as follows (see VIII, § 8, p. 306-307).

Let $\mathfrak{L}_0$ be the language $\mathfrak{L}$. Suppose the language $\mathfrak{L}_n$ is defined and E_n is the set of all existential formulas in $\mathfrak{L}_n$ which are not in $\mathfrak{L}_{n-1}$. Then $\mathfrak{L}_{n+1}$ is

the extension of $\mathfrak{L}_n$, obtained by adding a set Ψ_n of new functors such that there is a one-to-one mapping ψ^n from E_n onto Ψ_n with the property: if a formula α in E_n contains exactly m free individual variables, then its image ψ^n_α is an m-argument functor. The language $\mathfrak{L}'$ is the union of all the languages $\mathfrak{L}_n$, i.e. $\mathfrak{L}'$ is obtained from $\mathfrak{L}$ by adding the union Ψ of sets $\Psi_0, \Psi_1, \Psi_2, \ldots$ of functors.

The set E' of all existential formulas in $\mathfrak{L}'$ is the union of disjoint sets $E_0, E_1, E_2, \ldots$ and consequently all the functions ψ^n determine together a one-to-one mapping ψ from E' onto Ψ. For every existential formula α in $\mathfrak{L}'$ with m free individual variables, i.e. a formula of the form

$$\bigcup_\xi \beta(\xi, x_1, \ldots, x_m), \tag{1}$$

the image ψ_α of α, is an m-argument functor. According to VIII, § 8, the formula

$$\beta(\psi_\alpha(x_1 \ldots x_m), x_1, \ldots, x_m) \tag{2}$$

will be denoted by α'.

18.1. *Let R be a topological realization of $\mathfrak{L}$ in a set $J \neq 0$ and in the algebra $\mathfrak{G}(X)$ of all open subsets of a topological space X, and let p be a fixed point in X. The realization R can be extended to a realization R_p (in J and $\mathfrak{G}(X)$) of the rich extension $\mathfrak{L}'$ of $\mathfrak{L}$ in such a way that, for every existential formula α in $\mathfrak{L}'$, and for every valuation v,*

$$\text{if} \quad p \in \alpha_{R_p}(v), \quad \text{then} \quad p \in \alpha'_{R_p}(v). \tag{3}$$

We shall define, by induction, a sequence of realizations R_n of the languages $\mathfrak{L}_n$ in J and $\mathfrak{G}(X)$ such that R_{n+1} is an extension of R_n, and R_0 is the realization R. Suppose that R_n has been defined. To define R_{n+1} it suffices to define $\varphi_{R_{n+1}}$ for every m-argument functor φ in Ψ_n $(m = 0, 1, 2, \ldots)$. We have $\varphi = \psi_\alpha$ for an existential formula α of the form (1) where $\beta(x, x_1, \ldots, x_m)$ is a formula in $\mathfrak{L}_n$. By the induction hypothesis, α_{R_n} and β_{R_n} are already defined. We shall use the notation from VI, § 6, p. 227-228, for values of the mappings α_{R_n} and β_{R_n}. Let $j_1, \ldots, j_m$ be any elements in J. If $p \notin \alpha_{R_n}(j_1, \ldots, j_m)$, we define $\varphi_{R_{n+1}}(j_1, \ldots, j_m)$ in an arbitrary way. If

$$p \in \alpha_{R_n}(j_1, \ldots, j_m) = \bigcup_{j \in J} \beta_{R_n}(j, j_1, \ldots, j_m), \tag{4}$$

then there exists a $j_0 \in J$ such that $p \in \beta_{R_n}(j_0, j_1, \ldots, j_m)$. Then we define

$$\varphi_{R_{n+1}}(j_1, \ldots, j_m) = j_0.$$

By the definition,

$$p \in \beta_{R_n}(\varphi_{R_{n+1}}(j_1, \ldots, j_m), j_1, \ldots, j_m) = \alpha'_{R_{n+1}}(j_1, \ldots, j_m). \tag{5}$$

Since R_{n+1} is an extension of R_n $(n = 0, 1, 2, \ldots)$, all the realizations $R_0, R_1, R_2, \ldots$ determine together a realization R_p of $\mathfrak{L}'$ which, by (4) and (5), has property (3).

Theorem 18.1 just proved is stronger than theorem VIII, 8.1. To obtain VIII, 8.1 from 18.1 it suffices to assume that X is a one-point space.

§ 19. Elimination of initial quantifiers from axioms of an intuitionistic theory. In VIII, 17.3 we proved that every classical theory of the first order has an inessential extension which is an open theory. The analogous statement for intuitionistic theories is not true. In fact, let $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$ be a consistent intuitionistic theory which has no semantic model. The theory $\mathfrak{T}$ cannot be extended to a consistent open theory $\mathfrak{T}' = \{\mathfrak{L}', \mathfrak{C}_\iota, \mathfrak{A}'\}$. For if such an extension $\mathfrak{T}'$ existed, then $\mathfrak{T}'$ would have a semantic model R' on account of 16.6, and the restriction R of R' to the language $\mathfrak{L}$ would be a semantic model for $\mathfrak{T}$.

In this section we shall prove an intuitionistic analogue (theorem 19.5 below) of VIII, 17.3 which is, however, weaker than theorem VIII, 17.3. Our considerations in VIII, § 17 were based on the fact that for every formula α (in particular, for every axiom of the theory under consideration) there exists a prenex formula β such that

$$(1) \qquad (\alpha \Rightarrow \beta) \quad \text{and} \quad (\beta \Rightarrow \alpha)$$

are classical tautologies. This statement is not true for intuitionistic logic: for some formulas α there exists no prenex formula β such that implications (1) are intuitionistic tautologies (an example will be quoted at the end of this section). Therefore our result is restricted to intuitionistic theories $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\iota, \mathfrak{A}\}$ such that every axiom in $\mathfrak{A}$ is a prenex formula. Also under this restriction the result is weaker than VIII, 17.3. We shall prove that every intuitionistic theory with prenex axioms has an extension which is inessential for open formulas only. The proof will be preceded by the following definitions and lemmas.

Let $\mathfrak{L}$ be a formalized language of the first order and let $\mathfrak{L}'$ be the rich extension of $\mathfrak{L}$, defined in § 18. For every existential formula α in $\mathfrak{L}'$, the symbol α' will have the same meaning as in § 18. If R is any realization of $\mathfrak{L}$ in a set $J \neq 0$ and the algebra $\mathfrak{G}(X)$ of all open subsets of a topological space X, and if $p \in X$, then R_p will denote an extension of R to a realization of $\mathfrak{L}'$ such that R_p has property (3) from theorem 18.1.

Let α be a formula in $\mathfrak{L}'$ such that the first sign in α is a quantifier. If this quantifier is existential, then α' will be called the formula obtained from α by *elimination of the first quantifier*. If the first sign in α is the universal quantifier, i.e. if α is of the form

$$(2) \qquad \bigcap_\xi \beta(\xi, x_1, \ldots, x_m),$$

then each formula of the form

$$\beta(x, x_1, \ldots, x_m), \tag{3}$$

where the free individual variable x does not appear in α, will be called a formula obtained from α by elimination of the first quantifier.

19.1. *If γ is a formula obtained from α by elimination of the first quantifier, then*

(i) *α belongs to $\mathfrak{C}_{\mathfrak{L}'}(\gamma)$;*

(ii) *for every topological realization R of $\mathfrak{L}$ in a set $J \neq 0$ and in the algebra $\mathfrak{G}(X)$ of open subsets of a topological space X, and for every point $p \in X$, if*

$$p \in \alpha_{R_p}(v) \quad \textit{for every valuation } v\,,$$

then

$$p \in \gamma_{R_p}(v) \quad \textit{for every valuation } v\,.$$

The proof of (i) is the same as the proof of part (i) in the analogous theorem VIII, 17.1.

If α is an existential formula, then (ii) follows directly from 18.1.

If α is of the form (2), then γ is the formula (3) and

$$\alpha_{R_p}(j_1, \ldots, j_m) = \mathbf{I} \bigcap_{j \in J} \beta_{R_p}(j, j_1, \ldots, j_m)$$

by VI, § 6, (4) and IV, 7.5. Here $\bigcap$ denotes the set-theoretical intersection. If $p \in \alpha_{R_p}(j_1, \ldots, j_m)$ for all $j_1, \ldots, j_m \in J$, then also $p \in \beta_{R_p}(j, j_1, \ldots, j_m)$ for all $j, j_1, \ldots, j_m \in J$. This completes the proof of (ii).

Every formula α in $\mathfrak{L}'$ is of the form

$$\Xi^1_{\xi_1} \ldots \Xi^n_{\xi_n} \beta(\xi_1, \ldots, \xi_n, x_1, \ldots, x_m) \tag{4}$$

where $\Xi^1, \ldots, \Xi^n$ are quantifiers and $\beta(y_1, \ldots, y_n, x_1, \ldots, x_m)$ is a formula whose first sign is not a quantifier. The case $n = 0$ is admitted.

A formula α^* is said to be obtained from formula (4) by *elimination of all initial quantifiers* if there exists a sequence

$$\alpha_0, \alpha_1, \ldots, \alpha_n \tag{5}$$

such that

1) α_0 is the formula (4);

2) α_{i+1} is obtained from α_i by elimination of the first quantifiers $(i = 0, \ldots, n-1)$;

3) α_n is the formula α^*.

By definition, the first sign of α^* is not a quantifier. More precisely, α^* is a substitution of $\beta(y_1, \ldots, y_n, x_1, \ldots, x_m)$. For instance, if α is the formula

$$\bigcap_{\xi_1} \bigcup_{\eta_1} \ldots \bigcap_{\xi_r} \bigcup_{\eta_r} \beta(x, \xi_1, \eta_1, \ldots, \xi_r, \eta_r)$$

then α^* is of the form

$$\beta(x, x_1, \varphi_1(x\, x_1), \ldots, x_r, \varphi_r(x\, x_1 \ldots x_r)) .$$

Let α be any formula (4) in $\mathfrak{L}'$. Note that

(6) if α is a prenex formula, then α^* is an open formula;

(7) if $\bar{\alpha}$ is a closure of α, then α^* is also the formula obtained from $\bar{\alpha}$ by elimination of all initial quantifiers.

19.2. *If α^* is obtained from α by elimination of all initial quantifiers, then*

(i) *α belongs to $\mathfrak{C}_{\mathfrak{L}'}(\alpha^*)$;*

(ii) *for every topological realization R of $\mathfrak{L}$ in a set $J \neq 0$ and in the algebra $\mathfrak{G}(X)$ of open subsets of a topological space X, and for every point $p \in X$, if*

$$p \in \alpha_{R_p}(v) \quad \text{for every valuation } v,$$

then

$$p \in \alpha^*_{R_p}(v) \quad \text{for every valuation } v.$$

Theorem 19.2 follows directly from 19.1.

Now let $\mathcal{A}$ be a set of formulas in the language $\mathfrak{L}$. Assign, to every formula α in $\mathcal{A}$, a formula α^* (in $\mathfrak{L}'$) obtained from α by elimination of all initial quantifiers. Let $\mathcal{A}^*$ be the set of all formulas α^* where α is in $\mathcal{A}$. Let $\mathfrak{L}^*$ be the least sublanguage of $\mathfrak{L}'$ such that $\mathfrak{L}^*$ is an extension of $\mathfrak{L}$ and $\mathfrak{L}^*$ contains all the formulas in $\mathcal{A}^*$. The intuitionistic theory $\mathfrak{T}^* = \{\mathfrak{L}^*, \mathfrak{C}_i, \mathcal{A}^*\}$ is said to be obtained from the intuitionistic theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_i, \mathcal{A}\}$ by *elimination of all initial quantifiers in axioms* of $\mathfrak{T}$.

19.3. *The theory $\mathfrak{T}^*$ is an extension of $\mathfrak{T}$.*

This follows directly from 19.2, (i).

19.4. *Let $\alpha_1, \ldots, \alpha_n$ be any formulas in $\mathcal{A}$, let β be any formula in $\mathfrak{L}$ and let $s_1, \ldots, s_n$ be any substitutions in $\mathfrak{L}^*$. If the implication*

$$(\bigcap_{i=1}^{n} s_i \alpha_i^* \Rightarrow \beta) \tag{8}$$

is an intuitionistic tautology, then the implication

$$(\bigcap_{i=1}^{n} \bar{\alpha}_i \Rightarrow \beta) \tag{9}$$

is also an intuitionistic tautology, $\bar{\alpha}_1, \ldots, \bar{\alpha}_n$ being closures of $\alpha_1, \ldots, \alpha_n$ respectively.

Denote formulas (8) and (9) by γ and δ respectively.

Suppose that δ is not an intuitionistic tautology, i.e. it is not derivable in $\mathcal{S}_i = \{\mathfrak{L}, \mathfrak{C}_i, 0\}$. By 4.1, there exists a topological realization R of $\mathfrak{L}$

in a set J and in the algebra $\mathfrak{G}(X)$ of open subsets of a topological space $X \neq 0$, and a valuation v^0: $V \to J$ such that

$$\delta_R(v^0) \neq X . \tag{10}$$

Since the $\overline{a}_i$ are closed formulas, the functions $\overline{a}_{iR}$ are constant, i.e. they are fixed open subsets of X. Inequality (10) means that

$$\text{the set } \bigcap_{i=1}^n \overline{a}_{iR} \text{ is not contained in the set } \beta_R(v^0),$$

i.e. there exists a point $p \in X$ such that

$$p \in \overline{a}_{iR} \quad \text{for} \quad i = 1, \ldots, n \tag{11}$$

and

$$p \notin \beta_R(v^0) . \tag{12}$$

Take the extension R_p of R. It follows from (7), (11) and 19.2, (ii) that

$$p \in a^*_{iR_p}(v) \quad \text{for every valuation } v$$

and $i = 1, \ldots, n$. In particular

$$p \in a^*_{iR_p}(s_{iR_p} v^0)$$

where, according to VI, § 3, p. 218, for any substitution v and any valuation $v = \{v_x\}_{x \in V}$ the symbol $s_{iR_p} v^0$ denotes the valuation $\{s_i x_{R_p}(v^0)\}_{x \in V}$. By VI, 6.7,

$$a^*_{iR_p}(s_{iR_p} v^0) = s_i a^*_{iR_p}(v^0) .$$

Since R_p is an extension of R, we have (see VI, 6.4)

$$\beta_{R_p}(v^0) = \beta_R(v^0) .$$

Thus

$$p \in (\bigcap_{i=1}^n s_i a^*_i)_{R_p}(v^0) \quad \text{and} \quad p \notin \beta_{R_p}(v^0) .$$

This proves that the set $(\bigcap_{i=1}^n s_i a^*_i)_{R_p}(v^0)$ is not a subset of $\beta_{R_p}(v^0)$, i.e. that

$$\gamma_{R_p}(v^0) \neq X .$$

By 4.1, γ is not an intuitionistic tautology.

19.5. *If all formulas in $\mathcal{A}$ are prenex formulas, then the theory $\mathcal{C}^* = \{\mathfrak{L}, \mathfrak{C}_i, \mathcal{A}^*\}$ obtained from the theory $\mathcal{C} = \{\mathfrak{L}, \mathfrak{C}_i, \mathcal{A}\}$ by elimination of all initial quantifiers in axioms is open.*

For every open formula β in $\mathfrak{L}$, β is a theorem in $\mathcal{C}$ if and only if β is a theorem in $\mathcal{C}^*$.

Consequently, $\mathcal{C}$ is consistent if and only if $\mathcal{C}^*$ is consistent [1].

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[1] Rasiowa [6], [7]. The above proof is a modification of the proof due to Rasiowa [7].

The first statement follows directly from (6).

By 19.3, in order to prove the second statement it suffices to show that if an open formula β in $\mathfrak{L}$ is a theorem in $\mathfrak{T}^*$, then β is also a theorem in $\mathfrak{T}$. By the deduction theorem 17.1, if β is an open theorem in $\mathfrak{T}^*$, then there exist some formulas $a_1^*, \ldots, a_n^*$ in $\mathfrak{A}^*$ and substitutions s_i in $\mathfrak{L}^*$ such that implication (8) is an intuitionistic tautology. By 19.4, implication (9) also is an intuitionistic tautology. By the deduction theorem 9.2, β is a theorem in $\mathfrak{T}$.

The third statement follows directly from the second one.

As an application of 19.5 we shall prove the following generalization of theorem 16.6.

19.6. *Every consistent intuitionistic theory* $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_i, \mathfrak{A}\}$ *such that all axioms in* $\mathfrak{A}$ *are prenex formulas has a semantic model in an enumerable set* [1].

By 19.5, the open theory $\mathfrak{T}^*$ is consistent. By 16.6, $\mathfrak{T}^*$ has a semantic model R^* in the enumerable set of all terms of $\mathfrak{T}^*$. The restriction R of R^* to the language $\mathfrak{L}$ is a semantic model for $\mathfrak{T}$, on account of 19.3.

Now we are able to give an example of a formula a such that there exists no prenex formula β such that implications (1):

$$(a \Rightarrow \beta) \quad \text{and} \quad (\beta \Rightarrow a)$$

are intuitionistic tautologies. Let ϱ be a one-argument predicate and let a be the formula $-\bigcap_\xi (\varrho(\xi) \cup -\varrho(\xi))$. Suppose that there exists a prenex formula β such that the two implications are intuitionistic tautologies. Then the theories $\{\mathfrak{L}, \mathfrak{C}_i, [a]\}$ and $\{\mathfrak{L}, \mathfrak{C}_i, [\beta]\}$ are equivalent. On the other hand, the first theory has no semantic model (see the proof of 11.7). The second theory has a semantic model by 19.6. Since equivalent theories have the same models, we get a contradiction.

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[1] This remark was communicated to us by J. Łoś.

CHAPTER XI

THE POSITIVE LOGIC AND THE MODAL LOGIC

§ 1. Introduction. In Chapter V we fixed a type of formalized languages of the zero and first order. We also introduced a general definition of logic as a mapping which, to every formalized language $\mathfrak{L}$, assigns a consequence operation in $\mathfrak{L}$. We gave a general scheme (L) of constructing logics. The scheme (L) is based on the choice of a set I of formulas in an auxiliary language of zero order. All substitutions of formulas in I are assumed to form the set $\mathscr{A}_\mathrm{I}$ of logical axioms in any language $\mathfrak{L}$ of the zero or first order. The consequence operation in $\mathfrak{L}$ is determined by the set $\mathscr{A}_\mathrm{I}$ of logical axioms and the assumed rules of inference (see V, § 10, (S)). Thus, in the scheme (L) adopted, the logic is determined uniquely by the set I, and the set R of the rules of inference.

We have applied the scheme (L) to define classical logic and the intuitionistic logic. The classical logic plays a special part in metamathematical investigations because it is a formalization of the logical means used in intuitive mathematics. Intuitionistic logic was introduced by changing the set I assumed for classical logic. Of course, by other modifications of the set I we can obtain various logics different from the classical logic and intuitionistic logic. Usually they are weaker than classical logic. Many examples of such non-classical logics have been examined by logicians [1].

Other non-classical logics can be introduced by modifying the definition of formalized languages of the zero and first order. Usually these modifications consist in a change of the set of logical connectives. In Chapters V-X we assumed that all languages contain the disjunction sign $\cup$, the conjunction sign $\cap$, the implication sign $\Rightarrow$, and the negation sign $-$. By omitting one or more of these signs we can obtain other types of formalized languages. An example of a non-classical logic based on languages with the connectives $\cup$, $\cap$, $\Rightarrow$ only will be discussed in §§ 2-6.

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[1] For instance, the minimal logic of Johansson [1] the ν-logic of Rasiowa and Sikorski [3], the 3-valued logic of Łukasiewicz [1], the m-valued logics of Łukasiewicz and Tarski [1] (see also Rosser and Turquette [1]), etc.

It is called the *positive logic* (1). There are also types of formalized languages containing more connectives than $\cup, \cap, \Rightarrow, -$ or different connectives from those mentioned above (2). An example of a non-classical logic based on languages containing more connectives than $\cup, \cap, \Rightarrow, -$ will be discussed in §§ 7-11. It is called the modal *logic* (3). Of course, the introduction of new logical connectives may be connected with adopting some new rules of inference.

Similarly to the case of the two-valued and intuitionistic logic, the discussion of any type of non-classical logics leads to the examination of an adequate type of abstract algebras. Usually these algebras are lattices with some additional operations. We shall see that the examination of positive and modal logics can be reduced to the examination of relatively pseudocomplemented lattices and topological Boolean algebras, respectively.

The theory of the positive and modal logics, which are the subject of this Chapter, could be developed as widely as the theory of the classical and intuitionistic logics presented in this book. However, we shall not discuss the positive and modal logics in detail because the general methods of investigation are similar to those of the classical and intuitionistic logics. We shall sketch only the completeness theorems for the positive and modal logics, some related questions and the connection with the classical or intuitionistic logic.

The symbols $\mathbf{C}$ and $\mathbf{C}_i$ will always denote the classical consequence operation defined in V § 11 and the intuitionistic consequence operation defined in IX § 1, respectively.

§ 2. The positive logic. In §§ 2-6, the letters $\mathfrak{L}_0$ and $\mathfrak{L}$ will exclusively denote formalized languages (of zero order and the first order, respectively) defined as in V § 6 and V § 3 respectively, with the following modification:

The set of all connectives contains only the disjunction sign $\cup$, *the conjunction sign* $\cap$ *and the implication sign* $\Rightarrow$.

As we stated in § 1, formalized languages $\mathfrak{L}_0, \mathfrak{L}$ of the above type constitute the domain of the definition of a non-classical logic called the *positive logic*. Namely, the positive logic is the mapping $\mathbf{C}_\pi$ which assigns a consequence operation in every language $\mathfrak{L}_0$ and $\mathfrak{L}$ of the above type

(1) Hilbert and Bernays [1], vol. I.

(2) For instance, the constructive logic with strong negation of Nelson [2] and Марков [1], examined by Воробьев [1], [2] and—from an algebraic point of view—by Rasiowa [9], [10] and Białynicki-Birula and Rasiowa [1].

(3) Lewis and Langford [1]. In this book by a *modal logic* we understand the logic which corresponds to the system S.4 of the modal propositional calculus in the terminology of Lewis and Langford.

by means of the general scheme V, § 11, (L) (see p. 187) where I is the set formed of tautologies (t_1)-(t_9) only (see V, § 6, p. 168). The admitted rules of inference are the same as in the case of classical or intuitionistic logic: *modus ponens* in the case of any language $\mathfrak{L}_0$ of zero order, and the rules (r_1)-(r_6) from V § 8 in the case of any language $\mathfrak{L}$ of the first order.

By the general convention in V, § 11, p. 188, the positive consequence operation $C_{\pi\mathfrak{L}_0}$ (or: $C_{\pi\mathfrak{L}}$) assigned by C_π in a language $\mathfrak{L}_0$ (or: $\mathfrak{L}$) will be simply denoted by C_π if no ambiguity is possible. Thus, for any set S of formulas in $\mathfrak{L}_0$ or $\mathfrak{L}$, $C_\pi(S)$ will denote the set of all positive consequences of S in the language $\mathfrak{L}_0$ or $\mathfrak{L}$, respectively. The deductive systems

$$\mathcal{S}_{0\pi} = \{\mathfrak{L}_0, C_\pi\}, \qquad \mathcal{S}_\pi = \{\mathfrak{L}, C_\pi\} \tag{1}$$

will be called respectively: the *positive propositional calculus* based on $\mathfrak{L}_0$ and the *positive predicate calculus* based on $\mathfrak{L}$.

Besides the positive propositional and predicate calculi (1) we shall also examine *positive theories of zero order*

$$\{\mathfrak{L}_0, C_\pi, \mathcal{A}\} \tag{2}$$

where $\mathcal{A}$ is any set of formulas in $\mathfrak{L}_0$, and *positive theories of the first order*

$$\{\mathfrak{L}, C_\pi, \mathcal{A}\} \tag{3}$$

where $\mathcal{A}$ is any set of formulas in $\mathfrak{L}$. According to the general convention in V, § 10, p. 186, formulas in the set $C_\pi(\mathcal{A})$ are called *theorems* of the theory (2) or (3) respectively.

Of course, $\mathcal{S}_{0\pi}$ and $\mathcal{S}_\pi$ will be identified with positive theories $\{\mathfrak{L}_0, C_\pi, 0\}$ and $\{\mathfrak{L}, C_\pi, 0\}$ respectively. Formulas in $C_\pi(0)$ are said to be *derivable in* $\mathcal{S}_{0\pi}$ or $\mathcal{S}_\pi$ respectively, or formulas *positively derivable.*

We shall apply to formalized languages $\mathfrak{L}_0$ and $\mathfrak{L}$ all the results obtained in Chapter VI. Of course, all definitions and theorems in Chapter VI now require obvious modifications caused by avoiding the negation sign in the alphabets of the languages $\mathfrak{L}_0$ and $\mathfrak{L}$. For instance, the set F of all formulas in $\mathfrak{L}_0$ or $\mathfrak{L}$ should now be understood as an algebra with three operations only:

$$\{F, \cup, \cap, \Rightarrow\}, \tag{4}$$

etc. The reader can easily verify that all results in Chapter VI remain valid, avoiding those directly concerning operations corresponding to the negation sign.

In particular (see VI, 10.3, VI, 10.4), if $\mathfrak{T}$ is a positive formalized theory of the zero or first order, then the relation $\approx$ defined in F as follows:

(5) $\alpha \approx \beta$ if and only if $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are theorems in $\mathfrak{T}$

is a congruence in the algebra (4) of all formulas in $\mathfrak{C}$. The quotient algebra $F/\approx$ will be denoted by $\mathfrak{A}(\mathfrak{C})$ and called the *algebra of the theory* $\mathfrak{C}$, according to VI, § 10. Directly from VI, 10.3-10.4 follows

2.1. *The algebra* $\mathfrak{A}(\mathfrak{C})$ *is a relatively pseudo-complemented lattice. Moreover, for any formulas* α, β,

$$\|\alpha\| \cup \|\beta\| = \|(\alpha \cup \beta)\|, \quad \|\alpha\| \cap \|\beta\| = \|(\alpha \cap \beta)\|, \tag{6}$$
$$\|\alpha\| \Rightarrow \|\beta\| = \|(\alpha \Rightarrow \beta)\|.$$

The relation

$$\|\alpha\| \leqslant \|\beta\| \tag{7}$$

holds if and only if $(\alpha \Rightarrow \beta)$ *is a theorem in* $\mathfrak{C}$. *The element* $\|\alpha\|$ *is the unit* V *of* $\mathfrak{A}(\mathfrak{C})$ *if and only if* α *is a theorem in* $\mathfrak{C}$.

If $\mathfrak{C}$ is a positive theory of the first order and T is the set of all terms in $\mathfrak{C}$, then for any formula $\beta(x)$

$$\|\bigcup_\xi \beta(\xi)\| = \bigcup_{\tau \in T} \|\beta(\tau)\|, \tag{Q}$$
$$\|\bigcap_\xi \beta(\xi)\| = \bigcap_{\tau \in T} \|\beta(\tau)\|$$

in the relatively pseudo-complemented lattice $\mathfrak{A}(\mathfrak{C})$ (see VI, 11.2). Then we shall always conceive of $\mathfrak{A}(\mathfrak{C})$ as a generalized algebra $\{\mathfrak{A}(\mathfrak{C}), \cup, \cap, \Rightarrow, \bigcup|\mathfrak{Q}, \bigcap|\mathfrak{Q}\}$ where $\bigcup$ and $\bigcap$ are respectively the infinite join and meet. As the common domain $\mathfrak{Q}$ of $\bigcup$ and $\bigcap$ in $\mathfrak{A}(\mathfrak{C})$ we assume the class of all indexed sets $\{\|\beta(\tau)\|\}_{\tau \in T}$, where $\beta(x)$ is any formula in $\mathfrak{L}$. This generalized algebra is called the *quantifier algebra of* $\mathfrak{C}$, or briefly the $\mathfrak{Q}$*-algebra of* $\mathfrak{C}$.

§ 3. Positive theories of zero order. In this section we shall examine a fixed language of zero order $\mathfrak{L}_0$ of the type described in § 2.

Let

$$\{A, \cup, \cap, \Rightarrow\}$$

be an abstract algebra with three binary operations $\cup, \cap, \Rightarrow$. According to VI, § 2, every element $v = \{v_a\}_{a \in V_0} \in A^{V_0}$ where V_0 is the set of all propositional variables in $\mathfrak{L}_0$ is called a valuation in A. By VI, § 2, p. 211, each formula α in $\mathfrak{L}_0$ determines a mapping

$$\alpha_A \colon A^{V_0} \to A. \tag{1}$$

In §§ 3-4 we shall always assume that algebra (1) is a relatively pseudo-complemented lattice. Sometimes we shall take as A the algebra $\mathfrak{A}(\mathfrak{C})$ of a formalized theory $\mathfrak{C} = \{\mathfrak{L}_0, \mathfrak{C}_\pi, \mathcal{A}\}$. We recall that the valuation

$$v^0 = \{\|\alpha\|\}_{\alpha \in V_0} \tag{2}$$

in $\mathfrak{A}(\mathfrak{C})$ is called the *canonical valuation* for $\mathfrak{C}$. By VI, 2.6

$$\alpha_{\mathfrak{A}(\mathfrak{C})}(v^0) = \|\alpha\| . \tag{3}$$

A valuation $v \in A^{V_0}$ is said to be a *model* for a set $\mathcal{A}$ of formulas if

$$\alpha_A(v) = 1 \quad \text{for every formula } \alpha \text{ in } \mathcal{A} .$$

In particular (the case in which $\mathcal{A}$ is formed of one formula only), the valuation v is said to be a *model* for a formula α if $\alpha_A(v) = 1$. A valuation v is said to be a *model* for a theory $\mathfrak{C} = \{\mathfrak{L}_0, C_\pi, \mathcal{A}\}$ provided v is a model for the set $\mathcal{A}$ of axioms of $\mathfrak{C}$.

A formula α is said to be *valid* in a relatively pseudo-complemented lattice A provided every valuation $v \in A^{V_0}$ is a model for α, i.e. if the mapping α_A is identically equal to the unit 1 of A.

A formula is said to be *positively valid* or to be a *positive propositional tautology* if it is valid in every relatively pseudo-complemented lattice.

For instance, all logical axioms, i.e. all the formulas in $\mathfrak{L}_0$ which are of the form (T_1)-(T_9) (see p. 188) are positive propositional tautologies (the proof is the same as in the case of intuitionistic propositional calculi—see IX, 2.5).

3.1. *For every formula β in a positive formalized theory $\mathfrak{C} = \{\mathfrak{L}_0, C_\pi, \mathcal{A}\}$, the following conditions are equivalent:*

(i) *β is a theorem of $\mathfrak{C}$;*

(ii) *every model for $\mathfrak{C}$ is a model for β.*

In fact, if a valuation $v \in A^{V_0}$ is a model for $\mathfrak{C}$ and β is a theorem in $\mathfrak{C}$, then v is a model for β since the class of all formulas α for which $\alpha_A(v) = 1$ contains all axioms in $\mathcal{A}$ and all logical axioms and, it is closed under *modus ponens* (the exact proof is analogous to that of VII, 7.1 and IX, 8.1). On the other hand, the canonical valuation v^0 for $\mathfrak{C}$ is a model for $\mathfrak{C}$ by 2.1 and (3). If β is not a theorem in $\mathfrak{C}$, then v^0 is not a model for β, by the same argument (see the analogous proof of the corresponding parts of theorems VII, 7.8 and IX, 8.5).

Now denote by $\mathfrak{L}_0^*$ the language obtained from $\mathfrak{L}_0$ by including the negation sign $-$ in the alphabet of $\mathfrak{L}_0$. Thus $\mathfrak{L}_0^*$ is a language (of zero order) of the type investigated in Chapters VII and IX. All formulas in $\mathfrak{L}_0$ are formulas in $\mathfrak{L}_0^*$. Any set $\mathcal{A}$ of formulas in $\mathfrak{L}_0$ uniquely determines two theories of zero order: the positive theory $\mathfrak{C} = \{\mathfrak{L}_0, C_\pi, \mathcal{A}\}$ and the intuitionistic theory $\mathfrak{C}^* = \{\mathfrak{L}_0^*, C_\iota, \mathcal{A}\}$.

3.2. *For every formula β in $\mathfrak{L}_0$, the following conditions are equivalent:*

(i) *β is a theorem in $\mathfrak{C} = \{\mathfrak{L}_0, C_\pi, \mathcal{A}\}$;*

(ii) *β is a theorem in $\mathfrak{C}^* = \{\mathfrak{L}_0^*, C_\iota, \mathcal{A}\}$.*

In other words,

$$C_{\pi\mathfrak{L}_0}(\mathcal{A}) = C_{i\mathfrak{L}_0^*}(\mathcal{A}) \cap F_0 \tag{4}$$

where F_0 denote the set of all formulas in $\mathfrak{L}_0$.

Since all formulas in $\mathfrak{L}_0$ of the form (T_1)-(T_9), i.e. all logical axioms of the positive logic, are also logical axioms of the intuitionistic logic, we have $C_{\pi\mathfrak{L}_0}(\mathcal{A}) \subset C_{i\mathfrak{L}_0^*}(\mathcal{A})$. Thus (i) implies (ii).

Suppose now that a formula β in $\mathfrak{L}_0$ is not a theorem in $\mathfrak{T}$. By 3.1, there exists a relatively pseudo-complemented lattice $\boldsymbol{A}$ and a valuation $v \in \boldsymbol{A}^{V_0}$ such that v is a model for $\mathfrak{T}$ but v is not a model for β, i.e. $\beta_{\boldsymbol{A}}(v) \neq V$. By IV 1.3, the lattice $\boldsymbol{A}$ can be extended to a pseudo-Boolean algebra $\boldsymbol{A}^*$ by adding a zero element Λ, the operations $\cup$, $\cap$, $\Rightarrow$ in $\boldsymbol{A}$ coinciding with those in $\boldsymbol{A}^*$ restricted to $\boldsymbol{A}$. Thus we can interpret the valuation v as a valuation in $\boldsymbol{A}^*$. It is easy to see that

$$\alpha_{\boldsymbol{A}^*}(v) = \alpha_{\boldsymbol{A}}(v) \quad \text{for every formula } \alpha \text{ in } \mathfrak{L}_0.$$

Hence it follows that v, interpreted as a valuation in $\boldsymbol{A}^*$, is a model for $\mathfrak{T}^*$ and that v is not a model for β. By IX 8.1, β is not a theorem in $\mathfrak{T}^*$.

Theorem 3.2 shows that the examination of positive theories $\mathfrak{T}$ of zero order can easily be reduced to the examination of the corresponding intuitionistic theories $\mathfrak{T}^*$ of zero order. The lattice-theoretical explanation of this fact consists in the possibility of extending any relatively pseudo-complemented lattice to a pseudo-Boolean algebra with preservation of the operations $\cup$, $\cap$, $\Rightarrow$ (see IV 1.3).

§ 4. The positive propositional calculus. Assuming as $\mathcal{A}$ the empty set in theorem 3.1 we get the following *completeness theorem for the positive propositional calculus* $\mathcal{S}_{0\pi} = \{\mathfrak{L}_0, C_\pi\}$ ($\mathfrak{L}_0$ always denotes a language of zero order; of the type described in § 2):

4.1. *For every formula β in $\mathfrak{L}_0$, the following conditions are equivalent:*

(i) *β is derivable in $\mathcal{S}_{0\pi}$;*

(ii) *β is a positive propositional tautology.*

We recall that every formula in $\mathfrak{L}_0$ is also a formula of the language $\mathfrak{L}_0^*$ obtained from $\mathfrak{L}_0$ by adding the negation sign $-$. The following theorem is a direct consequence of 3.2 (the case in which $\mathcal{A}$ is the empty set) by 4.1 and IX 3.1.

4.2. *For every formula β in $\mathfrak{L}_0$ the following conditions are equivalent:*

(i) *β is a positive propositional tautology;*

(ii) *β is an intuitionistic propositional tautology.*

Roughly speaking, the examination of the positive propositional calculus can be reduced to the examination of the part of the intuition-

istic propositional calculus composed of formulas without the negation sign. As an application we mention here the following theorem which follows directly from 4.2 and the analogous statement IX, 6.1.

4.3. *A disjunction $(\alpha \cup \beta)$ in $\mathfrak{L}_0$ is a positive propositional tautology if and only if one of the formulas α, β is a positive propositional tautology.*

It also follows from 4.2 and IX, 4.1 that all formulas of the form (T_0), (T_{13}), (T_{14}), (T_{15}), (T_{16}) (see p. 388) are positive propositional tautologies. It follows from 4.2 and IX, 4.2 that the tautology

$$(T_{30}) \qquad (((a \Rightarrow b) \Rightarrow a) \Rightarrow a)$$

where a, b are different propositional variables, is not a positive tautology.

The following theorem can be proved in the same way as the analogous statements VII, 4.1 and IX, 7.1.

4.4. *The algebra $\mathfrak{A}(\mathfrak{S}_{0\pi})$ of the positive propositional calculus is free in the class of all relatively pseudo-complemented lattices $\{\boldsymbol{A}, \cup, \cap, \Rightarrow\}$, the class of all elements $\|\alpha\|$ where α is any propositional variable being a set of free generators.*

§ 5. Positive theories of the first order. In this and in the next section we shall examine a fixed language of the first order $\mathfrak{L}$, of the type described in § 2. To facilitate our consideration we shall always assume that the language $\mathfrak{L}$ is enumerable, similarly as in Chapter X (see p. 411).

The definition of a realization R of $\mathfrak{L}$ in a set $J \neq 0$ and a complete algebra

$$(1) \qquad \{\boldsymbol{A}, \cup, \cap, \Rightarrow, \bigcup, \bigcap\}$$

with three binary operations $\cup, \cap, \Rightarrow$ and two infinite operations $\bigcup, \bigcap$ is the same as in VI, § 6: it is a mapping which assigns, to every m-argument functor φ in $\mathfrak{L}$, a function $\varphi_R\colon J^m \to J$, and which assigns, to every m-argument predicate ϱ in $\mathfrak{L}$, a function $\varrho_R\colon J^m \to \boldsymbol{A}$. By VI, § 6, given any realization R of $\mathfrak{L}$ in J and $\boldsymbol{A}$, every formula α in $\mathfrak{L}$ uniquely determines a mapping

$$(2) \qquad \alpha_R\colon J^V \to \boldsymbol{A}$$

where V is the set of all free individual variables in $\mathfrak{L}$. Points $v = \{v_x\}_{x \in V}$ in J^V are called *valuations* in J. According to VI, § 7, we shall also admit realizations in incomplete generalized algebras (1) under the hypothesis that all infinite operations appearing in the inductive definition of α_R are always feasible. In §§ 5 and 6 we shall always suppose that $\boldsymbol{A}$ is a relatively pseudo-complemented lattice.

In particular, we can assume as algebra (1) the Q-algebra $\mathfrak{A}(\mathfrak{T})$ of a positive theory $\mathfrak{T} = \{\mathfrak{L}, C_\pi, \mathcal{A}\}$. The definition of the canonical

realization R^0 of $\mathfrak{L}$ in the set T of all terms in $\mathfrak{L}$ and in the algebra $\mathfrak{A}(\mathfrak{C})$ is the same as in VI § 7, p. 239. We recall the fundamental property of the canonical realization R^0 (see VI 7.4): for every formula α in $\mathfrak{L}$,

$$a_{R^0}(v) = \|v'\alpha\| \in \mathfrak{A}(\mathfrak{C}) \tag{3}$$

where $v'\alpha$ denotes the result of the substitution v in α. In particular, for the identity valuation i we have

$$a_{R^0}(\mathrm{i}) = \|\alpha\| . \tag{3'}$$

A realization R of $\mathfrak{L}$ in $J \neq 0$ and in a relatively pseudo-complemented lattice $\boldsymbol{A}$ is said to be a *model* for a set $\mathcal{A}$ of formulas in $\mathfrak{L}$ (or: a *model for the theory* $\mathfrak{C} = \{\mathfrak{L}, C_\pi, \mathcal{A}\}$) if for every α in $\mathcal{A}$ and for every valuation v: $V \to J$

$$a_R(v) = \mathit{V} . \tag{4}$$

In particular, R is a model for a formula α (the case where $\mathcal{A}$ consists of one formula only), if for every valuation v: $V \to J$ condition (4) holds. In this case we also say that α is *valid* in R.

A formula is said to be *positively valid* or to be a *positive predicate tautology* if it is valid in every realization R in every $J \neq 0$ and in every relatively pseudo-complemented lattice $\boldsymbol{A}$.

For instance, all logical axioms, i.e. all the formulas in $\mathfrak{L}$ which are of the form (T_1)-(T_9) (see p. 188) are positive predicate tautologies (the proof is the same as in the case of intuitionistic predicate calculi—see X, 2.3).

5.1. *For every formula β in a positive formalized theory $\mathfrak{C} = \{\mathfrak{L}, C_\pi, \mathcal{A}\}$, the following conditions are equivalent:*

(i) *β is a theorem in $\mathfrak{C}$;*

(ii) *β is valid in every model for $\mathfrak{C}$.*

In fact, if a realization R is a model for $\mathfrak{C}$, and β is a theorem in $\mathfrak{C}$, then β is also valid in R since the class of all formulas α which are valid in R contains all axioms in $\mathcal{A}$ and all logical axioms, and it is closed under all the rules of inference (r_1)-(r_6) (the exact proof is analogous to that of VIII, 2.2 and X, 2.2). On the other hand, the canonical realization R^0 of $\mathfrak{C}$ in T and $\mathfrak{A}(\mathfrak{C})$ is a model for $\mathfrak{C}$ by 2.1 and (3). If β is not a theorem in $\mathfrak{C}$, then $\beta_{R^0}(\mathrm{i}) = \|\beta\| \neq \mathit{V}$ by (3') and 2.1. Thus β is not valid in R (compare the analogous proof of the corresponding parts of theorems VIII, 3.5 and X, 3.7).

Denote now by $\mathfrak{L}^*$ the language obtained from $\mathfrak{L}$ by including the negation sign $-$ in the alphabet of $\mathfrak{L}$. By definition, $\mathfrak{L}^*$ is a language (of the first order) of the type investigated in Chapters VIII and X.

All formulas in $\mathfrak{L}$ are formulas in $\mathfrak{L}^*$. Any set $\mathcal{A}$ of formulas in $\mathfrak{L}$ uniquely determines two theories of the first order: the positive theory $\mathfrak{T} = \{\mathfrak{L}, C_\pi, \mathcal{A}\}$ and the intuitionistic theory $\mathfrak{T}^* = \{\mathfrak{L}^*, C_\iota, \mathcal{A}\}$.

5.2. *For every formula β in $\mathfrak{L}$, the following conditions are equivalent:*

(i) *β is a theorem in $\mathfrak{T} = \{\mathfrak{L}, C_\pi, \mathcal{A}\}$;*

(ii) *β is a theorem in $\mathfrak{T}^* = \{\mathfrak{L}^*, C_\iota, \mathcal{A}\}$.*

In other words,

$$C_{\pi\mathfrak{L}}(\mathcal{A}) = C_{\iota\mathfrak{L}^*}(\mathcal{A}) \cap F \tag{5}$$

where F denotes the set of all formulas in $\mathfrak{L}$.

Since all formulas in $\mathfrak{L}$ of the form (T_1)-(T_9), i.e. all logical axioms of positive logic, are also logical axioms of the intuitionistic logic, we have $C_{\pi\mathfrak{L}}(\mathcal{A}) \subset C_{\iota\mathfrak{L}^*}(\mathcal{A})$. Thus (i) implies (ii).

Suppose now that a formula β in $\mathfrak{L}$ is not a theorem in $\mathfrak{T}$. By 5.1 there exists a relatively pseudo-complemented lattice $\boldsymbol{A}$, a set $J \neq 0$ and a valuation $v_0 \in J^V$ such that $\beta_R(v_0) \neq V$. By IV, 1.3, the lattice $\boldsymbol{A}$ can be extended to a pseudo-Boolean algebra $\boldsymbol{A}^*$ by adding a zero element Λ. The operations $\cup$, $\cap$, $\Rightarrow$, and $\bigcup$, $\bigcap$ in $\boldsymbol{A}$ coincide with those in $\boldsymbol{A}^*$ restricted to $\boldsymbol{A}$. Thus we can interpret R as a realization R^* in the pseudo-Boolean algebra $\boldsymbol{A}^*$. It is easy to see that

$$a_{R^*}(v) = a_R(v)$$

for every formula a in $\mathfrak{L}$ and for every valuation v. Hence it follows that R^* is a model for the intuitionistic theory $\mathfrak{T}^*$. Since $\beta_{R^*}(v_0) = \beta_R(v_0) \neq V$, formula β is not a theorem in $\mathfrak{T}^*$, by X, 2.2.

Theorem 5.2 shows that the examination of positive theories $\mathfrak{T}$ of the first order can easily be reduced to an examination of the corresponding intuitionistic theories $\mathfrak{T}^*$ of the first order.

§ 6. The positive predicate calculus. Assuming as $\mathcal{A}$ the empty set in theorem 5.1 we obtain the following *completeness theorem for the positive predicate calculus* $\mathfrak{S}_\pi = \{\mathfrak{L}, C_\pi\}$ ($\mathfrak{L}$ always denotes a language of the first order, of the type described in § 2).

6.1. *For every formula β in $\mathfrak{L}$, the following conditions are equivalent:*

(i) *β is derivable in $\mathfrak{S}_\pi$;*

(ii) *β is a positive predicate tautology.*

We recall that every formula in $\mathfrak{L}$ is also a formula of the language $\mathfrak{L}^*$ obtained from $\mathfrak{L}$ by adding the negation sign $-$. The following theorem is a direct consequence of 5.2 (the case in which $\mathcal{A}$ is the empty set) by 6.1 and X, 4.1.

6.2. *For every formula β in $\mathfrak{L}$, the following conditions are equivalent:*

(i) *β is a positive predicate tautology,*

(ii) *β is an intuitionistic predicate tautology* [1].

Roughly speaking, the examination of the positive predicate calculus can be reduced to an examination of the part of the intuitionistic predicate calculus composed of formulas without the negation sign. As an application we mention here the following theorem which follows directly from 6.2 and the analogous statements X, 8.1-8.2.

6.3. *A disjunction $(\alpha \cup \beta)$ in $\mathfrak{L}$ is a positive predicate tautology if and only if one of the formulas α, β is a positive predicate tautology. An existential formula $\bigcup_\xi \beta(\xi)$ is a positive predicate tautology if and only if there exists a term τ such that $\beta(\tau)$ is a positive predicate tautology.*

As another application we get the following theorem which results directly from 6.1 and X, 6.1.

6.4. *An open formula in $\mathfrak{L}$ is a positive predicate tautology if and only if it is a substitution in a positive propositional tautology.*

The following theorem can be proved in the same way as the analogous statement in VIII, 24.1 and X, 5.1.

6.5. *The algebra $\mathfrak{A}(\mathfrak{S}_\pi)$ of the positive predicate calculus is a generalized free algebra for the class of all complete relatively pseudo-complemented lattices* § 5 (1). *The set of all elements $\|\alpha\|$ where α is any elementary formula is a set of free generators.*

§ 7. The modal logic. In §§ 7-11, the letters $\mathfrak{L}_0$ and $\mathfrak{L}$ will exclusively denote formalized languages (of zero order and of the first order, respectively) defined as in V, § 6 and V, § 3 respectively, with the following modification.

The set of binary connectives contains the disjunction sign $\cup$, the conjunction sign $\cap$ and the implication sign $\Rightarrow$; the set of unary connectives contains the negation sign — and another sign denoted by **I** *and called the necessity sign.*

By V, § 6, (f') and V, § 3, (f_2), if α is a formula in $\mathfrak{L}_0$ or $\mathfrak{L}$, then $\mathbf{I}\alpha$ is also a formula in $\mathfrak{L}_0$ or $\mathfrak{L}$, respectively. The formula $\mathbf{I}\alpha$ should be read: *it is necessary that α* [2].

Formalized languages $\mathfrak{L}_0$, $\mathfrak{L}$ of the type just mentioned will be called *modal languages* of the zero or first order.

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[1] Rasiowa and Sikorski [3].

[2] The formula $\mathbf{I}(\alpha \Rightarrow \beta)$ is usually denoted $(\alpha \prec \beta)$ and called the *strict implication* of formulas α and β (Lewis and Langford [1]). The modal logic is sometimes called the *logic of strict implication.*

As we stated in § 1, modal languages $\mathfrak{L}_0$, $\mathfrak{L}$ constitute the domain of the definition of a non-classical logic $\mathfrak{C}_\mu$ called the *modal logic*. The modal logic is defined by means of the general scheme (L) from V, § 11, with the following modification: to the admitted rules of inferences we add the following rule of inference:

$$(r_\mu) \qquad \frac{(\alpha \Rightarrow \beta)}{(\mathbf{I}\alpha \Rightarrow \mathbf{I}\beta)}$$

in any modal language of the zero or first order.

The auxiliary language mentioned in scheme (L) should be modal. The set (I) of formulas mentioned in (L) is composed of all formulas (t_1)-(t_{12}) from V, § 6 (p. 168) and of the following formulas:

$$(m_1) \qquad ((\mathbf{I}a \cap \mathbf{I}b) \Rightarrow \mathbf{I}(a \cap b)),$$

$$(m_2) \qquad (\mathbf{I}a \Rightarrow a),$$

$$(m_3) \qquad (\mathbf{I}a \Rightarrow \mathbf{II}a),$$

$$(m_4) \qquad \mathbf{I}(a \cup -a).$$

Thus, in any modal language $\mathfrak{L}_0$ of zero order (in any modal language $\mathfrak{L}$ of the first order) the set $\mathscr{A}_\mathrm{I}$ of logical axioms is formed of all formulas of the form (T_1)-(T_{12}) from V, § 11, p. 188, where α, β, γ are any formulas in $\mathfrak{L}_0$ (in $\mathfrak{L}$) and from all formulas of the form

$$(M_1) \qquad ((\mathbf{I}\alpha \cap \mathbf{I}\beta) \Rightarrow \mathbf{I}(\alpha \cap \beta)),$$

$$(M_2) \qquad (\mathbf{I}\alpha \Rightarrow \alpha),$$

$$(M_3) \qquad (\mathbf{I}\alpha \Rightarrow \mathbf{II}\alpha),$$

$$(M_4) \qquad \mathbf{I}(\alpha \cup -\alpha),$$

where α, β are any formulas in $\mathfrak{L}_0$ (in $\mathfrak{L}$).

By definition, the modal logic [1] $\mathfrak{C}_\mu$ assigns, to every modal language $\mathfrak{L}_0$ of zero order (for every modal language $\mathfrak{L}$ of the first order) a consequence operation $\mathfrak{C}_{\mu\mathfrak{L}_0}$ in $\mathfrak{L}_0$ ($\mathfrak{C}_{\mu\mathfrak{L}}$ in $\mathfrak{L}$) defined as follows: for any set S of formulas in that language, the set $\mathfrak{C}_{\mu\mathfrak{L}_0}(S)$ (the set $\mathfrak{C}_{\mu\mathfrak{L}}(S)$) is the least set containing all formulas in S and all the logical axioms in $\mathscr{A}_\mathrm{I}$ just mentioned, and closed under the rules of inference (r_1) and (r_μ) (rules of inference (r_1)-(r_6) and (r_μ)). According to the general convention in V, § 11, p. 188, we shall usually omit the subscripts $\mathfrak{L}_0$ and $\mathfrak{L}$ in the above notation if no ambiguity is possible.

[1] The formalization of the modal logic admitted in this book differs from this of Lewis and Langford [1], but it is equivalent.

The deductive systems

$$\mathbb{S}_{0\mu} = \{\mathfrak{L}_0, \mathbb{C}_\mu\}, \quad \mathbb{S}_\mu = \{\mathfrak{L}, \mathbb{C}_\mu\} \tag{1}$$

will be called respectively: the *modal propositional calculus* based on $\mathfrak{L}_0$ and the *modal predicate calculus* [1] based on $\mathfrak{L}$.

Besides the modal propositional and predicate calculi (1) we shall also examine *modal theories of zero order*

$$\{\mathfrak{L}_0, \mathbb{C}_\mu, \mathcal{A}\} \tag{2}$$

where $\mathcal{A}$ is any set of formulas in $\mathfrak{L}_0$, and *modal theories of the first order*

$$\{\mathfrak{L}, \mathbb{C}_\mu, \mathcal{A}\} \tag{3}$$

where $\mathcal{A}$ is any set of formulas in $\mathfrak{L}$. According to the general definition in V, § 10, p. 186, formulas in the set $\mathbb{C}_\mu(\mathcal{A})$ are called *theorems* of theory (2) or (3), respectively.

Of course, $\mathbb{S}_{0\mu}$ and $\mathbb{S}_\mu$ will be identified with the modal theories $\{\mathfrak{L}_0, \mathbb{C}_\mu, 0\}$ and $\{\mathfrak{L}, \mathbb{C}_\mu, 0\}$ respectively. Formulas in $\mathbb{C}_\mu(0)$ are said to be *derivable in* $\mathbb{S}_{0\mu}$ or $\mathbb{S}_\mu$ respectively, or *formulas modally derivable.*

We shall apply to modal languages $\mathfrak{L}_0$ and $\mathfrak{L}$ all the results obtained in Chapter VI. Of course, all definitions and theorems in Chapter VI now require obvious modifications caused by adding the necessity sign **I** to the alphabet of languages $\mathfrak{L}_0$ and $\mathfrak{L}$. Consequently, a unary operation **I** should be added to the set of operations in all algebras considered in Chapter VI. In particular, the set F of all formulas in $\mathfrak{L}_0$ or $\mathfrak{L}$ should now be understood as an algebra

$$\{F, \cup, \cap, \Rightarrow, -, \mathbf{I}\} \tag{4}$$

with three binary operations $\cup, \cap, \Rightarrow$ and two unary operations $-$ and **I**. The reader can easily verify that all results in Chapter VI remain valid after these modifications.

In particular (see VI, 10.6), if $\mathfrak{C}$ is a modal theory of the zero or first order, then the relation $\approx$, defined in F as follows:

(5) $\alpha \approx \beta$ if and only if $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are theorems in $\mathfrak{C}$,

is a congruence with respect to the operations $\cup, \cap, \Rightarrow, -$. If $\alpha \approx \beta$ i.e. $(\alpha \Rightarrow \beta)$ and $(\beta \Rightarrow \alpha)$ are theorems in $\mathfrak{C}$, then also formulas $(\mathbf{I}\alpha \Rightarrow \mathbf{I}\beta)$ and $(\mathbf{I}\beta \Rightarrow \mathbf{I}\alpha)$ are theorems in $\mathfrak{C}$ by the rule of inference (r_μ), i.e. $\mathbf{I}\alpha \approx \mathbf{I}\beta$. This proves that the equivalence relation $\approx$ is also a congruence with respect to the operation **I**, i.e. it is a congruence in the algebra (4). By I, § 4, the set $\mathfrak{A}(\mathfrak{C}) = F/\approx$ can be conceived of as an algebra

$$\{\mathfrak{A}(\mathfrak{C}), \cup, \cap, \Rightarrow, -, \mathbf{I}\}$$

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[1] Rasiowa [2]. For other systems of modal predicate calculi, see Barcan [1] and Carnap [2].

called the *algebra of the theory* $\mathcal{C}$. The operation $\mathbf{I}$ in $\mathfrak{A}(\mathcal{C})$ is defined by the equality $\mathbf{I}\|a\| = \|\mathbf{I}a\|$ (see I, § 4, (8)).

7.1. (1) *The algebra* $\mathfrak{A}(\mathcal{C})$ *is a topological Boolean algebra. For any formulas* α, β

$$\begin{gathered}\|\alpha\| \cup \|\beta\| = \|(\alpha \cup \beta)\|, \quad \|\alpha\| \cap \|\beta\| = \|(\alpha \cap \beta)\| \\ \|\alpha\| \Rightarrow \|\beta\| = \|(\alpha \Rightarrow \beta)\|, \quad -\|\alpha\| = \|-\alpha\|, \quad \mathbf{I}\|\alpha\| = \|\mathbf{I}\alpha\|.\end{gathered} \tag{6}$$

Moreover,

$$\|\alpha\| \leqslant \|\beta\| \text{ if and only if } (\alpha \Rightarrow \beta) \text{ is a theorem in } \mathcal{C}. \tag{7}$$

For every formula α,

$$\|\alpha\| = V \text{ if and only if } \alpha \text{ is a theorem in } \mathcal{C}, \tag{8}$$

$$\|\alpha\| \neq \Lambda \text{ if and only if } \alpha \text{ is irrefutable in } \mathcal{C}. \tag{9}$$

The topological Boolean algebra $\mathfrak{A}(\mathcal{C})$ *is non-degenerate if and only if the theory* $\mathcal{C}$ *is consistent.*

We have to prove only that $\mathbf{I}$ is an interior operation in $\mathfrak{A}(\mathcal{C})$. The fact that $\mathfrak{A}(\mathcal{C})$ is a Boolean algebra and all the remaining parts of theorem 6.1 follow directly from VI, 10.4, 10.5, 10.6, 10.7, 10.10.

It follows from (7) and (r_μ) that

$$\|\alpha\| \leqslant \|\beta\| \quad \text{implies} \quad \mathbf{I}\|\alpha\| \leqslant \mathbf{I}\|\beta\|. \tag{10}$$

Since $\|\alpha\| \cap \|\beta\| \leqslant \|\alpha\|$ and $\|\alpha\| \cap \|\beta\| \leqslant \|\beta\|$, we have $\mathbf{I}(\|\alpha\| \cap \|\beta\|) \leqslant \mathbf{I}\|\alpha\|$, and $\mathbf{I}(\|\alpha\| \cap \|\beta\|) \leqslant \mathbf{I}\|\beta\|$. Consequently $\mathbf{I}(\|\alpha\| \cap \|\beta\|) \leqslant \mathbf{I}\|\alpha\| \cap \mathbf{I}\|\beta\|$. It follows directly from (7), (6) and (M_1) that $\mathbf{I}\|\alpha\| \cap \mathbf{I}\|\beta\| \leqslant \mathbf{I}(\|\alpha\| \cap \|\beta\|)$ which proves that

$$\mathbf{I}(\|\alpha\| \cap \|\beta\|) = \mathbf{I}\|\alpha\| \cap \mathbf{I}\|\beta\|,$$

i.e. $\mathbf{I}$ satisfies condition (i_1) from III § 1 (p. 93). It follows from (M_2) and (7) that

$$\mathbf{I}\|\alpha\| \leqslant \|\alpha\|, \tag{11}$$

i.e. $\mathbf{I}$ satisfies condition (i_2) from III, § 1. It follows from (M_3) and (7) that $\mathbf{I}\|\alpha\| \leqslant \mathbf{II}\|\alpha\|$. On the other hand, it follows from (11) and (10) that $\mathbf{II}\|\alpha\| \leqslant \mathbf{I}\|\alpha\|$. Thus

$$\mathbf{II}\|\alpha\| = \mathbf{I}\|\alpha\|$$

which proves condition (i_3) from III, § 1. Since $V = \|(\alpha \cup -\alpha)\|$ (see (8)), it follows from (M_4) and (8) that

$$\mathbf{I}V = V,$$

(1) Cf. McKinsey [1].

i.e. condition (i_4) from III, § 1, is also satisfied. This proves that **I** is an interior operation in the Boolean algebra $\mathfrak{A}(\mathfrak{T})$.

If $\mathfrak{T}$ is a modal theory of the first order and T is the set of all terms in $\mathfrak{T}$, then for any formula $\beta(x)$

$$\text{(Q)} \qquad \begin{aligned} \|\bigcup_\xi \beta(\xi)\| &= \bigcup_{\tau \in T} \|\beta(\tau)\|, \\ \|\bigcap_\xi \beta(\xi)\| &= \bigcap_{\tau \in T} \|\beta(\tau)\| \end{aligned}$$

in the Boolean algebra $\mathfrak{A}(\mathfrak{T})$ (see VI, 11.2). Then we shall conceive of $\mathfrak{A}(\mathfrak{T})$ as a generalized algebra

$$\text{(12)} \qquad \{\mathfrak{A}(\mathfrak{T}), \cup, \cap, \Rightarrow, -, \mathbf{I}, \bigcup|\mathfrak{Q}, \bigcap|\mathfrak{Q}\}$$

where $\bigcup$ and $\bigcap$ are respectively the infinite join and meet. As the common domain $\mathfrak{Q}$ of $\bigcup$ and $\bigcap$ in $\mathfrak{A}(\mathfrak{T})$ we assume only the class of all indexed sets

$$\{\|\beta(\tau)\|\}_{\tau \in T}$$

where $\beta(x)$ is any formula in $\mathfrak{L}$. In other words, $\bigcup$ and $\bigcap$ are restricted in $\mathfrak{A}(\mathfrak{T})$ only to infinite joins and meets (Q) corresponding to quantifiers. The generalized algebra (12) is called the *quantifier algebra of the theory* $\mathfrak{T}$, or briefly the Q-*algebra* of $\mathfrak{T}$. According to VI, § 11, by a Q-*homomorphism* from $\mathfrak{A}(\mathfrak{T})$ into any topological Boolean algebra we shall understand any topological homomorphism (see III, § 3) preserving all the infinite joins and meets (Q).

§ 8. Modal theories of zero order. Let $\mathfrak{L}_0$ be a modal language of zero order and let

$$\{A, \cup, \cap, \Rightarrow, -, \mathbf{I}\}$$

be an abstract algebra with three binary operations $\cup, \cap, \Rightarrow$ and two unary operations $-, \mathbf{I}$. According to VI, § 2, every element $v = \{v_a\}_{a \in V_0} \in A^{V_0}$ where V_0 is the set of all propositional variables in $\mathfrak{L}_0$ is called a valuation in A. By VI, § 2, every formula a in $\mathfrak{L}_0$ determines a mapping

$$\text{(1)} \qquad a_A \colon A^{V_0} \to A$$

(to the inductive definition of a_A in VI, § 2, (10), p. 212, we now have to add the identity: $(\mathbf{I}a)_A(v) = \mathbf{I}\big(a_A(v)\big)$ for every formula a and every valuation v).

In §§ 8-9 we shall always assume that algebra (1) is a non-degenerate topological Boolean algebra. Sometimes we shall assume as A the algebra $\mathfrak{A}(\mathfrak{T})$ of a consistent modal theory $\mathfrak{T}$ of zero order. We recall that the valuation

$$\text{(2)} \qquad v^0 = \{\|a\|\}_{a \in V_0}$$

in $\mathfrak{A}(\mathfrak{C})$ is called the *canonical valuation* for $\mathfrak{C}$ (see VI, § 2, p. 214). By VI, 2.6

$$a_{\mathfrak{A}(\mathfrak{C})}(v^0) = \|a\| . \tag{3}$$

A valuation $v \in A^{V_0}$ is said to be a *model* for a set $\mathcal{A}$ of formulas if

$$a_A(v) = V \quad \text{for every formula } a \text{ in } \mathcal{A} .$$

In particular (when $\mathcal{A}$ is composed of one formula only), the valuation v is said to be a model for a formula a if $a_A(v) = V$. A valuation v is said to be a *model* for a theory $\mathfrak{C} = \{\mathfrak{L}_0, C_\mu, \mathcal{A}\}$ provided v is a model for the set $\mathcal{A}$ of axioms of $\mathfrak{C}$.

A formula a is said to be *valid* in a topological Boolean algebra A provided every valuation $v \in A^{V_0}$ is a model for a. A formula is said to be *modally valid* or to be a *modal propositional tautology* if it is valid in every topological Boolean algebra.

For instance all logical axioms, i.e. all formulas in $\mathfrak{L}_0$ which are of the form (T_1)-(T_{12}) (see p. 188) or of the form (M_1)-(M_4) (see p. 470) are modal propositional tautologies. The proof of this fact for formulas (T_1)-(T_{12}) is exactly the same as the proof of the analogous statement for classical propositional calculi (see the proof of VII, 1.3). The validity of formulas (M_1)-(M_4) in any topological Boolean algebra follows easily from III, § 1, (i_1)-(i_4) (see also I, 12.2, (4)).

8.1. *If a is a theorem of a modal theory $\mathfrak{C} = \{\mathfrak{L}_0, C_\mu, \mathcal{A}\}$ of zero order, then every model v for $\mathfrak{C}$ in any topological Boolean algebra A is a model for a.*

This follows from the fact that the set of all formulas a in $\mathfrak{L}_0$ for which $a_A(v) = V$ contains all axioms in $\mathcal{A}$ and all logical axioms, and is closed under *modus ponens* (r_1) and the rule of inference (r_μ). The exact proof is the same as that of the analogous statement VII, 7.1, for classical theories. We explain here only why this set is closed under (r_μ). If $(a \Rightarrow \beta)_A(v) = V$, then $a_A(v) \leqslant \beta_A(v)$. Hence

$$\mathbf{I}a_A(v) \leqslant \mathbf{I}\beta_A(v), \quad \text{i.e.} \quad (\mathbf{I}a \Rightarrow \mathbf{I}\beta)_A(v) = V .$$

A model v for a theory $\mathfrak{C}$ in A is said to be *adequate* if, for every formula a, the equation $a_A(v) = V$ holds if and only if a is a theorem in $\mathfrak{C}$.

By a *topological model* we shall mean any model in the topological Boolean algebra $\mathfrak{B}(X)$ of all subsets of a topological space $X \neq 0$.

8.2. *For every modal theory $\mathfrak{C} = \{\mathfrak{L}_0, C_\mu, \mathcal{A}\}$ of zero order, the following conditions are equivalent:*

(i) *$\mathfrak{C}$ is consistent,*

(ii) *there exists an adequate model for $\mathfrak{C}$;*

(iii) *there exists an adequate topological model for* $\mathfrak{T}$;

(iv) *there exists a model for* $\mathfrak{T}$.

The proof is similar to that of the analogous statements VII, 7.7 and IX, 8.4 for classical and intuitionistic theories. If $\mathfrak{T}$ is consistent, then the canonical valuation (2) for $\mathfrak{T}$ is an adequate model for $\mathfrak{T}$ by (3) and 7.1, (8). Thus (i) implies (ii). Suppose that v is an adequate model for $\mathfrak{T}$ in a topological Boolean algebra $\boldsymbol{A}$. By III, 4.3, there exists a topological space X and a topological isomorphism h from $\boldsymbol{A}$ into $\mathfrak{B}(X)$. The superposition hv of h and v is an adequate model for $\mathfrak{T}$ in $\mathfrak{B}(X)$. Thus (ii) implies (iii). Clearly (iii) implies (iv). The proof that (iv) implies (i) is exactly the same as for classical or intuitionistic theories.

8.3. *For every formula* β *in a consistent modal theory* $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\mu, \mathcal{A}\}$ *of zero order, the following conditions are equivalent*:

(i) β *is a theorem in* $\mathfrak{T}$;

(ii) *every model for* $\mathfrak{T}$ *is a model for* β;

(iii) *every topological model for* $\mathfrak{T}$ *is a model for* β;

(iv) $\beta_{\mathfrak{A}(\mathfrak{T})}(v^0) = \mathbb{V}$ *for the canonical valuation* v^0.

The proof is similar to that of the analogous statements VII, 7.8 and IX, 8.5 for classical and intuitionistic theories.

(i) implies (ii) by 8.1. (ii) trivially implies (iii). Conversely, (iii) implies (ii) since every topological Boolean algebra can be considered as a subalgebra of $\mathfrak{B}(X)$ for a topological space X (see III, 4.3). (ii) implies (iv) since v^0 is a model for $\mathfrak{T}$ by (3) and 7.1, (8). Finally (iv) implies (i) by (3) and 7.1, (8).

As an application of 8.3 we shall prove the following theorem.

8.4. *A formula* β *in* $\mathfrak{L}_0$ *is a theorem of a modal theory* $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\mu, \mathcal{A}\}$ *if and only if the formula* $\mathbf{I}\beta$ *is a theorem in* $\mathfrak{T}$.

Let v be any model for $\mathfrak{T}$ in an algebra $\boldsymbol{A}$. If β is a theorem in $\mathfrak{T}$, then $\beta_{\boldsymbol{A}}(v) = \mathbb{V}$ by 8.3. By III, § 1, (i_4), we have $\mathbf{I}\beta_{\boldsymbol{A}}(v) = \mathbb{V}$. Thus every model v for $\mathfrak{T}$ is a model for $\mathbf{I}\beta$. By 8.3, $\mathbf{I}\beta$ is a theorem in $\mathfrak{T}$.

Conversely, suppose that $\mathbf{I}\beta$ is a theorem in $\mathfrak{T}$. For any model v for $\mathfrak{T}$ in a topological Boolean algebra $\boldsymbol{A}$ we have $\mathbf{I}\beta_{\boldsymbol{A}}(v) = \mathbb{V}$, by 8.3. By III, § 1, (i_2), $\beta_{\boldsymbol{A}}(v) = \mathbb{V}$. Thus every model v for $\mathfrak{T}$ is also a model for β. By 8.3, β is a theorem in $\mathfrak{T}$.

Denote now by $\mathfrak{L}_0^*$ the formalized language (of zero order) obtained from $\mathfrak{L}_0$ by avoiding the connective $\mathbf{I}$ in the alphabet of $\mathfrak{L}_0$. Thus $\mathfrak{L}_0^*$ is a language of the type investigated in Chapters VII and IX. The set of all formulas in $\mathfrak{L}_0^*$ coincides with the set of all those formulas in $\mathfrak{L}_0$ which do not contain the sign $\mathbf{I}$. Any set $\mathcal{A}$ of formulas in $\mathfrak{L}_0^*$ uniquely determines two theories of zero order: the modal theory $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\mu, \mathcal{A}\}$

and the classical theory $\mathfrak{T}^* = \{\mathfrak{L}_0^*, \mathfrak{C}, \mathcal{A}\}$. The following theorem states that $\mathfrak{T}$ is an inessential extension of $\mathfrak{T}^*$.

8.5. *For every formula β in $\mathfrak{L}_0^*$, the following conditions are equivalent:*

(i) *β is a theorem in $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\mu, \mathcal{A}\}$;*

(ii) *β is a theorem in $\mathfrak{T}^* = \{\mathfrak{L}_0^*, \mathfrak{C}, \mathcal{A}\}$.*

In other words,

$$\mathfrak{C}_{\mathfrak{L}_0^*}(\mathcal{A}) = \mathfrak{C}_\mu \mathfrak{L}_0(\mathcal{A}) \cap F_0^* \tag{4}$$

where F_0^* is the set of all formulas in $\mathfrak{L}_0^*$, $\mathcal{A} \subset F_0^*$.

It follows directly from the definition of $\mathfrak{C}_\mu$ and $\mathfrak{C}$ that $\mathfrak{C}_{\mathfrak{L}_0^*}(\mathcal{A}) \subset \mathfrak{C}_\mu \mathfrak{L}_0(\mathcal{A})$. Thus (ii) implies (i).

Suppose now that a formula β in $\mathfrak{L}_0^*$ is not a theorem in $\mathfrak{T}^*$. By VII, 7.8, there exists a model v_0 for $\mathfrak{T}^*$ in a non-degenerate Boolean algebra $\boldsymbol{A}$ such that $\beta_{\boldsymbol{A}}(v_0) \neq V$. The Boolean algebra $\boldsymbol{A}$ can be conceived of as a topological Boolean algebra, the interior operation in $\boldsymbol{A}$ being the identity mapping. For every formula α in $\mathfrak{L}_0^*$ and every valuation $v \in \boldsymbol{A}^{V_0}$, the value of $\alpha_{\boldsymbol{A}}(v)$ does not depend on the fact whether α is interpreted as a formula in $\mathfrak{L}_0^*$ and $\boldsymbol{A}$ as a Boolean algebra, or α is interpreted as a formula in $\mathfrak{L}_0$ and $\boldsymbol{A}$ is interpreted as a topological Boolean algebra just defined. Thus, using the second of the interpretations just mentioned, we assert that v_0 is a model for $\mathfrak{T}$ but not a model for β. By 8.1, β is not a theorem in $\mathfrak{T}$.

To establish a connection between intuitionistic theories and modal theories we have to define a mapping $\mathfrak{J}$ which, to every formula α in $\mathfrak{L}_0^*$, assigns a formula $\mathfrak{J}\alpha$ in $\mathfrak{L}_0$. The definition of $\mathfrak{J}$ is by induction on the length of formulas, as follows:

$\mathfrak{J}\alpha$ is the formula $\mathbf{I}\alpha$, for every propositional variable α;

$\mathfrak{J}(\alpha \cup \beta)$ is the formula $(\mathfrak{J}\alpha \cup \mathfrak{J}\beta)$;

$\mathfrak{J}(\alpha \cap \beta)$ is the formula $(\mathfrak{J}\alpha \cap \mathfrak{J}\beta)$;

$\mathfrak{J}(\alpha \Rightarrow \beta)$ is the formula $\mathbf{I}(\mathfrak{J}\alpha \Rightarrow \mathfrak{J}\beta)$;

$\mathfrak{J}-\alpha$ is the formula $\mathbf{I}-\mathfrak{J}\alpha$.

Here α and β denote any formulas in $\mathfrak{L}_0^*$.

If $\mathcal{A}$ is any set of formulas in $\mathfrak{L}_0^*$, then $\mathfrak{J}\mathcal{A}$ denotes the set of all formulas $\mathfrak{J}\alpha$ where α is in $\mathcal{A}$. By definition, $\mathfrak{J}\mathcal{A}$ is a set of formulas in $\mathfrak{L}_0$. Thus any set $\mathcal{A}$ of formulas in $\mathfrak{L}_0^*$ uniquely determines two theories of zero order: the intuitionistic theory $\mathfrak{T}^* = \{\mathfrak{L}_0^*, \mathfrak{C}_\iota, \mathcal{A}\}$ and the modal theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\mu, \mathfrak{J}\mathcal{A}\}$.

8.6. *For every formula β in $\mathfrak{L}_0^*$ the following conditions are equivalent:*

(i) *$\mathfrak{J}\beta$ is a theorem in $\mathfrak{T} = \{\mathfrak{L}_0, \mathfrak{C}_\mu, \mathfrak{J}\mathcal{A}\}$;*

(ii) *β is a theorem in $\mathfrak{T}^* = \{\mathfrak{L}_0^*, \mathfrak{C}_\iota, \mathcal{A}\}$.*

We recall that if $\boldsymbol{A}$ is a topological Boolean algebra, then $\mathfrak{G}(\boldsymbol{A})$ denotes the pseudo-Boolean algebra of all open elements in $\boldsymbol{A}$. The following identity is the basis for the proof of 8.5: for every formula α in $\mathfrak{L}_0^*$ and for every valuation $v \in \boldsymbol{A}^{V_0}$

$$a_{\mathfrak{G}(\boldsymbol{A})}(\mathbf{I}v) = \mathfrak{J}a_{\boldsymbol{A}}(v) \tag{5}$$

where $\mathbf{I}v$ is the superposition of v: $V_0 \to \boldsymbol{A}$ and $\mathbf{I}$: $\boldsymbol{A} \to \mathfrak{G}(\boldsymbol{A})$. Of course, $a_{\mathfrak{G}(\boldsymbol{A})}$ is the mapping defined in IX, § 2. The easy proof of (5) (based on theorem IV, 1.4) by induction on the length of α is omitted.

If $\mathfrak{J}\beta$ is not a theorem in $\mathfrak{C}$, then there exists a topological Boolean algebra $\boldsymbol{A}$ and a valuation $v \in \boldsymbol{A}^{V_0}$ which is a model for $\mathfrak{C}$ but not for $\mathfrak{J}\beta$, by 8.3. Assuming as α any formula in $\mathcal{A}$, we infer from (5) that the valuation $\mathbf{I}v \in \mathfrak{G}(\boldsymbol{A})^{V_0}$ is a model for the intuitionistic theory $\mathfrak{C}^*$ (in the sense defined in IX, § 2). Replacing α by β we infer from (5) that $\mathbf{I}v$ is not a model for β. By IX, 8.1, β is not a theorem in $\mathfrak{C}^*$.

Suppose now that β is not a theorem in $\mathfrak{C}^*$. By IX, 8.5 there exists a model v for $\mathfrak{C}^*$ in a pseudo-Boolean algebra such that v is not a model for β. By IV, 3.1, we may suppose that this pseudo-Boolean algebra is the algebra $\mathfrak{G}(\boldsymbol{A})$ of all open elements in a topological Boolean algebra $\boldsymbol{A}$. The valuation v: $V_0 \to \mathfrak{G}(\boldsymbol{A})$ can be considered as a valuation v: $V_0 \to \boldsymbol{A}$. Moreover $\mathbf{I}v = v$ since $\mathbf{I}g = g$ for all open elements g in $\boldsymbol{A}$. Assuming as α any formula in $\mathcal{A}$, we infer from (5) that v is a model for $\mathfrak{C}$. Replacing α by β we infer from (5) that v is not a model for $\mathfrak{J}\beta$. By 8.1, $\mathfrak{J}\beta$ is not any theorem in $\mathfrak{C}$.

It follows from 8.5 and 8.6 that the examination of classical and intuitionistic theories can always be reduced to the examination of some parts of corresponding modal theories.

§ 9. The modal propositional calculus. The following theorem, called the *completeness theorem for modal propositional calculi* $\mathfrak{S}_{0\mu} = \{\mathfrak{L}_0, \mathfrak{C}_\mu\}$, is the modal analogue of the completeness theorem VII, 2.1, for classical propositional calculi and theorem IX, 3.1, for intuitionistic propositional calculi.

9.1. *For every formula β in $\mathfrak{L}_0$, the following conditions are equivalent:*

(i) *β is derivable in $\mathfrak{S}_{0\mu} = \{\mathfrak{L}_0, \mathfrak{C}_\mu\}$;*

(ii) *β is a modal propositional tautology;*

(iii) *β is valid in every topological field of sets $\mathfrak{B}(X)$;*

(iv) *$\beta_{\mathfrak{A}(\mathfrak{S}_{0\mu})}(v^0) = \mathbb{V}$, where v^0 is the canonical valuation for $\mathfrak{S}_{0\mu}$;*

(v) *β is valid in every finite topological algebra;*

(vi) *β is valid in every topological Boolean algebra of a power $\leqslant 2^{2^r}$ where r is the cardinal of the set of all subformulas of β;*

(vii) *β is valid in the topological field $\mathfrak{B}(X)$ of all subsets of a dense-in-itself metric space $X \neq 0$* (1).

The proof is similar to that of IX, 3.2. The equivalence of (i), (ii), (iii), (iv) follows from 8.3 (the case in which $\mathcal{A}$ is the empty set of formulas).

Clearly (ii) implies (v) and (v) implies (vi).

We shall prove that (vi) implies (iv). Suppose that (iv) does not hold for a formula β, i.e. $\beta_{\mathfrak{A}(\mathfrak{S}_{0\mu})}(v^0) = \|\beta\| \neq V$. Suppose that β contains r subformulas. Let A_0 be the set composed of all $\|\alpha\|$ such that α is a subformula of β. Thus A_0 is a subset of $\mathfrak{A}(\mathfrak{S}_{0\mu})$ containing r elements. It follows from III, 4.4, that there exists a finite topological Boolean algebra A containing at most 2^{2^r} elements such that $A_0 \subset A$ and the operations in A are extensions of the operations in A_0. Moreover, A is a Boolean subalgebra of $\mathfrak{A}(\mathfrak{S}_{0\mu})$. Let v be the valuation in A such that

$$v_a = \|a\| \quad \text{for every propositional variable } a \text{ in } \beta.$$

It is easy to prove by induction on the length of a formula α that for any subformula α of β

$$\alpha_A(v) = \|\alpha\| \in A_0 .$$

In particular we have

$$\beta_A(v) = \|\beta\| \neq V ,$$

i.e. (vi) does not hold for β. Thus (vi) implies (iv). In consequence the conditions (i)-(vi) are equivalent.

Note that (iii) implies (vii). To complete the proof of 9.1 it suffices to show that (vii) implies (vi). In fact, suppose

$$\beta_A(v) \neq V \tag{1}$$

for a valuation v in a finite topological Boolean algebra A. Let $X \neq 0$ be a dense-in-itself metric space. By III, 8.3, there exists an open set $G \subset X$ and an isomorphism h_0 from A into the topological field $\mathfrak{B}(G)$. It follows from VI, 2.4, and from (1) that

$$\alpha_{\mathfrak{B}(G)}(h_0 v) = h_0 \alpha_A(v) \neq G \tag{2}$$

where the valuation $h_0 v$: $V_0 \to \mathfrak{B}(G)$ is the superposition of v: $V_0 \to A$ and h_0: $A \to \mathfrak{B}(G)$. On the other hand, by III, 3.1, the mapping

$$h(A) = A \cap G \quad \big(A \in \mathfrak{B}(X)\big)$$

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(1) McKinsey [1], McKinsey and Tarski [1], [3]. The first steps leading to establish a relationship between topology and the modal logic were made by Tang Tsao-Chen [1].

is a homomorphism from the topological field $\mathfrak{B}(X)$ onto the topological field $\mathfrak{B}(G)$. For every propositional variable a in V_0, let A_a be a subset of X such that

$$h(A_a) = h_0(v_a) .$$

Then $v' = \{A_a\}_{a \in V_0}$ is a valuation in $\mathfrak{B}(X)$ and the valuation hv' (i.e. the superposition of v': $V_0 \to \mathfrak{B}(X)$ and h: $\mathfrak{B}(X) \to \mathfrak{B}(G)$) is identical with the valuation $h_0 v$. Thus by (2) and VI, 2.4

$$h\left(a_{\mathfrak{B}(X)}(v')\right) = a_{\mathfrak{B}(G)}(hv') = a_{\mathfrak{B}(G)}(h_0 v) \neq G .$$

Consequently $a_{\mathfrak{B}(X)}(v') \neq X$, i.e. a is not valid in $\mathfrak{B}(X)$. This completes the proof of 9.1.

It follows directly from the equivalence of (i) and (vi) that in order to verify whether a formula β is derivable in $\mathfrak{S}_{0\mu}$ it is sufficient to calculate $\beta_A(v)$ in all finite topological Boolean algebras A with the cardinal $\leqslant 2^{2^r}$, where r is the cardinal of all subformulas of β. Thus we have a general decision procedure [1] for determining whether or not a formula β in $\mathfrak{S}_{0\mu}$ is derivable. In fact this method consists in verifying whether $\beta_A(v)$ is identically equal to $\mathbb{V}$ in every topological Boolean algebra A with the cardinal $\leqslant 2^{2^r}$. When β is a formula constructed of n propositional variables and A consists of m elements we have to calculate the value of $\beta_A(v)$ at m^n points $v \in A^{V_0}$ whose coordinates v_a, determined by the propositional variables a occurring in β, all have possible values in A.

9.2. *A formula β is a modal propositional tautology if and only if the formula $\mathbf{I}\beta$ is a modal propositional tautology.*

This follows directly from 8.4 and 9.1, (i)-(ii) (or it can be deduced immediately from III, § 1, (i_2), (i_4) and the definition of modal tautologies).

We recall (see § 8, p. 475) that $\mathfrak{L}_0^*$ denotes the language obtained from $\mathfrak{L}_0$ by elimination of the sign **I**. The following theorem follows directly from 8.5 (the case in which $\mathcal{A}$ is the empty set) and theorems 9.1, VII, 2.1.

9.3. *For every formula β in $\mathfrak{L}_0^*$, the following conditions are equivalent*:

(i) *β is a classical tautology*;

(ii) *β is a modal tautology* [2].

For every formula a in $\mathfrak{L}_0^*$, let $\mathfrak{J}a$ be the formula (in $\mathfrak{L}_0$) defined in § 8, p. 476. The following theorem follows immediately from 8.6 (the case in which $\mathcal{A}$ is the empty set) and from theorems 9.1, IX, 3.1.

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[1] McKinsey [1].

[2] Lewis and Langford [1].

9.4. *For every formula β in $\mathfrak{L}_0^*$, the following conditions are equivalent:*

(i) *β is an intuitionistic tautology;*

(ii) *$\mathfrak{I}\beta$ is a modal tautology* (1).

The intuitive meaning of 9.3 and 9.4 is that both the classical propositional calculus and the intuitionistic propositional calculus can be conceived as parts of the modal propositional calculus.

The following theorem analogous to IX, 6.1, holds for the modal propositional calculi.

9.5. *For any formulas α, β in $\mathfrak{L}_0$, the disjunction $(\mathbf{I}\alpha \cup \mathbf{I}\beta)$ is a modal tautology if and only if either α or β is a modal tautology* (2).

The proof is analogous to that of IX, 6.1.

If α is a modal propositional tautology, then $\mathbf{I}\alpha$ is a modal propositional tautology by 9.2 and so is $(\mathbf{I}\alpha \cup \mathbf{I}\beta)$. In the same way we prove that if β is a modal tautology, so is $(\mathbf{I}\alpha \cup \mathbf{I}\beta)$.

Suppose that $(\mathbf{I}\alpha \cup \mathbf{I}\beta)$ is a modal tautology. By III, 4.3, there exists an isomorphism h from $\mathfrak{A}(\mathfrak{S}_{0\mu})$ into the topological field $\mathfrak{B}(X)$ of all subsets of a topological space X. By VI, 2.4, for any formula γ

$$h\gamma_{\mathfrak{A}(\mathfrak{S}_{0\mu})}(v^0) = \gamma_{\mathfrak{B}(X)}(hv^0) \tag{3}$$

where v^0 is the canonical valuation in $\mathfrak{A}(\mathfrak{S}_{0\mu})$ (see § 8) and the valuation hv^0 is the superposition of v^0: $V_0 \to \mathfrak{A}(\mathfrak{S}_{0\mu})$ and h: $\mathfrak{A}(\mathfrak{S}_{0\mu}) \to \mathfrak{B}(X)$. By III, 5.1, X is an open subset of a strongly compact space X_0. By III, 3.1, the mapping $h_0(A) = X \cap A$ for $A \in \mathfrak{B}(X_0)$ is a topological homomorphism from $\mathfrak{B}(X_0)$ onto $\mathfrak{B}(X)$. For every propositional variable a, let S_a be a subset of X_0 such that $h_0(S_a) = hv_a^0 = h(\|a\|)$. The valuation $v = \{S_a\}_{a \in V_0}$ in $\mathfrak{B}(X_0)$ satisfies the equation $h_0 v = hv^0$. Since $(\mathbf{I}\alpha \cup \mathbf{I}\beta)$ is a modal propositional tautology, we have

$$\mathbf{I}\alpha_{\mathfrak{B}(X_0)}(v) \cup \mathbf{I}\beta_{\mathfrak{B}(X_0)}(v) = (\mathbf{I}\alpha \cup \mathbf{I}\beta)_{\mathfrak{B}(X_0)}(v) = X_0 .$$

Since X_0 is strongly compact, at least one of the summands is equal to X_0. Assume, e.g., that $\mathbf{I}\alpha_{\mathfrak{B}(X_0)}(v) = X_0$. Consequently $\alpha_{\mathfrak{B}(X_0)}(v) = X_0$. Hence

$$h\alpha_{\mathfrak{A}(\mathfrak{S}_{0\mu})}(v^0) = \alpha_{\mathfrak{B}(X)}(hv^0) = \alpha_{\mathfrak{B}(X)}(h_0 v) = h_0 \alpha_{\mathfrak{B}(X_0)}(v) = h_0(X_0) = X .$$

Since h is an isomorphism, the last identity implies that $\alpha_{\mathfrak{A}(\mathfrak{S}_{0\mu})}(v^0) = V$. This proves, by 9.1, that α is a modal tautology.

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(1) McKinsey and Tarski [3].

(2) McKinsey and Tarski [3] and [1]. The proof is a modification of the proof in the papers cited.

9.6. *The algebra* $\mathfrak{A}(S_{0\mu})$ *of the modal propositional calculus is free in the class of all topological Boolean algebras. The class of all elements* $\|a\|$ *where* a *is any propositional variables is a set of free generators* (1).

The proof of 9.6 is the same as that of analogous statements VII, 4.1 and IX, 7.1.

§ 10. Modal theories of the first order (2). In §§ 10-11 $\mathfrak{L}$ always denotes a fixed modal language of the first order. To facilitate our consideration we shall always assume that the language $\mathfrak{L}$ is enumerable, similarly as in Chapter X (see p. 411).

The definition of a realization R of $\mathfrak{L}$ in a set $J \neq 0$ and a complete algebra

$$\{\boldsymbol{A}, \cup, \cap, \Rightarrow, -, \mathbf{I}, \bigcup, \bigcap\} \tag{1}$$

with three binary operations $\cup$, $\cap$, $\Rightarrow$, two unary operations $-$, $\mathbf{I}$ and two infinite operations $\bigcup$, $\bigcap$ is the same as in VI, § 6: it is a mapping which assigns to every m-argument functor φ in $\mathfrak{L}$, a function $\varphi_R: J^m \to J$, and which assigns, to every m-argument predicate ϱ in $\mathfrak{L}$, a function $\varrho_R: J^m \to \boldsymbol{A}$. By VI § 6, given any realization R of $\mathfrak{L}$ in J and an algebra (1), every formula α in $\mathfrak{L}$ uniquely determines a mapping

$$\alpha_R: J^V \to \boldsymbol{A} \tag{2}$$

where V is the set of all free individual variables in $\mathfrak{L}$ (to the inductive definition of α_R on p. 229 we have to add, of course, the identity: $(\mathbf{I}\alpha)_R(v) = \mathbf{I}(\alpha_R(v))$). Points $v = \{v_x\}_{x \in V}$ in J^V are called *valuations* in J.

According to VI, § 7, we shall also admit realizations in incomplete generalized algebras (1) under the hypothesis that all infinite operations appearing in the inductive definition of α_R are always feasible.

In §§ 10-11 we shall always suppose that (1) is a non-degenerate topological Boolean algebra (3).

In particular, we can assume as algebra (1) the Q-algebra $\mathfrak{A}(\mathfrak{T})$ of a consistent modal theory $\mathfrak{T} = \{\mathfrak{L}, C_\mu, \mathcal{A}\}$ of the first order. The definition of the *canonical realization* R^0 of $\mathfrak{L}$ in the set T of all terms in $\mathfrak{L}$ and in the algebra $\mathfrak{A}(\mathfrak{T})$ is the same as in VI, § 7, p. 239. We recall the fundamental property of the canonical realization R^0: for every formula α in $\mathfrak{L}$,

$$\alpha_{R^0}(v) = \|v'\alpha\| \in \mathfrak{A}(\mathfrak{T}) \tag{3}$$

(1) Cf. McKinsey and Tarski [1].

(2) For an expository paper on modal theories of the first order, see Rasiowa [11].

(3) Realizations of modal languages of the first order in complete topological Boolean algebras were for the first time examined in Rasiowa [2].

On account of III 4.2 every realization in a topological Boolean algebra can be conceived as a realization in a complete topological Boolean algebra.

where $v'a$ denotes the result of the substitution v in a. In particular, for the identity valuation i we have

$$a_{R^0}(\mathrm{i}) = \|a\| . \tag{3'}$$

A realization R of $\mathfrak{L}$ in a set $J \neq 0$ and in a non-degenerate topological Boolean algebra is said to be a *model* for a set $\mathcal{A}$ of formulas (or: a *model* for the theory $\mathfrak{T} = \{\mathfrak{L}, C_\mu, \mathcal{A}\}$) provided that, for every a in $\mathcal{A}$ and for every valuation v: $V \to J$,

$$a_R(v) = 1 . \tag{4}$$

In particular, R is a model for a formula a, if for every v: $V \to J$ condition (4) holds. We also say in this case that a is *valid* in R.

A formula is said to be *modally valid*, or to be a *modal predicate tautology* if it is valid in every realization R in every $J \neq 0$ and in every topological Boolean algebra A in which it is possible to define R.

For instance, all logical axioms, i.e. all the formulas in $\mathfrak{L}$ which are of the form (T_1)-(T_{12}) or (M_1)-(M_4) (see p. 188 and p. 470) are modal predicate tautologies. The exact proof is based on the same arguments as the proof of the analogous statement for the modal propositional calculus (see p. 474) and VI, 6.9.

10.1. *If a is a theorem of a modal theory $\mathfrak{T} = \{\mathfrak{L}, C_\mu, \mathcal{A}\}$ of the first order, then every model R for $\mathfrak{T}$ in any topological Boolean algebra A is a model for a.*

The proof is based on the same arguments as the proof of the analogous statements 8.1 for modal theories of zero order, and VIII, 2.2, for classical theories of the first order.

A model R for a theory $\mathfrak{T} = \{\mathfrak{L}, C_\mu, \mathcal{A}\}$ is said to be *adequate* provided, for every formula a in $\mathfrak{L}$, a is a theorem in $\mathfrak{T}$ if and only if a is valid in R.

By a *topological realization* (or *topological model*) we shall mean any realization (or model) in the topological field $\mathfrak{B}(X)$ of all subsets of a topological space $X \neq 0$.

10.2 [1]. *For every modal theory $\mathfrak{T} = \{\mathfrak{L}, C_\mu, \mathcal{A}\}$ of the first order, the following conditions are equivalent:*

(i) *$\mathfrak{T}$ is consistent;*

(ii) *there exists a model for $\mathfrak{T}$;*

(iii) *there exists a model for $\mathfrak{T}$ in a complete topological Boolean algebra;*

(iv) *there exists a topological model for $\mathfrak{T}$;*

(v) *there exists an adequate enumerable topological model for $\mathfrak{T}$ in the topological field of sets $\mathfrak{B}(X_0)$ of all subsets of a set X_0 of irrational numbers*

[1] Rasiowa [2], [4], Rasiowa and Sikorski [3], Sikorski [7].

The proof is similar to that of the analogous theorem X 3.6 for intuitionistic theories.

First we shall prove that (i) implies (v). In fact, if $\mathfrak{T}$ is consistent, the Q-algebra $\mathfrak{A}(\mathfrak{T})$ is non-degenerate. By III, 10.1, there exists a topological Q-isomorphism from $\mathfrak{A}(\mathfrak{T})$ into the topological field of all subsets of a set X_0 of irrational numbers. Let R^0 be the canonical realization of $\mathfrak{L}$ in the enumerable set T of all terms and in $\mathfrak{A}(\mathfrak{T})$. The realization hR^0 (see VI, § 6, p. 232) is an adequate model for $\mathfrak{T}$ in the set T and $\mathfrak{B}(X_0)$. In fact, it follows from (3), 7.1, (8) and VI, 6.6 (or: VI, 7.3) that, for every axiom a in $\mathfrak{A}$,

$$a_{hR^0}(v) = h\big(a_{R^0}(v)\big) = h(\|v'a\|) = h(V) = X_0 .$$

Thus hR^0 is a model for $\mathfrak{T}$. On the other hand, if a formula a is valid in hR^0, then by (3′) and VI, 6.6

$$h(\|a\|) = h\big(a_{R^0}(i)\big) = a_{hR^0}(i) = X_0 . \tag{5}$$

Since h is an isomorphism, we have $\|a\| = V \in \mathfrak{A}(\mathfrak{T})$, i.e. a is a theorem in $\mathfrak{T}$ by 7.1, (8).

Obviously (v) implies (iv), (iv) implies (iii) and (iii) implies (ii). The proof that (ii) implies (i) is the same as that of the analogous statements VIII, 2.4 and X, 2.4 for classical and intuitionistic theories.

10.3. *For every formula β in a consistent modal theory $\mathfrak{T} = \{\mathfrak{L}, \mathfrak{C}_\mu, \mathfrak{A}\}$ of the first order, the following conditions are equivalent:*

(i) *β is a theorem of $\mathfrak{T}$;*

(ii) *β is valid in every model for $\mathfrak{T}$;*

(iii) *β is valid in every model for $\mathfrak{T}$ in any complete topological Boolean algebra;*

(iv) *β is valid in every topological model for $\mathfrak{T}$;*

(v) *β is valid in every model for $\mathfrak{T}$ in an enumerable set J_0 and in every topological field $\mathfrak{B}(X_0)$ where X_0 is a set of irrational numbers;*

(vi) *there exists an infinite set J such that β is valid in every model for $\mathfrak{T}$ in J and in every topological field $\mathfrak{B}(X_0)$ where X_0 is a set of irrational numbers;*

(vii) *$\beta_{R^0}(\mathrm{i}) = V$ for the canonical model R^0 for $\mathfrak{T}$ in $\mathfrak{A}(\mathfrak{T})$ and for the identity valuation* i.

The proof is similar to that of the analogous statement X, 3.7, for intuitionistic theories.

(i) implies (ii) by 10.1. Obviously (ii) implies (iii), (iii) implies (iv), and (iv) implies (v).

(v) implies (vii). To prove it, let h and X_0 have the same meaning as in the proof of 10.2. It follows easily from (3) that the canonical realization R^0 is a model for $\mathfrak{T}$. If a formula β is valid in hR^0, we infer from (5) that $\|\beta\| = \mathbb{V} \in \mathfrak{A}(\mathfrak{T})$, i.e. β is a theorem in $\mathfrak{T}$.

(vii) implies (i). This follows directly from (3′) and 7.1 (8).

(v) implies (vi) trivially. (vi) implies (v). The proof of the last statement is exactly the same as the proof of the analogous statement in X, 3.7.

10.4. *A formula β in $\mathfrak{L}$ is a theorem in a modal theory $\mathfrak{T} = \{\mathfrak{L}, C_\mu, \mathcal{A}\}$ if and only if the formula $\mathbf{I}\beta$ is a theorem in $\mathfrak{T}$.*

The proof based on 10.3 is similar to that of the analogous statement 8.4 for modal theories of zero order.

Denote now by $\mathfrak{L}^*$ the formalized language (of the first order) obtained from $\mathfrak{L}$ by rejection of the sign **I** in the alphabet of $\mathfrak{L}$. Thus $\mathfrak{L}^*$ is a language of the type investigated in Chapters VIII and X. The set of all formulas in $\mathfrak{L}^*$ coincides with the set of all formulas in $\mathfrak{L}$ which do not contain the sign **I**.

Any set $\mathcal{A}$ of formulas in $\mathfrak{L}$ uniquely determines two theories of the first order: the modal theory $\mathfrak{T} = \{\mathfrak{L}, C_\mu, \mathcal{A}\}$ and the classical theory $\mathfrak{T}^* = \{\mathfrak{L}^*, C, \mathcal{A}\}$. The following theorem stating that $\mathfrak{T}$ is an inessential extension of $\mathfrak{T}^*$ can be deduced from 10.1 and VIII, 5.5 exactly in the same way as the analogous statement 8.5 for theories of zero order was deduced from 8.1 and VII, 7.8.

10.5. *For every formula β in $\mathfrak{L}^*$, the following conditions are equivalent:*

(i) *β is a theorem in $\mathfrak{T} = \{\mathfrak{L}, C_\mu, \mathcal{A}\}$;*

(ii) *β is a theorem in $\mathfrak{T}^* = \{\mathfrak{L}^*, C, \mathcal{A}\}$.*

In other words,

$$C_{\mathfrak{L}^*}(\mathcal{A}) = C_{\mu\mathfrak{L}}(\mathcal{A}) \cap F^* \tag{6}$$

where F^* is the set of all formulas in $\mathfrak{L}^*$, $\mathcal{A} \subset F^*$.

To establish a connection between intuitionistic theories and modal theories of the first order we have to define a mapping $\mathfrak{J}$ which, to every formula α in $\mathfrak{L}^*$, assigns a formula $\mathfrak{J}\alpha$ in $\mathfrak{L}$. The definition of $\mathfrak{J}$ is by induction on the length of formulas, as follows:

$\mathfrak{J}\alpha$ is the formula $\mathbf{I}\alpha$ for every elementary formula α;

$\mathfrak{J}(\alpha \cup \beta)$ is the formula $(\mathfrak{J}\alpha \cup \mathfrak{J}\beta)$;

$\mathfrak{J}(\alpha \cap \beta)$ is the formula $(\mathfrak{J}\alpha \cap \mathfrak{J}\beta)$;

$\mathfrak{J}(\alpha \Rightarrow \beta)$ is the formula $\mathbf{I}(\mathfrak{J}\alpha \Rightarrow \mathfrak{J}\beta)$;

$\mathfrak{J}-\alpha$ is the formula $\mathbf{I}-\mathfrak{J}\alpha$;

$\mathfrak{J}\bigcup_\xi \alpha(\xi)$ is the formula $\bigcup_\xi \mathfrak{J}\alpha(\xi)$;

$\mathfrak{J}\bigcap_\xi \alpha(\xi)$ is the formula $\mathbf{I}\bigcap_\xi \mathfrak{J}\alpha(\xi)$.

If $\mathcal{A}$ is any set of formulas in $\mathfrak{L}^*$, then $\exists\mathcal{A}$ denotes the set of all formulas $\exists a$ where a is in $\mathcal{A}$. By definition, $\exists\mathcal{A}$ is a set of formulas in $\mathfrak{L}$. Thus any set $\mathcal{A}$ of formulas in $\mathfrak{L}^*$ uniquely determines two theories of the first order: the intuitionistic theory $\mathfrak{T}^* = \{\mathfrak{L}^*, C_\iota, \mathcal{A}\}$ and the modal theory $\mathfrak{T} = \{\mathfrak{L}, C_\mu, \exists\mathcal{A}\}$.

10.6. (1) *For every formula β in $\mathfrak{L}^*$, the following conditions are equivalent:*

(i) *$\exists\beta$ is a theorem in $\mathfrak{T} = \{\mathfrak{L}, C_\mu, \exists\mathcal{A}\}$;*

(ii) *β is a theorem in $\mathfrak{T}^* = \{\mathfrak{L}^*, C_\iota, \mathcal{A}\}$.*

The proof is similar to that of the analogous theorem 8.6 for theories of zero order. It is based on the following identity:

$$(\exists a)_R(v) = a_{\mathbf{I}R}(v) \tag{7}$$

for every realization R in a set $J \neq 0$ and a complete topological Boolean algebra $\boldsymbol{A}$, for every valuation v in J and for every formula a in $\mathfrak{L}^*$. Here the symbol $\mathbf{I}R$ obviously denotes the realization of $\mathfrak{L}^*$ in J and the algebra $\mathfrak{G}(\boldsymbol{A})$ of all open elements in $\boldsymbol{A}$, defined as follows:

$$\varphi_{\mathbf{I}R} = \varphi_R \quad \text{for every functor } \varphi \text{ in } \mathfrak{L}^*,$$

$$\varrho_{\mathbf{I}R} = \mathbf{I}\varrho_R \quad \text{for every predicate } \varrho \text{ in } \mathfrak{L}^*.$$

(see the similar notation in VI, § 6, p. 232). The mapping $a_{\mathbf{I}R}$ is here understood as in X, § 2, p. 414. The proof of (7)—by induction on the length of a—is omitted. It is based on theorems IV, 1.4 and IV, 7.5.

If $\exists\beta$ is not a theorem in $\mathfrak{T}$, then by **10.3** there exists a model R for $\mathfrak{T}$ in a set $J \neq 0$ and a complete topological Boolean algebra $\boldsymbol{A}$ and a valuation v_0 such that $(\exists\beta)_R(v_0) \neq V$. Assuming as a any formula in $\mathcal{A}$, we infer from (7) that $\mathbf{I}R$ is a model for $\mathfrak{T}^*$. Replacing a by β we infer from (7) that $\beta_{\mathbf{I}R}(v_0) \neq V$, i.e. β is not valid in $\mathbf{I}R$. By X, 2.2, β is not a theorem in $\mathfrak{T}^*$.

Suppose now that β is not a theorem in $\mathfrak{T}^*$. By X, 3.7, there exists a model R for the intuitionistic theory $\mathfrak{T}^*$ in a set $J \neq 0$ and the pseudo-Boolean algebra $\mathfrak{G}(X)$ of all open subsets of a topological space X, and a valuation v_0 such that $\beta_R(v_0) \neq V$ (here the symbol β_R has the meaning explained in X, § 2). The mapping R can also be understood as a realization R of $\mathfrak{L}$ in $J \neq 0$ and the topological field $\mathfrak{B}(X)$. Moreover $\mathbf{I}R = R$ since $\mathbf{I}G = G$ for every open set $G \subset X$. Assuming as a any formula in $\mathcal{A}$ we infer from (7) that the realization R of $\mathfrak{L}$ in J and $\mathfrak{B}(X)$ is a model

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(1) Rasiowa and Sikorski [3].

for the modal theory $\mathfrak{T}$. Replacing α by β we infer from (7) that $\exists\beta$ is not valid in the model R for $\mathfrak{T}$. By 10.1, $\exists\beta$ is not any theorem in $\mathfrak{T}$.

It follows from 10.5 and 10.6 that the examination of classical and intuitionistic theories can always be reduced to the examination of some parts of corresponding modal theories.

§ 11. The modal predicate calculus. The following theorem called the *completeness theorem for modal predicate calculi* $\mathbb{S}_\mu = \{\mathfrak{L}, C_\mu, 0\}$ is the modal analogue of the completeness theorem VIII, 6.1, for classical predicate calculi and theorem X, 4.1, for intuitionistic predicate calculi.

11.1 (1). *For every formula β in a modal language $\mathfrak{L}$ of the first order, the following conditions are equivalent:*

(i) *β is derivable in $\mathbb{S}_\mu = \{\mathfrak{L}, C_\mu\}$;*

(ii) *β is valid in every realization of $\mathfrak{L}$;*

(iii) *β is valid in every realization of $\mathfrak{L}$ in any complete topological Boolean algebra;*

(iv) *β is valid in every topological realization of $\mathfrak{L}$;*

(v) *β is valid in every realization of $\mathfrak{L}$ in an enumerable set J_0 and in every topological field $\mathfrak{B}(X)$ where X is a set of irrational numbers;*

(vi) *there is an infinite set J such that β is valid in every realization of $\mathfrak{L}$ in J and in every topological field $\mathfrak{B}(X)$ where X is a set of irrational numbers;*

(vii) *$\beta_{R^0}(\mathrm{i}) = V$ for the canonical model R^0 in $\mathfrak{A}(\mathbb{S}_\mu)$, and the identity valuation* i.

Theorem 11.1 follows directly from 10.3 (the case in which $\mathcal{A}$ is the empty set).

We shall complete theorem 11.1 with the following remark.

11.2 (2). *There exists a metric space X_0 such that any formula β in $\mathfrak{L}$ is a modal predicate tautology if and only if β is valid in every realization of $\mathfrak{L}$ in an enumerable set J and in $\mathfrak{B}(X_0)$.*

Theorem 11.2 follows directly from 10.2, (i), (v) and 11.1. It follows from 10.2 that we may additionally assume that X_0 is a set of irrational numbers. However, the space X_0 mentioned in 11.2 cannot be a complete metric space or a topologically complete space (see III, § 11). Since the space of all real numbers is complete and the space of all irrational numbers is topologically complete (see III, § 11, p. 118), they cannot be assumed as the space X_0 in 11.2.

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(1) Rasiowa [2], Rasiowa and Sikorski [3], and Sikorski [7].

(2) Sikorski [7]. See also Rasiowa and Sikorski [3].

In fact, consider the following formula (1)

$$-\mathbf{I}\bigcup_{\xi}(\varrho(\xi)\cap -\mathbf{I}-\mathbf{I}-\varrho(\xi)) \tag{1}$$

where ϱ is a one-argument predicate. It follows directly from III, § 11, (5) that (1) is valid in every realization R in any enumerable set and any topologically complete space. However, (1) is not a modal tautology since the example III, § 11, (6) shows that (1) is not valid in all realizations.

11.3. *A formula β is a modal tautology if and only if the formula $\mathbf{I}\beta$ is a modal tautology.*

This follows directly from 10.4 and 11.1 (or it can be deduced immediately from III, § 1, (i_2), (i_4) and the definition of modal tautologies).

We recall (see § 10, p. 484) that $\mathfrak{L}^*$ denotes the language obtained from $\mathfrak{L}$ by rejection of the sign **I**. The following theorem follows directly from 10.5 (the case in which $\mathcal{A}$ is the empty set) and 11.1, VIII, 6.1

11.4. *For every formula β in $\mathfrak{L}^*$, the following conditions are equivalent:*

(i) *β is a classical tautology;*

(ii) *β is a modal tautology.*

For every formula α in $\mathfrak{L}^*$, let $\mathfrak{I}\alpha$ be the formula (in $\mathfrak{L}$) defined in § 10, p. 484. The following theorem results immediately from 10.6, (the case in which $\mathcal{A}$ is the empty set), 11.1 and X, 4.1:

11.5. *For every formula β in $\mathfrak{L}^*$, the following conditions are equivalent:*

(i) *β is an intuitionistic tautology;*

(ii) *$\mathbf{I}\beta$ is a modal tautology* (2).

The intuitive meaning of 11.4 and 11.5 is that both the classical predicate calculus and the intuitionistic predicate calculus can be conceived of as parts of the model predicate calculus.

The following theorems are modal analogous to theorems X, 8.1 and X, 8.2 for the intuitionistic predicate calculus.

11.6. *For any formulas α, β in $\mathfrak{L}$ the disjunction $(\mathbf{I}\alpha\cup\mathbf{I}\beta)$ is a modal predicate tautology if and only if either α or β is a modal predicate tautology.*

11.7. *An existential formula $\bigcup_{\xi}\mathbf{I}\beta(\xi)$ in $\mathfrak{L}$ is a predicate modal tautology if and only if there exists a term τ in $\mathfrak{L}$ such that $\beta(\tau)$ is a modal predicate tautology* (3).

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(1) This counterexample was given by Rasiowa and Sikorski [3] using a remark in Kuratowski [3] (first volume, p. 323) on the Baire theorem.

(2) Rasiowa and Sikorski [3].

(3) Theorems 11.6 and 11.7 were proved by Rasiowa and Sikorski [5].

The proof of 11.6 and 11.7 is an easy modification of the proofs of X, 8.1 and X, 8.2 respectively. The direction of the modification was shown in the proof of the analogous theorem 9.5 (on the modal propositional calculus) which was obtained as a similar modification of the proof of the corresponding theorem IX, 6.1, on the intuitionistic propositional calculus.

11.8. *The* Q*-algebra* $\mathfrak{A}(\mathfrak{S}_\mu)$ *of the modal predicate calculus is a generalized free algebra in the class of all complete topological Boolean algebras. The class of all elements* $\|\alpha\|$ *where* α *is any elementary formula is a set of free generators.*

The proof of 11.8 is the same as that of the analogous statements VIII, 24.1 and X, 5.1.

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